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SEMIPARAMETRIC ESTIMATION IN THE GENERALIZED ADDITIVE–MULTIPLICATIVE MODEL

1. Introduction

Suppose that a time to failure $T_{\mathbf{x}(\cdot)}$ is a non-negative random variable with the survival function $S_{\mathbf{x}(\cdot)}(t) = \mathbf{P}\{T_{\mathbf{x}(\cdot)} > t\}$ which depends on a vector of *stresses or covariates*

$$\mathbf{x} : [0, +\infty) \to B \subset \mathbb{R}^m.$$

Lin and Ying (1996) considered the additive-multiplicative model: for all x from some set of stresses

$$\alpha_{\mathbf{x}(\cdot)}(t) = r[\mathbf{x}(t)]\alpha_0(t) + \rho(\mathbf{x}(t)),$$

where $\alpha_{\mathbf{x}(\cdot)}/S_{\mathbf{x}(\cdot)}$ is the hazard rate under the stress $\mathbf{x}(\cdot)$, $r > 0$ and ρ are some functions on B and α_0 is the baselibe hazard rate function. Particular cases of this model are the proportional hazards (Cox (1972)) and the additive hazards (see Lin and Ying (1994)) models.

Consider generalizations of this model.

The random variable

$$R^U = 1 - S_{\mathbf{x}(\cdot)}(T_{\mathbf{x}(\cdot)})$$

we'll call *the uniform resource* used till the failure $T_{\mathbf{x}(\cdot)}$ or briefly the *resourse*. It takes values in the interval [0,1) and does not depend on $\mathbf{x}(\cdot)$. Note that $T_{\mathbf{x}(\cdot)} = t$ if and only if $R^U = 1 - S_{\mathbf{x}(\cdot)}(t)$. So the number $1 - S_{\mathbf{x}(\cdot)}(t) \in [0, 1)$ is called *the uniform resource used until the moment t under the stress* $\mathbf{x}(\cdot)$. The concrete item which failed at the moment t under the stress $\mathbf{x}(\cdot)$ used $1 - S_{\mathbf{x}(\cdot)}(t)$ of the resource.

Instead of the uniform resource one can define a resource with any probability distribution, so we can consider a whole class of resources. Really, suppose that G is some fixed survival function, strictly decreasing and continuous on the support $[a, b]$, $-\infty \leqslant a < b \leqslant \infty$, $G(a) = 1$, $G(b) = 0$. Denote $H = G^{-1}$ the inverse function of G. The functional

$$f_{\cdot}^{G}(\cdot) : (\mathbf{x}(\cdot), t) \mapsto f_{\mathbf{x}(\cdot)}^{G}(t)$$

defined by

$$f^G_{\mathbf{x}(\cdot)}(t) = (H \circ S_{\mathbf{x}(\cdot)})(t),$$

is called the *G-transfer functional*. It takes values in $[a, b]$.

The survival function of the random variable

$$R^G = f^G_{\mathbf{x}(\cdot)}(T_{\mathbf{x}(\cdot)})$$

is G and does not depend on $\mathbf{x}(\cdot)$. If $[a, b] = \mathbf{R}^+$, the random variable R^G can be called the *G-resource* and the number $f^G_{\mathbf{x}(\cdot)}(t)$ is called the *G-resource used till the moment* t.

Following Bagdonavičius and Nikulin (1996) denote $\mathcal{G}$ the family of survival functions, continuous and decreasing on their supports. Consider the class of transfer functionals:

$$\mathcal{M} = \{f^G, \, G \in \mathcal{G}\}.$$

Definition. *The generalized G-additive-multiplicative model (GAM) holds on $E \subset B$ if there exist a transfer functional $f^G \in \mathcal{M}$, a positive function r and a function ρ on E, and also a survival function S_0 such that for all $\mathbf{x}(\cdot) \in E$*

$$\frac{\partial f^G_{\mathbf{x}(\cdot)}(t)}{\partial t} = r[\mathbf{x}(t)] \frac{\partial f^G_0(t)}{\partial t} + \rho(\mathbf{x}(t)) \tag{1}$$

with the initial conditions $f^G_0(0) = f^G_{\mathbf{x}(\cdot)}(0) = a$.

So the stress influences the rate of resource using as multiplicatively as additively. The equation (1) implies that

$$f^G_{\mathbf{x}(\cdot)}(t) = \int_0^t r[\mathbf{x}(\tau)] df^G_0(\tau) + \int_0^t \rho(\mathbf{x}(\tau)) d\tau. \tag{2}$$

In the particular case of the exponential resource

$$f^G_{\mathbf{x}(\cdot)}(t) = -\ln S_{\mathbf{x}(\cdot)}(t)$$

and we obtain the model of Lin and Ying (1994).

2. Estimation

Consider the GAM model with some specified G and an unknown baseline survival function S_0.

The functions $r[\mathbf{x}(\cdot)]$ and $\rho[\mathbf{x}(\cdot)]$ are parametrized as follows:

$$r[\mathbf{x}(t)] = e^{\boldsymbol{\beta}^T \mathbf{z}(t)}, \quad \rho[\mathbf{x}(t)] = \boldsymbol{\gamma}^T \mathbf{w}(t),$$

where $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^T$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_s)^T$ are vectors of unknown regression parameters, $\mathbf{x}^T(t) = (\mathbf{z}^T(t), \mathbf{w}^T(t))$. Denote $\boldsymbol{\theta}^T = (\boldsymbol{\beta}^T, \boldsymbol{\gamma}^T)$. So the following model is considered: for all $\mathbf{x}(\cdot) \in E$

$$S_{\mathbf{x}(\cdot)}(t) = G\left\{ \int_0^t e^{\boldsymbol{\beta}^T \mathbf{z}(\tau)} dH(S_0(\tau)) + \int_0^t \boldsymbol{\gamma}^T \mathbf{w}(\tau) d\tau \right\}. \qquad (3)$$

Suppose that n_i items are tested under the stress $\mathbf{x}_i(\cdot)$ $(i = 1, 2, \dots, k)$, $n = \sum_{i=1}^{k} n_i$. Denote T_{ij} $(i = 1, \dots, k;\ j = 1, \dots, n_i)$ the moments of failures (may be not all observed) and L_{ij} the censoring times. If $T_{ij} > L_{ij}$, the failure is not observed and the moment of censoring L_{ij} is known.

The data consist of the random vectors (X_{ij}, δ_{ij}), where $\delta_{ij} = I(T_{ij} \leqslant L_{ij})$ is the indicator of the event $(T_{ij} \leqslant L_{ij})$, $X_{ij} = T_{ij} \wedge L_{ij}$.

Denote

$$N_{ij}(t) = I\{X_{ij} \leqslant t,\ \delta_{ij} = 1\}, \quad Y_{ij}(t) = I\{X_{ij} \geqslant t\}.$$

Suppose that $\mathcal{F}_t$, $t \geqslant 0$ is the filtration generated by

$$\{N_{ij}(s), Y_{ij}(s),\ i = 1, \dots, k;\ j = 1, \dots, n_k;\ 0 \leqslant s \leqslant t\}$$

and Λ_{ij} are the compensators of the processes N_{ij} which are absolutely continuous:

$$\Lambda_{ij}(t) = \int_0^t \lambda_{ij}(s) ds.$$

Assume that censoring is independent, i.e., the random intensities λ_{ij} satisfy the *multiplicative intensity model*

$$\lambda_{ij}(t) = \alpha_i(t) Y_{ij}(t), \qquad (4)$$

where $\alpha_i(t) = \alpha_{\mathbf{x}_i}(t)$ are the hazard rates of the random variables T_{ij}.

Denote

$$N_i(t) = \sum_{j=1}^{n_i} N_{ij}(t), \quad N(t) = \sum_{i=1}^{k} N_i(t), \quad \lambda_i(t) = \alpha_i(t) Y_i(t),$$

$$Y_i(t) = \sum_{j=1}^{n_i} Y_{ij}(t), \quad Y(t) = \sum_{i=1}^{k} Y_i(t),$$

$$\Lambda_i(t) = \int_0^t \lambda_i(s)ds, \quad \Lambda(t) = \sum_{i=1}^{k} \Lambda_i(t),$$

$$M_i(t) = N_i(t) - \Lambda_i(t), \quad M(t) = \sum_{i=1}^{k} M_i(t), \quad J(s) = I(Y(s) > 0),$$

$$\alpha = -\frac{G'}{G}, \quad \psi = \alpha \circ H.$$

The Doob-Meyer decomposition $N = M + \Lambda$ and the equality (4) imply

$$\int_0^t \frac{J(u)(dN(u) - S_*^{(0)}(\boldsymbol{\theta}, u)du)}{S^{(0)}(\boldsymbol{\theta}, u)} = \int_0^t J(u)dH(S_0(u)) + \int_0^t \frac{J(u)dM(u)}{S^{(0)}(\boldsymbol{\theta}, u)},$$

where $S_i = S_{x_i(\cdot)}$ and

$$S^{(0)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} Y_i(u) e^{\boldsymbol{\beta}^T \mathbf{z}_i(u)} \psi(S_i(u)),$$

$$S_*^{(0)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} Y_i(u) \gamma^T \mathbf{w}_i(u) \psi(S_i(u)).$$

M is a martingale, therefore under some mild assumtions

$$\mathbf{E} \int_0^t \frac{J(u)(dN(u) - S_*^{(0)}(\boldsymbol{\theta}, u)du)}{S^{(0)}(\boldsymbol{\theta}, u)} = \mathbf{E} \int_0^t J(u)dH(S_0(u)). \tag{5}$$

If $Y(t) > 0$, then

$$\int_0^t J(u)dH(S_0(u)) = H(S_0(t)). \tag{6}$$

The equialities (5) and (6) imply an estimator for $S_0(t)$. Really, the survival functions S_i can be estimated by the Kaplan–Meier estimators

$$\widehat{S}_i(t) = \prod_{u \leqslant t} \left(1 - \frac{\Delta N_i(u)}{Y_i(u)}\right)$$

where $\Delta N_i(u) = N_i(u) - N_i(u-0)$. So the equalities (5) and (6) imply that a reasonable estimator for $H(S_0(t))$ (still depending on $\boldsymbol{\theta}$) is

$$H(\widehat{S}_0(t)) = \int_0^t J(u) \frac{dN(u) - tS_*^{(0)}(\boldsymbol{\theta}, u)du}{\widetilde{S}^{(0)}(\boldsymbol{\theta}, u)},$$

where

$$\widetilde{S}^{(0)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} Y_i(u) e^{\boldsymbol{\beta}^T \mathbf{z}_i(u)} \psi(\widehat{S}_i(u)),$$

$$\widetilde{S}_*^{(0)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} Y_i(u) \boldsymbol{\gamma}^T \mathbf{w}_i(u) \psi(\widehat{S}_i(u)).$$

The estimator $\widetilde{\boldsymbol{\theta}}$ of the parameter $\boldsymbol{\theta}_0$ can be obtained by maximizing the observed likelihood function

$$\begin{aligned} L(\boldsymbol{\theta}) = \prod_{i=1}^{k} \prod_{j=1}^{n_i} \Bigg[G \left\{ \int_0^{X_{ij}-} e^{\boldsymbol{\beta}^T \mathbf{z}_i(u)} d(H \circ \widehat{S}_0(u) + \int_0^{X_{ij}-} \boldsymbol{\gamma}^T \mathbf{w}_i(u) du \right\} \\ - G \left\{ \int_0^{X_{ij}} e^{\boldsymbol{\beta}^T \mathbf{z}_i(u)} d(H \circ \widehat{S}_0(u) \int_0^{X_{ij}} \boldsymbol{\gamma}^T \mathbf{w}_i(u) du \right\} \Bigg]^{\delta_{ij}} \\ \times \left[G \left\{ \int_0^{X_{ij}} e^{\boldsymbol{\beta}^T \mathbf{z}_i(u)} d(H \circ \widehat{S}_0(u) \int_0^{X_{ij}} \boldsymbol{\gamma}^T \mathbf{w}_i(u) du \right\} \right]^{1-\delta_{ij}} . \end{aligned}$$

Another, more simple method of estimation can be obtained in the following way.

Denote $\tau^* = \max_{i,j} X_{ij}$ the time of observation of the processes N and Y. Suppose that at the non-random moment τ all the experiments are censored, so $\tau^* \leqslant \tau$. Denote $\tau_i^* = \max_j X_{ij}$ the times of observation of the processes N_i and Y_i. We propose to estimate the parameter $\boldsymbol{\theta}$ by solving the estimating equations

$$\mathbf{U}(\boldsymbol{\theta}, \tau) = 0,$$

where

$$\mathbf{U}(\boldsymbol{\theta}, t) = \sum_{i=1}^{k} \int_0^t \mathbf{x}_i(u) \big\{ dN_i(u) - Y_i(u)\psi(\widehat{S}_i(u-))$$
$$\times \left[e^{\boldsymbol{\beta}^T \mathbf{z}_i(u)} dH(\widehat{S}_0(u)) + \boldsymbol{\gamma}^T \mathbf{w}_i(u) du \right] \big\}.$$

Note that the value of $\widehat{\mathbf{S}}_i(u-)$ for $u > \tau_i^*$ is not important here since $Y_i(u) = 0$ for $u > \tau_i^*$. These equations generalise the estimating equations of Lin and Ying (1994) for the additive hazards model (taking $\psi(t) \equiv 1$, $\boldsymbol{\beta} = \mathbf{0}$), the partial score function for the proportional hazards model (taking $\psi(t) \equiv 1$, $\boldsymbol{\gamma} = \mathbf{0}$), the estimating equation of Lin and Ying (1996) for the additive-multiplicative model (taking $\psi(t) \equiv 1$.

Denote

$$\widetilde{\mathbf{S}}^{(1)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} \mathbf{x}_i(u) Y_i(u) e^{\boldsymbol{\beta}^T \mathbf{z}_i(u)} \psi(\widehat{S}_i(u-)),$$
$$\widetilde{\mathbf{S}}^{(11)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} \mathbf{z}_i(u) Y_i(u) e^{\boldsymbol{\beta}^T \mathbf{z}_i(u)} \psi(\widehat{S}_i(u-)),$$
$$\widetilde{\mathbf{S}}^{(12)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} \mathbf{w}_i(u) Y_i(u) e^{\boldsymbol{\beta}^T \mathbf{z}_i(u)} \psi(\widehat{S}_i(u-)),$$
$$\widetilde{\mathbf{S}}_*^{(1)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} \mathbf{x}_i(u) Y_i(u) \boldsymbol{\gamma}^T \mathbf{w}_i(u) \psi(\widehat{S}_i(u-),$$

$$\widetilde{\mathbf{E}}(\theta, u) = \frac{\widetilde{\mathbf{S}^{(1)}}(\theta, u)}{\widetilde{S}^{(0)}(\theta, u)}, \quad \widetilde{\mathbf{E}}_1(\boldsymbol{\theta}, u) = \frac{\widetilde{\mathbf{S}^{(11)}}(\theta, u)}{\widetilde{S}^{(0)}(\theta, u)},$$
$$\widetilde{\mathbf{E}}_2(\theta, u) = \frac{\widetilde{\mathbf{S}^{(12)}}(\theta, u)}{\widetilde{S}^{(0)}(\theta, u)}.$$

Then the function $\mathbf{U}(\boldsymbol{\theta}, t)$ can be written

$$\mathbf{U}(\boldsymbol{\theta}, t) = \sum_{i=1}^{k} \int_0^t J(u) \{ \mathbf{x}_i(u) - \widetilde{\mathbf{E}}(\boldsymbol{\theta}, u) \}$$
$$\times \{ dN_i(u) - Y_i(u) \psi(\widehat{S}_i(u-)) \boldsymbol{\gamma}^T \mathbf{w}_i(u) du \}. \tag{7}$$

Denote $\widehat{\boldsymbol{\theta}}^T = (\widehat{\boldsymbol{\beta}}^T, \widehat{\boldsymbol{\gamma}}^T)$ the estimator satisfying the equation (7). Note that the estimator $\widehat{\boldsymbol{\gamma}}$ is an explicit function of $\widehat{\boldsymbol{\beta}}$:

$$\widehat{\boldsymbol{\gamma}} = \left[\sum_{i=1}^{k}\int_0^\tau J(u)(\mathbf{w}_i^T(u) - \widetilde{\mathbf{E}}_2(\widehat{\boldsymbol{\beta}}, u))\mathbf{w}_i(u)Y_i(u)\psi(\widehat{\mathbf{S}}_i(u-))du\right]^{-1} \times \left[\sum_{i=1}^{k}\int_0^\tau J(u)(\mathbf{w}_i(u) - \widetilde{\mathbf{E}}_2(\widehat{\boldsymbol{\beta}}, u))dN_i(u)\right].$$

The estimator of the baseline survival function is

$$\widetilde{S}_0(t) = G\left\{\int_0^t J(u)\frac{dN(u) - \widetilde{S}_*^{(0)}(\widehat{\boldsymbol{\theta}}, u)du}{\widetilde{S}^{(0)}(\widehat{\boldsymbol{\theta}}, u)}\right\} \tag{8}$$

and the estimator of the survival function under any stress $\mathbf{x}(\cdot) \in E$ is

$$\begin{aligned}\widehat{S}_{\mathbf{x}(\cdot)}(t) &= G\left\{\int_0^t e^{\widehat{\boldsymbol{\beta}}^T\mathbf{z}(u)}dH(\widehat{S}_0(u)) + \int_0^t \widehat{\boldsymbol{\gamma}}^T\mathbf{w}(u)du\right\} \\ &= G\left\{\int_0^t e^{\widehat{\boldsymbol{\beta}}^T\mathbf{z}(u)}\frac{dN(u) - \widetilde{S}_*^{(0)}(\widehat{\boldsymbol{\theta}}, u)du}{\widetilde{S}^{(0)}(\widehat{\boldsymbol{\theta}}, u)} \int_0^t \widehat{\boldsymbol{\gamma}}^T\mathbf{w}(u)du\right\}.\end{aligned} \tag{9}$$

3. Asymptotic properties of estimators

Denote $\|\mathbf{A}\| = \sup\limits_{i,j}|a_{ij}|$ the norm of any matrix $\mathbf{A}$, $\boldsymbol{\theta}_0$ – the true value of the parameter $\boldsymbol{\theta}$,

$$\mathbf{S}^{(1)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k}\mathbf{x}_i(u)Y_i(u)e^{\boldsymbol{\beta}^T\mathbf{z}_i(u)}\psi(S_i(u)),$$

$$\mathbf{S}^{(11)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k}\mathbf{z}_i(u)Y_i(u)e^{\boldsymbol{\beta}^T\mathbf{z}_i(u)}\psi(S_i(u)),$$

$$\mathbf{S}_*^{(1)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k}\mathbf{x}_i(u)Y_i(u)\psi(S_i(u))\boldsymbol{\gamma}^T\mathbf{w}_i(u), \quad \mathbf{E}(\boldsymbol{\theta}, u) = \frac{\mathbf{S}^{(1)}(\boldsymbol{\theta}, u)}{S^{(0)}(\boldsymbol{\theta}, u)},$$

$$\mathbf{E}_1(\boldsymbol{\theta}, u) = \frac{\mathbf{S}^{(11)}(\boldsymbol{\theta}, u)}{S^{(0)}(\boldsymbol{\theta}, u)}, \quad \mathbf{S}_*^{(2)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} \mathbf{x}_i(u)\mathbf{w}_i^T(u)Y_i(u)\psi(S_i(u)),$$

$$\mathbf{S}^{(2)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} \mathbf{x}_i(u)\mathbf{z}_i^T(u)Y_i(u)e^{\boldsymbol{\beta}^T\mathbf{z}_i(u)}\psi(S_i(u)).$$

Assumptions A.

a) *for all* $i = 1, 2, \ldots, k$ *there exist nonnegative functions* y_i, *continuous on their supports* $[0, \tau_i]$, *and constants* $l_i \in]0, 1[$ *such that, as* $n \to \infty$,

$$\frac{n_i}{n} \to l_i, \quad \sup_{0 \leqslant t < \infty} \left| \frac{Y_i(t)}{n_i} - y_i(t) \right| \xrightarrow{\mathbf{P}} 0;$$

b) $H(S_0(\tau)) < \infty$;

c) ψ *is a positive continuously differentiable on* $]0, 1[$ *and* ψ, ψ'/ψ *are bounded in the neighborhood of* 1.

d) $\max\limits_{i,j} \sup\limits_{t \in [0,\tau]} |x_{ij}(t)| \leqslant C_1$;

e) $\min\limits_{1 \leqslant i \leqslant k} S_i(\tau) > 0$.

Denote

$$s^{(0)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} l_i y_i(u)e^{\boldsymbol{\beta}^T\mathbf{z}_i(u)}\psi(S_i(u)),$$

$$s^{(1)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} l_i \mathbf{x}_i(u)e^{\boldsymbol{\beta}^T\mathbf{z}_i(u)}\psi(S_i(u)),$$

$$\mathbf{s}_*^{(1)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} l_i \mathbf{x}_i(u)y_i(u)\psi(S_i(u))\boldsymbol{\gamma}^T\mathbf{w}_i(u),$$

$$s_*^{(11)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} l_i \mathbf{w}_i(u)y_i(u)\psi(S_i(u)),$$

$$\mathbf{s}^{(11)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} l_i \mathbf{z}_i(u)y_i(u)e^{\boldsymbol{\beta}^T\mathbf{z}_i(u)}\psi(S_i(u)), \quad \mathbf{e}(\boldsymbol{\theta}, u) = \frac{\mathbf{s}^{(1)}(\boldsymbol{\beta}, u)}{s^{(0)}(\boldsymbol{\beta}, u)},$$

$$\mathbf{e}_1(\boldsymbol{\theta}, u) = \frac{\mathbf{s}^{(11)}(\boldsymbol{\beta}, u)}{s^{(0)}(\boldsymbol{\beta}, u)}, \quad \mathbf{s}_*^{(2)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} l_i \mathbf{x}_i(u)\mathbf{w}_i(u)y_i(u)\psi(S_i(u)),$$

$$\mathbf{s}^{(2)}(\boldsymbol{\theta}, u) = \sum_{i=1}^{k} l_i \mathbf{x}_i(u)\mathbf{z}_i^T(u)y_i(u)e^{\boldsymbol{\beta}^T\mathbf{z}_i(u)}\psi(S_i(u)).$$

We always suppose that $0/0 = 0$.

Under assumptions A we have that there exists a neighborhood B of $\boldsymbol{\theta}_0$ such that for $m = 0, 1, 2, 11$:

$$\sup_{\boldsymbol{\theta}\in B, t\in[0,\tau]} \left\| \frac{1}{n}\mathbf{S}^{(m)}(\boldsymbol{\theta}, t) - \mathbf{s}^{(m)}(\boldsymbol{\theta}, t) \right\| \xrightarrow{\mathbf{P}} 0,$$

$$\sup_{\boldsymbol{\theta}\in B, t\in[0,\tau]} ||\mathbf{E}(\boldsymbol{\theta}, t) - \mathbf{e}(\boldsymbol{\theta}, t)|| \xrightarrow{\mathbf{P}} 0,$$

$$\sup_{\boldsymbol{\theta}\in B, t\in[0,\tau]} ||\mathbf{E}_1(\boldsymbol{\theta}, t) - \mathbf{e}_1(\boldsymbol{\theta}, t)|| \xrightarrow{\mathbf{P}} 0 \text{ as } n \to \infty,$$

$$\sup_{\boldsymbol{\theta}\in B, t\in[0,\tau]} ||\mathbf{E}(\boldsymbol{\theta}, t)|| \leqslant C_1, \qquad \sup_{\boldsymbol{\theta}\in B, t\in[0,\tau]} ||\mathbf{E}_1(\boldsymbol{\theta}, t)|| \leqslant C_1.$$

Denote $\mathbf{A}^{\otimes 2} = \mathbf{A}\mathbf{A}^T$ for any matrix $\mathbf{A}$.

Theorem 1. *Assume conditions A hold. Then, as $n \to \infty$*

$$n^{-1/2}\mathbf{U}(\boldsymbol{\theta}_0, \tau) \xrightarrow{D} N(\mathbf{0}, \boldsymbol{\Sigma}(\boldsymbol{\theta}_0, \tau)), \tag{10}$$

where

$$\boldsymbol{\Sigma}(\boldsymbol{\theta}_0, t)) = \sum_i \int^{t\wedge\tau_i} \Bigg\{ \mathbf{x}_i(u) - \mathbf{e}(\boldsymbol{\theta}_0, u) + \int_u^{t\wedge\tau_i} (\mathbf{x}_i(s) - \mathbf{e}(\boldsymbol{\theta}_0, s)) \times$$
$$\psi'(S_i(s))S_i(s)\frac{y_i(s)}{y_i(u)} \left[e^{\boldsymbol{\beta}_0^T \mathbf{z}_i(s)} dH(S_0(s)) + \boldsymbol{\gamma}_0^T \mathbf{w}_i(s) ds \right] \Bigg\}^{\otimes 2} y_i(u)\alpha_i(u)du. \tag{11}$$

Proof. The Doob–Meyer decomposition and the equality (19) imply that the function $U(\boldsymbol{\theta}_0, t)$ can be written in the following from:

$$\mathbf{U}(\boldsymbol{\theta}_0, t) = \sum_i \int_0^t J(s)(\mathbf{x}_i(s) - \mathbf{E}(\boldsymbol{\theta}_0, s))dM_i(s) -$$

$$\sum_i \int_0^t J(s)(\mathbf{x}_i(s) - \mathbf{E}(\boldsymbol{\theta}_0, s))\{\psi(\widehat{S}_i(s) - \psi(S_i(s))\}Y -$$

$$-i(s)\{e^{\boldsymbol{\beta}_0^T \mathbf{z}_i(s)} dH(S_0(s)) + \boldsymbol{\gamma}_0^T \mathbf{w}_i(s) ds\} -$$

$$\sum_i \int_0^t J(s)(\widetilde{\mathbf{E}}(\boldsymbol{\theta}_0, s) - \mathbf{E}(\boldsymbol{\theta}_0, s))dM_i(s)+ \tag{12}$$

$$\int_0^t J(s)(\widetilde{\mathbf{E}}(\boldsymbol{\theta}_0, s) - \mathbf{E}(\boldsymbol{\theta}_0, s))(\widetilde{S}^{(0)}(\boldsymbol{\theta}_0, s) - S^{(0)}(\boldsymbol{\theta}_0, s))dH(S_0(s))+$$

$$\int_0^t J(s)(\widetilde{\mathbf{E}}(\boldsymbol{\theta}_0, s) - \mathbf{E}(\boldsymbol{\theta}_0, s))\{\psi(\widehat{S}_i(s) - \psi(S_i(s))\}\boldsymbol{\gamma}_0^T \mathbf{w}_i(s)Y_i(s)ds =$$

$$\mathbf{U}^{(1)}(\boldsymbol{\theta}_0, s) + \mathbf{U}^{(2)}(\boldsymbol{\theta}_0, t) + \Delta_1(\boldsymbol{\theta}_0, t) + \Delta_2(\boldsymbol{\theta}_0, t) + \Delta_3(\boldsymbol{\theta}_0, t).$$

We consider the functions S_i on $[0, \tau_i]$. Under the assumptions a) and b)

$$\sqrt{n_i}(\widehat{S}_i - S_i) \xrightarrow{D} -S_i U_i$$

on $D[0, \tau_i]$, where U_i is a Gaussian martingale with $U_i(0) = 0$ and

$$\mathbf{Cov}\,(U_i(s_1), U_i(s_2)) = \sigma_i^2(s_1 \wedge s_2),$$

where

$$\sigma_i^2(t) = \int_0^t \frac{\alpha_i(u)}{y_i(u)}\, du.$$

ψ is compactly differentiable at S_i, so by functional delta method we obtain that

$$\sqrt{n_i}(\psi(\widehat{S}_i) - \psi(S_i)) \quad \text{and} \quad \psi'(S_i)\sqrt{n_i}(\widehat{S}_i - S_i)$$

have the same limit distribution and

$$\delta_i = \sqrt{n_i}(\psi(\widehat{S}_i) - \psi(S_i)) - \psi'(S_i)\sqrt{n_i}(\widehat{S}_i - S_i) \xrightarrow{\mathbf{P}} 0$$

uniformly on $[0, \tau_i]$. The random variable $\sqrt{n_i}(\widehat{S}_i(t-) - S_i(t))$ (see Fleming and Harrington (1991)) can be written in the form

$$\sqrt{n_i}(\widehat{S}_i(t-) - S_i(t)) = -S_i(t) \int_0^{t-} \frac{\widehat{S}_i(u-)}{S_i(u)} \sqrt{n_i}\frac{J_i(u)}{Y_i(u)}\, dM_i(u) + \sqrt{n_i}\, B_i(t), \tag{13}$$

where

$$\sup_{0\leqslant t\leqslant \tau_i} \sqrt{n_i}\, B_i(t) \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n_i \to \infty.$$

So

$$\mathbf{U}^{(2)}(\boldsymbol{\theta}_0, t) = \sum_i \int_0^t J(s)(\mathbf{x}_i(s) - \mathbf{E}(\boldsymbol{\theta}_0, s))\psi'(S_i(s))S_i(s)\times$$

$$\int_0^{s-} \frac{\widehat{S}_i(u-)}{S_i(u)Y_i(u)}\, dM_i(u)Y_i(s)\left[e^{\boldsymbol{\beta}_0^T\mathbf{z}_i(s)}dH(S_0(s)) + \boldsymbol{\gamma}^T\mathbf{w}_i(s)ds\right] -$$

$$\sum_i \frac{1}{\sqrt{n_i}} \int_0^t J(s)(\mathbf{x}_i(s) - \mathbf{E}(\boldsymbol{\theta}_0, s))\delta_i(s)Y_i(s)\times$$

$$\left[e^{\boldsymbol{\beta}_0^T\mathbf{z}_i(s)}dH(S_0(s)) + \boldsymbol{\gamma}^T\mathbf{w}_i(s)ds\right] -$$

$$\sum_i \int_0^t J(s)(\mathbf{x}_i(s) - \mathbf{E}(\boldsymbol{\theta}_0, s))\psi'(S_i(s))B_i(s)Y_i(s)\times$$

$$\left[e^{\boldsymbol{\beta}_0^T\mathbf{z}_i(s)}dH(S_0(s)) + \boldsymbol{\gamma}^T\mathbf{w}_i(s)ds\right] =$$

$$\mathbf{U}^{(3)}(\boldsymbol{\theta}_0, t) + \Delta_4(\boldsymbol{\theta}_0, t) + \Delta_5(\boldsymbol{\theta}_0, t).$$

Note that in this formula we can take $t \wedge \tau_i$ in the place of t in the upper limits of the integrals since $Y_i(s) = 0$ for $s > \tau_i$ in the expression of $\mathbf{U}^{(2)}(\boldsymbol{\theta}_0, t)$ (see (22)). So all the terms are defined in the last formula. Se note

$$\mathbf{U}^*(\boldsymbol{\theta}_0, t) = \mathbf{U}^{(1)}(\boldsymbol{\theta}_0, t) + \mathbf{U}^{(3)}(\boldsymbol{\theta}_0, t).$$

So

$$\frac{1}{\sqrt{n}}\mathbf{U}(\boldsymbol{\theta}_0, t) = \frac{1}{\sqrt{n}}\mathbf{U}^*(\boldsymbol{\theta}_0, t) + \frac{1}{\sqrt{n}}\Delta(\boldsymbol{\theta}_0, t),$$

where $\Delta = \Delta_1 + \ldots + \Delta_5$ and

$$\frac{1}{\sqrt{n}}\mathbf{U}^*(\boldsymbol{\theta}_0, t)) = \frac{1}{\sqrt{n}}\sum_i \int_0^t J(u)\Big\{\mathbf{x}_i(u) - \mathbf{E}(\boldsymbol{\theta}_0, t) + \frac{\widehat{S}_i(u-)}{S_i(u)}\,\frac{1}{Y_i(u)}\times$$

$$\Big[\int_u^t J(s)(\mathbf{x}_i(s)-\mathbf{E}(\boldsymbol{\theta}_0,s))\psi'(S_i(s))S_i(s)Y_i(s)\times \tag{14}$$

$$\big(e^{\boldsymbol{\beta}_0^T\mathbf{z}_i(s)}dH(S_0(s))+\boldsymbol{\gamma}^T\mathbf{w}_i(s)ds\big)\Big]\Big\}dM_i(u)$$

We'll show that $\frac{1}{\sqrt{n}}\mathbf{U}^*$ converges in distribution to a Gaussian martingale on $D[0,\tau]$ and $\frac{1}{\sqrt{n}}\Delta$ converges in probability to zero uniformly on $[0,\tau]$.

The predictable covariation process for $\frac{1}{\sqrt{n}}\mathbf{U}^*$ is

$$\langle n^{-1/2}\mathbf{U}^*, n^{-1/2}\mathbf{U}^*\rangle(\boldsymbol{\theta}_0,t)=\frac{1}{n}\sum_0^t J(u)\Big\{\mathbf{x}_i(u)-\mathbf{E}(\boldsymbol{\theta}_0,u)+$$

$$\frac{\widehat{S}_i(u-)}{S_i(u)}\,\frac{1}{Y_i(u)}\Big[\int_0^t(\mathbf{x}_i(s)-\mathbf{E}(\boldsymbol{\theta}_0,s))\psi'(S_i(s))S_i(s)Y_i(s)\times$$

$$\big(e^{\boldsymbol{\beta}_0^T\mathbf{z}_i(s)}dH(S_0(s))+\boldsymbol{\gamma}_0^T\mathbf{w}_i(s)ds\big)\Big]\Big\}^{\otimes 2}$$

$$Y_i(u)\psi(S_i(u))\big[e^{\boldsymbol{\beta}_0^T\mathbf{z}_i(u)}dH(S_0(u))+\boldsymbol{\gamma}_0^T\mathbf{w}_i(u)du\big].$$

Under assumptions of the theorem

$$\langle n^{-1/2}\mathbf{U}^*, n^{-1/2}\mathbf{U}^*\rangle(\boldsymbol{\theta}_0,t)\xrightarrow{\mathbf{P}}\boldsymbol{\Sigma}(\boldsymbol{\theta}_0,t).$$

Denote

$$H_{ij}(u)=J(u)\big\{x_{ij}(u)-E_j(\boldsymbol{\theta}_0,t)+\frac{\widehat{S}_i(u-)}{S_i(u)}\,\frac{1}{Y_i(u)}\times$$

$$\int_0^t(x_{ij}(s)-E_j(\boldsymbol{\theta}_0,s))\psi(S_i(s))S_i(s)Y_i(s)\times$$

$$\big(e^{\boldsymbol{\beta}_0^T\mathbf{z}_i(s)}dH(S_0(u))+\boldsymbol{\gamma}_0^T\mathbf{w}_i(s)ds\big)\big\}.$$

where $E_j(\boldsymbol{\theta}_0,s))$ is a ith component of the vector $\mathbf{E}(\boldsymbol{\theta}_0,s)$. We can apply Rebolledo's theorem (see Andersen et al. (1993), p. 83) if

$$\frac{1}{n}\sum_i\int_0^t H_{ij}^2(u)I\left\{\frac{1}{\sqrt{n}}\big|H_{ij}(u)\big|>\varepsilon\right\}Y_i(u)\alpha_i(s)ds\xrightarrow{bP}0.$$

But it is obvious as by assumptions of the theorem $H_{ij}(u)$ and $Y_i(u)/n$ are bounded on $[0,\tau]$ and

$$\int_0^\tau \alpha_i(s)ds < \infty.$$

By Rebolledo's theorem

$$n^{-1/2}\mathbf{U}^*(\boldsymbol{\theta}_0,\tau) \xrightarrow{D} N(\mathbf{0}, \boldsymbol{\Sigma}(\boldsymbol{\theta}_0,\tau)).$$

The term $\frac{1}{\sqrt{n}}\Delta$ converges in probability to zero. Really, consider $\frac{1}{\sqrt{n}}\Delta_1(\boldsymbol{\theta}_0,t)$. Applying Lenglart's inequality (see Andersen et al (1993)) we obtain for any $\alpha > 0$ and $\varepsilon > 0$

$$\mathbf{P}\left\{\frac{1}{\sqrt{n}}\Delta_1(\boldsymbol{\theta}_0,t) \geqslant \varepsilon\right\} =$$

$$\mathbf{P}\left\{\frac{1}{\sqrt{n}}\int_0^t J(u)(\widetilde{E}_j(\boldsymbol{\theta}_0,u) - E_j(\boldsymbol{\theta}_0,u))dM_i(u) \geqslant \varepsilon\right\} \leqslant \frac{\delta}{\varepsilon^2}+$$

$$\mathbf{P}\left\{\frac{1}{n}\int_0^t J(u)(\widetilde{E}_j(\boldsymbol{\theta}_0,u) - E_j(\boldsymbol{\theta}_0,u))^2\times\right.$$

$$\left.(S^{(0)}(\boldsymbol{\theta}_0,u)dH(S_0(u)) + S_*^{(0)}(\boldsymbol{\theta}_0,u))du) \geqslant \alpha\right\}.$$

As

$$\widetilde{\mathbf{S}}^{(m)}(\boldsymbol{\theta}_0,u)/n \xrightarrow{\mathbf{P}} \mathbf{s}^{(m)}(\boldsymbol{\theta}_0,u)$$

uniformly on $[0,\tau]$ we have

$$\widetilde{E}_j(\boldsymbol{\theta}_0,u) - E_j(\boldsymbol{\theta}_0,u) \xrightarrow{\mathbf{P}} 0$$

uniformly onm $[0,\tau]$. Taking into account the assumption b), we obtain

$$\frac{1}{n}\int_0^t (\widetilde{E}_j(\boldsymbol{\theta}_0,u) - E_j(\boldsymbol{\theta}_0,u))^2 S^{(0)}(\boldsymbol{\theta}_0,u)dH(S_0(u)) \xrightarrow{\mathbf{P}} 0.$$

So the right side of (3) can be made infinitely small when n is sufficiently large.

Consider the term $\frac{1}{\sqrt{n}}\Delta_2(\boldsymbol{\theta}_0, t)$. As

$$\widetilde{E}_j(\boldsymbol{\theta}_0, u) - E_j(\boldsymbol{\theta}_0, u) \xrightarrow{\mathbf{P}} 0$$

uniformly on $[0, \tau]$, we obtain

$$\frac{1}{\sqrt{n}}(\widetilde{S}^{(0)}(\boldsymbol{\theta}_0, u) - S^{(0)}(\boldsymbol{\theta}_0, u) \xrightarrow{\mathbf{P}} 0 - \sum_i y_i(u)\sqrt{l_i}\, e^{\beta_0^T \mathbf{x}_i(u)} S_i(u)\psi'(S_i(u))U_i$$

uniformly on $[0, \tau]$ and $H(S_0(\tau)) < \infty$, we have $\Delta_2/\sqrt{n} \xrightarrow{\mathbf{P}} 0$.

Convergence of

$$\frac{1}{\sqrt{n}}(\Delta_3(\boldsymbol{\theta}_0, t) + \Delta_4(\boldsymbol{\theta}_0, t) + \Delta_5(\boldsymbol{\theta}_0, t))$$

in probability to zero uniformly on $0, \tau]$ is obvious. The proof is complete.

Denote

$$\boldsymbol{\Sigma}_{11}(\boldsymbol{\theta}, t) = \int_0^t \mathbf{v}(\boldsymbol{\theta}, u) s^{(0)}(\boldsymbol{\theta}, u) dH(S_0(u)),$$

$$\boldsymbol{\Sigma}_{11}(\boldsymbol{\theta}, t) = \int_0^t \mathbf{s}_*^{(2)}(u) - \mathbf{e}(\boldsymbol{\beta}, u) s_*^{(11)T}(u)) du,$$

$$\boldsymbol{\Sigma}_1(\boldsymbol{\theta}, t) = (\boldsymbol{\Sigma}_{11}(\boldsymbol{\theta}, t), \boldsymbol{\Sigma}_{12}(\boldsymbol{\theta}, t)),$$

where $\mathbf{v} = \mathbf{s}^{(2)}/s^{(0} - \mathbf{e}\mathbf{e}_1^T$.

Theorem 2. *Assume conditions* A *hold and the matrix* $\boldsymbol{\Sigma}_1(\boldsymbol{\theta}_0, \tau)$ *is nonsingular. Then there exists a neighborhood of* $\boldsymbol{\theta}_0$ *within which, with probability tending to* 1 *as* $n \to \infty$, *the root* $\widehat{\boldsymbol{\theta}}$ *of* $\mathbf{U}(\boldsymbol{\theta}, \tau) = \mathbf{0}$ *is uniquely defined and*

$$n^{1/2}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{D} N(0, \boldsymbol{\Sigma}_2(\boldsymbol{\theta}_0, \tau)), \tag{15}$$

where

$$\boldsymbol{\Sigma}_2(\boldsymbol{\theta}_0, \tau) = \boldsymbol{\Sigma}_1^{-1}(\boldsymbol{\theta}_0, \tau)\boldsymbol{\Sigma}(\boldsymbol{\theta}_0, \tau)\boldsymbol{\Sigma}_1^{-1}(\boldsymbol{\theta}_0, \tau).$$

Proof. Using the Taylor expantion of $n^{-1/2}\mathbf{U}(\boldsymbol{\theta}, \tau)$ around $\boldsymbol{\theta}_0$ in $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$, we obtain

$$n^{1/2}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) = \left(-\frac{1}{n}\frac{\partial U(\boldsymbol{\theta}^*, \tau)}{\partial \boldsymbol{\theta}}\right)^{-1} n^{-1/2}\mathbf{U}(\boldsymbol{\theta}_0, \tau), \tag{16}$$

where θ^* is on the line segment between $\widehat{\boldsymbol\theta}$ and $\boldsymbol\theta_0$. The Theorem 1 implies, that the convergence (15) is proved if we prove that

$$-\frac{1}{n}\frac{\partial \mathbf{U}(\boldsymbol\theta_0,\tau)}{\partial \boldsymbol\theta} \xrightarrow{\mathbf{P}} \boldsymbol\Sigma_1(\boldsymbol\theta_0,\tau)$$

and $\widehat{\boldsymbol\theta}\overset{\mathbf{P}}{}\boldsymbol\theta_0$. Denote

$$\widetilde{\mathbf{V}} = \widetilde{\mathbf{S}}^{(2)}/\widetilde{S}^{(0)} - \widetilde{\mathbf{S}}^{(1)}\widetilde{\mathbf{S}}^{(11)T}/\widetilde{S}^{(0)2}.$$

We have

$$-\frac{1}{n}\frac{\partial \mathbf{U}(\boldsymbol\theta_0,\tau)}{\partial \boldsymbol\theta} = \frac{1}{n}\sum_{i=1}^{k}\times$$

$$\int_{i=1}^{k} \widetilde{\mathbf{V}}(\boldsymbol\theta_0,s)(dN_i(s)-Y_i(s)\psi(\widehat{S}_i(s))\boldsymbol\gamma_0^T\mathbf{w}_i(s)ds) \xrightarrow{\mathbf{P}} \boldsymbol\Sigma_{11}(\boldsymbol\theta_0,\tau).$$

This convergence can be easily obtained by writing $N_i(s)$ as a sum of the martingale M_i and the compensator and applying Lenglart's inequality. Analogously

$$-\frac{1}{n}\frac{\partial \mathbf{U}(\boldsymbol\theta_0,\tau)}{\partial \boldsymbol\gamma} \xrightarrow{\mathbf{P}} \boldsymbol\Sigma_{12}(\boldsymbol\theta_0,\tau).$$

So

$$-\frac{1}{n}\frac{\partial \mathbf{U}(\boldsymbol\theta_0,\tau)}{\partial \boldsymbol\theta} \xrightarrow{\mathbf{P}} \boldsymbol\Sigma_1(\boldsymbol\theta_0,\tau). \tag{17}$$

The rest can be proved as in Bagdonavičius and Nikulin (1997b, Theorem 2).

Now we'll consider the asymptotic distribution of the baseline survival function. Denote

$$\mathbf{C}_1(\boldsymbol\beta_0,t) = \int_0^t \mathbf{e}_1(\boldsymbol\theta_0,u)dH(S_0(u)), \quad \mathbf{C}_2(\boldsymbol\theta_0,t) = \int_0^t \mathbf{e}_*(\boldsymbol\theta_0,u)du,$$

$$\mathbf{e}_* = \mathbf{s}_*^{(11)}/s^{(0)}, \quad \mathbf{C}^T(\boldsymbol\theta_0,t) = (\mathbf{C}_1^T(\boldsymbol\theta_0,t),\mathbf{C}_2^T(\boldsymbol\theta_0,t)),$$

$$\sigma_1^2(\boldsymbol\theta_0,t) = \int_0^t \frac{s^{(0)}(\boldsymbol\theta_0,u)dH(S_0(u)) + s_*^{(0)}(\boldsymbol\theta_0,u)du}{(s^{(0)})^2(\boldsymbol\theta_0,u)},$$

$$\mathbf{A}(\boldsymbol{\theta}_0, t) = \sum_i \int_0^{t\wedge\tau_i} \{x_i(u) - \mathbf{e}(\boldsymbol{\theta}_0 u) +$$

$$\int_u^{t\wedge\tau_i} (\mathbf{x}_i(s) - \mathbf{e}(\boldsymbol{\theta}_0, s)\psi'(S_i(s))S_i(s)\frac{y_i(s)}{y_i(u)} \times$$

$$(e^{\boldsymbol{\beta}_0^T \mathbf{z}_i(s)} dH(S_0(s)) + \boldsymbol{\gamma}_0^T \mathbf{w}_i(s)ds)\} \times$$

$$l_i y_i(u)\psi(S_i(u)(e^{\boldsymbol{\beta}_0^T \mathbf{z}_i(u)} dH(S_0(u)) + \boldsymbol{\gamma}_0^T \mathbf{w}_i(u)du).$$

Theorem 3. *Under assumptions of the Theorem 1 for all $t \in [0, \tau]$*

$$n^{1/2}\{H(\widetilde{S}_0(t)) - H(S_0(t))\} \xrightarrow{D} N(0, \sigma_2^2(\boldsymbol{\theta}_0, t)) \text{ as } n \to \infty,$$

where

$$\sigma_2^2(\boldsymbol{\theta}_0, t) = \sigma_1^2(\boldsymbol{\theta}_0, t))$$

$$+\mathbf{C}^T(\boldsymbol{\theta}_0, t))\boldsymbol{\Sigma}_2(\boldsymbol{\theta}_0, \tau)\mathbf{C}(\boldsymbol{\theta}_0, t) - 2\mathbf{C}(\boldsymbol{\theta}_0, t)\boldsymbol{\Sigma}_1^{-1}(\boldsymbol{\theta}_0, \tau)\mathbf{A}(\boldsymbol{\theta}_0, t).$$

Proof. By Taylor expansion around $\boldsymbol{\theta}_0$ the random process $\sqrt{n}\{H(\widetilde{S}_0(t)) - H(S_0(t))\}$ can be written in the following manner:

$$\sqrt{n}\{H(\widetilde{S}_0(t)) - H(S_0(t))\} =$$

$$n^{1/2}\left\{\int_0^t \frac{J(u)}{\widetilde{S}^{(0)}\widehat{(\boldsymbol{\theta}, u)}}(dN(u) - \widetilde{S}_*^{(0)}(\widehat{\boldsymbol{\theta}}, u)du) - H(S_0(t))\right\} =$$

$$n^{1/2}\Big\{\int_0^t \frac{J(u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}_0, u)}(dN(u) - \widetilde{S}_*^{(0)}(\boldsymbol{\theta}_0, u)du) -$$

$$\int_0^t J(u)\frac{\widetilde{\mathbf{E}}_1^T(\boldsymbol{\theta}_0, u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}^*, u)}(dN(u)(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) +$$

$$\int_0^t J(u)\frac{\widetilde{\mathbf{E}}_1^T(\boldsymbol{\theta}_0, u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}^*, u)}\widetilde{S}_*^{(0)}(\boldsymbol{\theta}_0, u)du(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) -$$

$$\int_0^t J(u)\frac{\widetilde{\mathbf{S}}_*^{(11)}(\boldsymbol{\theta}_0,u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}^*,u)}\,du(\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_0)-H(S_0(t))\Big\}.$$

So

$$\sqrt{n}\{H(\widehat{S}_0(t))-H(S_0(t))\}=n^{1/2}\times$$

$$\Big\{-\int_0^t J(u)\frac{\widetilde{\mathbf{E}}_1^T(\boldsymbol{\theta}^*,u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}^*,u)}\,S^{(0)}(\boldsymbol{\theta}_0,u)d(H\circ S_0)(u)\times$$

$$(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)-\int_0^t J(u)\frac{\widetilde{\mathbf{S}}_*^{(11)}(\boldsymbol{\theta}_0,u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}^*,u)}\,du(\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_0)+$$

$$\int_0^t J(u)\frac{dM(u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}_0,u)}-\int_0^t J(u)\frac{\widetilde{\widetilde{S}}^{(0)}(\boldsymbol{\theta}_0,u)-S^{(0)}(\boldsymbol{\theta}_0,u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}_0,u)}\,d(H\circ S_0)(u)$$

$$-\int_0^t J(u)\frac{\widetilde{\widetilde{S}}^{(0)}(\boldsymbol{\theta}_0,u)-S^{(0)}(\boldsymbol{\theta}_0,u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}_0,u)}\,du+ \tag{19}$$

$$\int_0^t J(u)\frac{\widetilde{\mathbf{E}}_1^T(\boldsymbol{\theta}^*,u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}^*,u)}\widetilde{S}_*^{(0)}(\boldsymbol{\theta}^*,u)-S^{(0)}(\boldsymbol{\theta}_0,u)du(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)-$$

$$\int_0^t J(u)\frac{\widetilde{\mathbf{E}}_1^T(\boldsymbol{\theta}^*,u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}^*,u)}\,dM(u)(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)+\int_0^t(J(u)-1)d(H\circ S_0)(u)\Big\}=\sum_{i=1}^{8}B_i,$$

where $\boldsymbol{\theta}^*$ is on the line segment between $\widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}_0$. By the assumptions of the theorem

$$\int_0^t J(u)\frac{\widetilde{\mathbf{E}}_1^T(\boldsymbol{\theta}^*,u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}^*,u)}S^{(0)}(\boldsymbol{\theta}^*,u)dH(S^{(0)}(u))\xrightarrow{\mathbf{P}}\mathbf{C}_1(\boldsymbol{\theta}_0,t)$$

and

$$\int_0^t J(u)\frac{\widetilde{\mathbf{S}}_*^{11}(\boldsymbol{\theta}^*,u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}^*,u)}\,du\xrightarrow{\mathbf{P}}\int_0^t\frac{\widetilde{\mathbf{s}}_*^{11}(\boldsymbol{\theta}^*,u)}{s^{(0)}(\boldsymbol{\theta}^*,u)}\,du=\mathbf{C}_2(\boldsymbol{\theta}_0,t)$$

uniformly on $[0,\tau]$. Taking into account (17), we obtain

$$n^{1/2}(\widehat{\boldsymbol\theta}-\boldsymbol\theta_0)=\boldsymbol\Sigma_1^{-1}(\boldsymbol\theta_0,\tau)n^{-1/2}\mathbf{U}(\boldsymbol\theta_0,\tau)+\Delta_1,$$

where $\Delta_1 \xrightarrow{\mathbf{P}} 0$. So the sum of the first two terms in the right side of (19) can be written

$$B_1+B_2=-\mathbf{C}^T(\boldsymbol\theta_0,t)\boldsymbol\Sigma_1^{-1}(\boldsymbol\theta_0,\tau)n^{-1/2}\mathbf{U}(\boldsymbol\theta_0,\tau)+\Delta_2,$$

where $\Delta_2 \xrightarrow{\mathbf{P}} 0$ uniformly on $[0,\tau]$. It is obvious that $\sum\limits_{i=4}^{8} B_i \xrightarrow{\mathbf{P}} 0$ uniformly on $[0,\tau]$. So

$$\sqrt{n}(H(\widetilde S_0(t))-H(S_0(t)))-\mathbf{C}^T(\boldsymbol\theta_0,t)\boldsymbol\Sigma_1^{-1}(\boldsymbol\theta_0,\tau)n^{-1/2}\mathbf{U}(\boldsymbol\theta_0,\tau)+B_3+\Delta_3,$$

where

$$\Delta_3 \xrightarrow{\mathbf{P}} 0 \text{ uniformly on } [0,\tau].$$

By the Theorem 1 (cf. the equality (14)):

$$n^{-1/2}\mathbf{U}(\boldsymbol\theta_0,\tau)=n^{-1/2}\sum_{i=1}^{k}\int_0^{\tau}\mathbf{H}_i(\boldsymbol\theta_0,u)dM_i(u)+\Delta_4,$$

where

$$\mathbf{H}_i(\boldsymbol\theta_0,u)=J(u)\Bigg\{\mathbf{x}_i(u)-\mathbf{E}(\boldsymbol\theta_0,u)+\frac{\widehat S_i(u-)}{S_i(u)}\frac{1}{Y_i(u)}\times$$

$$\Bigg[\int_0^t J(s)(\mathbf{x}_i(s)-\mathbf{E}(\boldsymbol\theta_0,s))\psi'(S_i(s))S_i(s)Y_i(s)\times$$

$$(e^{\boldsymbol\beta_0^T\mathbf{z}_i(s)}dH(S_0(s))+\boldsymbol\gamma_0^T\mathbf{w}_i(s)ds)\Bigg]\Bigg\}$$

and $\Delta_4 \xrightarrow{\mathbf{P}} 0$. So

$$\langle B_3, n^{-1/2}\mathbf{U}\rangle(\boldsymbol\theta_0,t)=$$

$$\sum_{i=1}^{n}\int_0^t\frac{\mathbf{H}_i(\boldsymbol\theta_0,u)}{\widetilde S^{(0)}(\boldsymbol\theta_0,u)}\psi(S_i(u))Y_i(u)\times$$

$$(e^{\boldsymbol\beta_0^T\mathbf{z}_i(s)}dH(S_0(u))+\boldsymbol\gamma_0^T\mathbf{w}_i(u)du)\to\mathbf{A}(\boldsymbol\theta_0,t).$$

The predictable variation for B_3 is

$$\langle B_3, B_3\rangle(\boldsymbol{\theta}_0, t) =$$

$$n\int_0^t J(s)\frac{S^{(0)}(\boldsymbol{\theta}_0, s)dH(S_0(s)) + S_*^{(0)}(\boldsymbol{\theta}_0, s)ds}{\widetilde{S}^{(0)2}(\boldsymbol{\theta}_0, s)} \xrightarrow{\mathbf{P}} \sigma_1^2(\boldsymbol{\theta}_0, t).$$

Denote

$$H_{1i}(\boldsymbol{\theta}_0, u) = -\mathbf{C}^T(\boldsymbol{\theta}_0, t)\boldsymbol{\Sigma}_1^{-1}(\boldsymbol{\theta}_0, \tau)\mathbf{H}_i(\boldsymbol{\theta}_0, u) + nJ(u)/\widetilde{S}^{(0)}(\boldsymbol{\theta}_0, u),$$

then

$$\sqrt{n}(H(\widetilde{S}_0(t)) - H(S_0(t))) = n^{-1/2}\sum_{i=1}^{k}\int_0^t H_{1i}(\boldsymbol{\theta}_0, u)dM_i(u) + \Delta,$$

where

$$\Delta = \Delta(\boldsymbol{\theta}_0, t) \xrightarrow{\mathbf{P}} 0 \text{ uniformly on } [0, \tau].$$

By assumptions of the theorem for all $\varepsilon > 0$

$$n^{-1}\sum_{i=1}^{k}\int_0^t H_{1i}^2(\boldsymbol{\theta}_0, u)I\{|H_{1i}(\boldsymbol{\theta}_0, u)| > \varepsilon\}Y_i(u)\alpha_i(u)du \xrightarrow{\mathbf{P}} 0.$$

The asymptotic normality of $\sqrt{n}(H(\widehat{S}_0(y) - H(S_0(t))$ follows from the theorem of Rebolledo. The proof is complete.

Consider the asymptotic distribution of $\widehat{S}_{\mathbf{x}(\cdot)}$.

Denote

$$\mathbf{C}_{1\mathbf{x}}(\boldsymbol{\theta}_0, t) = \int_0^t e^{\boldsymbol{\beta}_0^T\mathbf{z}(u)}(\mathbf{e}_1^T(\boldsymbol{\theta}_0, u) - \mathbf{z}^T(u))d(H \circ S_0)(u),$$

$$\mathbf{C}_{2\mathbf{x}}(\boldsymbol{\theta}_0, t) = \int_0^t e^{\boldsymbol{\beta}_0^T\mathbf{z}(u)}\mathbf{e}_*(\boldsymbol{\theta}_0, u)du,$$

$$\mathbf{C}_{\mathbf{x}}^T(\boldsymbol{\theta}_0, t) = (\mathbf{C}_{1\mathbf{x}}^T(\boldsymbol{\theta}_0, t), \mathbf{C}_{2\mathbf{x}}^T(\boldsymbol{\theta}_0, t)), \;\; \sigma_{\mathbf{x}1}^2(\boldsymbol{\theta}_0, t) = \int_0^t e^{2\boldsymbol{\beta}_0^T\mathbf{x}(u)}d\sigma_1^2(\boldsymbol{\theta}_0, u),$$

$$\mathbf{A}_{\mathbf{x}}(\boldsymbol{\theta}_0, t) = \int_0^t e^{\boldsymbol{\beta}_0^T\mathbf{x}(u)}d\mathbf{A}(\boldsymbol{\theta}_0, u), \;\; g = G'.$$

Theorem 4. *Under assumptions of the Theorem* 1

$$\sqrt{n}(\widehat{S}_{\mathbf{x}(\cdot)}(t) - S_{\mathbf{x}(\cdot)}(t)) \xrightarrow{D} N(0, \sigma^2_{\mathbf{x}}) \text{ as } n \to \infty,$$

where

$$\sigma^2_{\mathbf{x}}(\boldsymbol{\theta}_0, t) = \{\sigma^2_{\mathbf{x}1}(\boldsymbol{\theta}_0, t) + \mathbf{C}^T_{\mathbf{x}}(\boldsymbol{\theta}_0, t)\boldsymbol{\Sigma}_2(\boldsymbol{\theta}_0, \tau)\mathbf{C}_{\mathbf{x}}(\boldsymbol{\theta}_0, t) -$$
$$2\mathbf{C}_{\mathbf{x}}(\boldsymbol{\theta}_0, t)\boldsymbol{\Sigma}_1^{-1}(\boldsymbol{\beta}_0, \tau)\mathbf{A}_{\mathbf{x}}(\boldsymbol{\theta}_0, t)\} g^2(H(S_{\mathbf{x}}(t))).$$

Proof. By Taylor expansion around $\boldsymbol{\theta}_0$ we obtain (similarly as in the proof of the Theorem 3):

$$n^{1/2}(H(\widehat{S}_{\mathbf{x}(\cdot)}(t)) - H(S_{\mathbf{x}(\cdot)}(t))) =$$

$$-\sqrt{n}\int_0^t J(u)e^{\boldsymbol{\beta}^{*T}\mathbf{z}(u)} \frac{\widetilde{\mathbf{E}}_1^T(\boldsymbol{\theta}^*, u) - \mathbf{z}^T(u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}^*, u)} S^{(0)}(\boldsymbol{\theta}_0, u) d(H \circ S_0)(u)(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) -$$

$$-\sqrt{n}\int_0^t J(u)e^{\boldsymbol{\beta}^{*T}\mathbf{z}(u)} \frac{\widetilde{\mathbf{S}}_*^{(11)T}(u)}{\widetilde{S}^{(0)}(\boldsymbol{\theta}^*, u)} (\widehat{\gamma} - \gamma_0) +$$

$$\sqrt{n}\int_0^t J(u)e^{\boldsymbol{\beta}^{*T}\mathbf{z}(u)} \frac{e^{\boldsymbol{\beta}_0^T\mathbf{x}(u)}}{\widetilde{S}^{(0)}(\boldsymbol{\theta}^*, u)} dM(u) + \Delta(\boldsymbol{\theta}_0, t),$$

where $\boldsymbol{\theta}^*$ is on the line segment between $\widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}_0$ and $\Delta \to 0$ uniformly on $[0, \tau]$.

The proof of the asymptotical normality of $\sqrt{n}(H\widehat{S}_{\mathbf{x}(\cdot)}(t)) - H(S_{\mathbf{x}(\cdot)}(t)))$ is similar to the proof of the asymptotical normality of $\sqrt{n}(H(\widehat{S}_0(t)) - H(S_0(t)))$ in the Theorem 3. So we skip it. The asymptotical normality of $\sqrt{n}(\widehat{S}_{\mathbf{x}(\cdot)}(t) - (S_{\mathbf{x}(\cdot)}(t))$ is obtained by the functional delta method. The proof is complete.

Remark 1. Replacing all the theoretical quantities by empirical ones in the Theorems 1–4, we obtain consistent estimates for the limiting covariance matrice of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$ and the limiting variances of

$$\sqrt{n}(H(\widetilde{S}_0(t)) - H(S_0(t)) \quad \text{and} \quad \sqrt{n}(\widehat{S}_{\mathbf{x}(\cdot)}(t) - S_{\mathbf{x}(\cdot)}(t)).$$

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Bagdonovičius V., Nikulin M. S. Semiparametric estimation in the generalized additive-multiplicative model.

The generalization of additive-multiplicative model of Lin and Ying is considered. The estimators of regression parameters, baseline survival function and survival function under any stress are proposed. The asymptotic properties of estimators are investigated.

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