**V. N. Solev and L. Gerville-Reache**

# THE ESTIMATION OF A FUNCTION BEING OBSERVED WITH A STATIONARY ERROR

ABSTRACT. We suppose that we observe a process $y(t)$ when $t \in [-T, T]$,

$$y(t) \;=\; s(t) \;+\; x(t) \qquad (t \in [-T, T]),$$

where $s$ is an unknown function (which we must estimate), $x$ is a stationary noise. We compare the accuracy of the least-squares estimator s$^*$ with the accuracy of the best linear unbiased estimator s$^\star$.

## §1. THE OBSERVATIONS WITH A STATIONARY NOISE

Let $x(t)$ be a stationary in the wide scence real-valued process with zero mean ($\mathbf{E}\, x(t) \;=\; 0$) and spectral density $f$.

That is $x$ is such a function

$$x \;:\; R^1 \longrightarrow L^2(d\mathbf{P}),$$

that

$$\mathbf{E}\, x(t) \cdot \overline{x(s)} \;=\; \int_{-\infty}^{\infty} exp\{i(t-s)u\}\, f(u)\, du,$$

where $L^2(d\mathbf{P})$ the $L^2$ – space, generated by a probability measure $\mathbf{P}$.

We denote by $x[\varphi]$ "the everage value"

$$x[\varphi] \;=\; \int_{-\infty}^{\infty} x(t)\,\varphi(t)\, dt \;=\; \big(\varphi,\, x\big),$$

where $\big(\cdot,\cdot\big)$ is the inner produch in a $L^2$ – space, induced by Lebesgue measure on $R^1$. So we have

$$\mathbf{E}\, x[\varphi] \cdot \overline{x[\psi]} \;=\; \int_{-\infty}^{\infty} \hat{\varphi}(u) \cdot \overline{\hat{\psi}(u)}\, f(u)\, du \;=\; \big(\hat{\varphi},\, \hat{\psi}\,\big)_f,$$

This research was suppoted in part by Russion Foundation for Basic Research, grant 96-01-00684.

where $\hat{\varphi}$ is the Fourier transformation of function $\varphi$,

$$\hat{\varphi}(v) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \varphi(u) exp\{ivu\}\, du;$$

$(\cdot,\cdot)_f$, $||\cdot||_f$ are the inner product and norm in the $L^2$ −space generated by the measure $f(u)du$.

For some applications and the theoretical problems we need to consider a case, when the functional $x(t)$ – "the value of the random function $x$ in a point $t$", can't be correctly defined. So we have to introduce generalized process, when we know only such everage values $x[\varphi]$.

Let $x[\cdot]$ be a stationary in the wide scence generalized process with the spectral density (s.d.) $f$ that is (see in [1]) $x$ is an operator : $D \to L^2(d\mathbf{P})$, from the space $D$ of infinitely differentiable functions with the compact support to the $L^2$ space, generated by a probability measure $\mathbf{P}$ such that

$$\mathbf{E}\, x[\varphi] \cdot \overline{x[\psi]} = \int_{-\infty}^{\infty} \hat{\varphi(u)} \cdot \overline{\hat{\psi}(u)}\, f(u)du = \big(\hat{\varphi}, \hat{\psi}\big)_f \quad \big(\varphi, \psi \in D\big). \quad (1)$$

We assume that process $x$ is real-valued that is random variables $x[\varphi]$ are real for any real-valued function $\varphi \in D$. We suppose that

$$\mathbf{E}x[\varphi] = 0, \quad \varphi \in D \quad (2)$$

We may define on the linear set $D$ the Hilbert structure generated by the norm $||\cdot||_x$, $||\varphi||_x = ||\hat{\varphi}||_f$ and the process $x$ may be extended to the linear set $D(x)$ of all locally integrable functions $\varphi \in L^2_{\mathrm{loc}}$ such that $\hat{\varphi} \in L^2_f$ (which is contained in the closure of $D$ in the Hilbert space above).

Further we assume that process $x$ is defined on $D(x)$ and denote by

$$D_T = D_T(x) = \{\varphi : \varphi \in D(x), \operatorname{supp}\varphi \subset [-T, T]\}.$$

We suppose that we observe a process $y[\varphi]$ when $\varphi \in D_T$,

$$y[\varphi] = s[\varphi] + x[\varphi] \qquad (\varphi \in D_T), \quad (3)$$

where $s[\cdot]$ is unknown functional: $D_T \to \mathbf{C}^1$ (which we must estimate), $x[\cdot]$ is a stationary noise.

## §2. The regular functionals

Let $\mathbf{H}$ be a Hilbert space of functions defined on $\mathbf{R}^1$ with the inner product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$ and let $\mathbf{L}$ be a subspace of space $\mathbf{H}$ . We assume that $D_T \subset \mathbf{H}$ and $\overline{D_T} = \mathbf{H}$, where $\overline{D_T}$ is the closure in $\mathbf{H}$ of $D_T$.

A functional $g: D_T \to \mathbf{C}^1$; is called regular if there exists such element $\mathbf{g} \in \mathbf{H}$ that

$$g[\varphi] = \langle\varphi, \mathbf{g}\rangle, \text{ for all } \varphi \in D_T$$

We denote by $\boldsymbol{v}$ the operator, which is defined on the linear subset of all the regular functionals by the relation: $vg = \mathbf{g}$ where $\mathbf{g}$ is such the element of $\mathbf{H}$ that $g[\cdot] = \langle\cdot, \mathbf{g}\rangle$.

Further we assume that unknown functional $s$ is regular and denote $\mathbf{s} = vs$, We suppose that the subspace $\mathbf{L}$ is known and $\dim(\mathbf{L}) = m < \infty$.

The random variable $x[\varphi], \varphi \in D_T$, is some function $\xi(\omega) = \xi_\varphi(\omega) = x[\varphi]$ from the space $L^2(d\mathbf{P})$. It can be possible that for some $\omega$ random functional $x[\cdot]$ is not regular. But a projection (in some scence which will be defined later) of $x$ on some finite-dimentional subspaces may be regular a.s.

## §3. The projection on subspace

Let $\mathbf{L}, \mathbf{L}^*$ be such subspaces of space $\mathbf{H}$ that

$$\mathbf{L} \cap \mathbf{L}^* = \{\mathbf{0}\} \qquad \textit{and} \qquad \mathbf{L} + \mathbf{L}^* = \mathbf{H}. \tag{4}$$

The projection of functional $g: D_t \to \mathbf{C}^1$, on the space $\mathbf{L}$ in parallel to the supspace $\mathbf{L}^*$ is called such functional $P(\mathbf{L}, \mathbf{L}^*)g$ (if it exists) that

$$P(\mathbf{L}, \mathbf{L}^*)g\,[\varphi] = \begin{cases} g[\varphi], & \text{if} \quad \varphi \in \mathbf{L}_* \\ \mathbf{0}, & \text{if} \quad \varphi \in \mathbf{L}^\perp. \end{cases} \tag{5}$$

(where we defined by $\mathbf{M}^\perp$ the orthogonal complement in space $\mathbf{H}$ to the subspace $\mathbf{M}$ and we put $\mathbf{L}_* = (\mathbf{L}^*)^\perp$. The projection (5) is well defined if functional g is defined on $\mathbf{L}_*$.

It is easily to see that if a functional $g$ is defined on subspace $\mathbf{L}_*$ then

$$P(\mathbf{L}, \mathbf{L}^*)g\,(\cdot) = \langle\cdot, \mathbf{h}\rangle, \quad \text{where} \qquad \mathbf{h} = \sum_{j=1}^m \overline{g(\psi_j)} \cdot \varphi_j, \tag{6}$$

where $m = \dim(\mathbf{L})$, $\{\varphi_j, j = 1, ..., m\}, \{\psi_j, j = 1, ..., m\}$ are such orthogonal basises of subspaces $\mathbf{L}$ and $\mathbf{L}_*$, that $\langle\varphi_i, \psi_j\rangle = \delta_{ij}$.

In the case, when $\mathbf{L}^* = \mathbf{L}^\perp$ the elenent $P(\mathbf{L})g = P(\mathbf{L}, \mathbf{L}^\perp)g$ is called the orthogonal projection of functional $g$ on the subspace $\mathbf{L}$

If function $g$ has the regular representation $g(\cdot) = \langle \cdot, \mathbf{g} \rangle$ it is evident that

$$P(\mathbf{L}, \mathbf{L}^*)g\ (\cdot) = \langle \cdot, \mathbf{P}(\mathbf{L}, \mathbf{L}^*)\mathbf{g} \rangle,$$

where $\mathbf{P}(\mathbf{L}, \mathbf{L}^*)$ is the corresponding projector in the space $\mathbf{H}$. That is

$$\mathbf{P}(\mathbf{L}, \mathbf{L}^*)\mathbf{h} = \begin{cases} \mathbf{h}, & \text{if} \quad \mathbf{h} \in \mathbf{L} \\ \mathbf{0}, & \text{if} \quad \mathbf{h} \in \mathbf{L}^*. \end{cases} \tag{7}$$

For simplicity we shall right $\mathbf{P}(\mathbf{L}, \mathbf{L}^*)g$ instead of $\upsilon\, P(\mathbf{L}, \mathbf{L}^*)g$. So for example we have by definition

$$\mathbf{P}(\mathbf{L}, \mathbf{L}^*)g = \mathbf{P}(\mathbf{L}, \mathbf{L}^*)\mathbf{g}, \quad \text{where} \quad \mathbf{g} = \upsilon g$$

## §4. Linear estimators. Unbiased estimators

Consider the problem of the linear estimating unknown function $\mathbf{s} \in \mathbf{H}$ based upon observations

$$y[\varphi] = s[\varphi] + x[\varphi] \qquad (\varphi \in D_T, D_T \subset \mathbf{H}),$$

(see in details [2], [3], [5]). We assume that $s[\varphi] = \langle \varphi, \mathbf{s} \rangle$, unknown function $\mathbf{s}$ belongs to the finite-dimentional subspase $\mathbf{L}$ of the space $\mathbf{H}$ and subspace $\mathbf{L}$ is known. We suppose that $x[\cdot]$ is a stationary generalized process with the spectral density $f$.

Let $\{\varphi_j, j = 1, ..., m\}$ be a basis of $\mathbf{L}$ and

$$\mathbf{s}(\cdot) = \sum_{j=1}^{m} \theta_j \varphi_j(\cdot).$$

We consider linear estimators $\hat{\mathbf{s}}$ for unknown function $\mathbf{s}$ based on observations $y[\varphi]$, $\varphi \in D_T$, that is such a kind of estimators, which may be represented in the form

$$\hat{\mathbf{s}}(\cdot) = \sum_{j=1}^{m} \hat{\theta}_j \varphi_j(\cdot), \tag{8}$$

where $\hat{\theta_j} \in \mathbf{Y}_{-T}^T$ $(j = 1, ..., m)$ and we denote

$$\mathbf{Y}_a^b = \overline{sp}\{y[\varphi], \quad \operatorname{supp} \varphi \in [a, b]\}.$$

It is clear that

$$\hat{\theta}_j = \overline{y[\psi_j]} = \overline{s[\psi_j]} + \overline{x[\psi_j]}, \quad \text{for some} \quad \psi_j \in D_T \quad (j = 1, ..., m)$$

and

$$\overline{s[\psi_j]} = \langle \mathbf{s},\, \psi_j \rangle \quad (j = 1, ..., m).$$

The linear estimator $\hat{\mathbf{s}}$ is called the unbiased estimator if $\mathbf{E}\,\hat{\mathbf{s}} = \mathbf{s}$ for any $\mathbf{s} \in \mathbf{L}$. Because of $\mathbf{E}\,y[\varphi] = s[\varphi]$ (see (2) ), if an estimator (8) is the unbiased estimator, then

$$\sum_{j=1}^{m} \mathbf{E}\,\hat{\theta}_j \varphi_j(\cdot) = \sum_{j=1}^{m} \langle \mathbf{s},\, \psi_j \rangle \varphi_j(\cdot) = \mathbf{s}(\cdot),$$

for any $\mathbf{s} \in \mathbf{L}$. So if an estimator $\hat{\mathbf{s}}$ is the unbiased estimator then $\langle \varphi_i,\, \psi_j \rangle = \delta_{ij} \quad (j = 1, ..., m)$. Therefore we conclude that a linear unbiased estimator $\hat{\mathbf{s}}$ in (8) has a representation

$$\hat{\mathbf{s}}(\cdot) = [\mathbf{P}(\mathbf{L}, \mathbf{L}^*) y]\,(\cdot), \tag{9}$$

where subspace $\mathbf{L}^* = \mathbf{L}^*(\hat{\mathbf{s}})$ is defined by the relation

$$\mathbf{L}^* = \mathbf{H} \ominus \mathbf{L}_*,$$

and subspace $\mathbf{L}_* = \mathbf{L}_*(\hat{\mathbf{s}})$ is defined by the relation

$$\mathbf{L}_* = \overline{sp}\{\psi_j, j = 1, ..., m\}.$$

We have to assume that

$$\mathbf{L}_* \subset D_T.$$

It is clear that

$$\mathbf{L} + \mathbf{L}^* = \mathbf{H}, \quad \mathbf{L} \cap \mathbf{L}^* = \mathbf{0}, \tag{10}$$

because $\langle \varphi_j,\, \psi_j \rangle = \delta_{ij} \quad (j = 1, ..., m)$. We put $\mathbf{P} = \mathbf{P}(\mathbf{L}, \mathbf{L}^*)$. It is easily to see that the conjugate operator $\mathbf{P}^*$ is a projector on subspace $\mathbf{L}_*$ in parallel of subspace $\mathbf{L}^{\perp} = \mathbf{H} \ominus \mathbf{L}$ :

$$\mathbf{P}^*\,\mathbf{h} = \begin{cases} \mathbf{h}, \text{if} & \mathbf{h} \in \mathbf{L}_* \\ \mathbf{0}, \text{if} & \mathbf{h} \in \mathbf{L}^{\perp}, \end{cases} \tag{11}$$

where the conjugate operater $B^*$ is defined by the relation

$$\langle B\,\cdot\,, \cdot\,\rangle = \langle\,\cdot\,, B^*\,\cdot\,\rangle.$$

## §5. The accuracy of the estimation

Suppose that an estimator $\hat{\mathbf{s}}$ is the linear unbiased estimator, $\hat{\mathbf{s}} = \mathbf{P}y$ where $\mathbf{P} = \mathbf{P}_{\hat{\mathbf{s}}} = \mathbf{P}(\mathbf{L}, \mathbf{L}^*)$ for some subspace $\mathbf{L}^*$ such that (10) is provided. The accuracy of the estimation we will measure by the value

$$\boldsymbol{\sigma}_T^2(\hat{\mathbf{s}},\ \mathbf{s}) \ = \ \mathbf{E}\,||\hat{\mathbf{s}} \ - \ \mathbf{s}||^2$$

We denote by $\mathbf{R}_x$ the covariation operator of process $x$ :

$$\mathbf{E}\,\{\langle x[h_1], x[h_2]\rangle\} \ = \ \langle R_x\ h_1, h_2\rangle, \quad h_1, h_2 \ \in \ \mathbf{H}.$$

Because of (1) we have $\hat{\mathbf{s}} \ - \ \mathbf{s} \ = \ \mathbf{P}x$ and (see (6))

$$[\mathbf{P}x](\cdot) = \sum_{j=1}^{m} \overline{x[\psi_j]} \cdot \varphi_j(\cdot),$$

for some orthonormal basis $\{\varphi_j,\ j \ = \ 1, ..., m\}$ of the subspace $\mathbf{L}$ and such basis $\{\psi_j,\ j = 1, ..., m\}$ of the space $\mathbf{L}^*$ that $\langle\varphi_i,\ \psi_j\rangle = \delta_{ij}$ $(i, j = 1, ..., m)$. Therefore

$$\boldsymbol{\sigma}_T^2(\hat{s},\ s) \ = \ \mathbf{E}\,||\mathbf{P}x||^2 \ = \ \mathbf{E}\,||\sum_{j=1}^{m} \overline{x[\psi_j]}\cdot\varphi_j(\cdot)||^2 \ = \ \sum_{j=1}^{m} \mathbf{E}\,|x[\psi_j]|^2. \quad (12)$$

So we have

$$\boldsymbol{\sigma}_T^2(\hat{s},\ s) \ = \ \sum_{j=1}^{m}\langle R_x\psi_j, \psi_j\rangle \ =$$

$$\sum_{j=1}^{m}\langle R_x P^*\varphi_j, P^*\varphi_j\rangle \ = \ \sum_{j=1}^{m}\langle \mathbf{P} R_x P^*\varphi_j,\ \varphi_j\rangle.$$

Because of $\mathbf{P}^*h = \mathbf{0}$ for $h \in \mathbf{L}^\perp$ we obtain

$$\boldsymbol{\sigma}_T^2(\hat{s},\ s) \ = tr\mathbf{P}\mathbf{R}_x\mathbf{P}^*, \quad (13)$$

where $trA$ is the trace of an operator $A$.

## §6. The best linear unbiased estimator

We consider also another Hilbert structure on $\mathbf{H}$ . Let $\mathbf{H}_\star$ be a Hilbert space with the inner product

$$\langle\varphi, \psi\rangle_\star \ = \ \langle\mathbf{R}_x\varphi, \psi\rangle \ = \ \mathbf{E}x[\varphi]\overline{x[\psi]}$$

and the norm $||\cdot||_\star^2 = \langle\cdot,\cdot\rangle_\star$.

We note that without loss of generality we can assume that

$$\mathbf{R}_x\varphi \neq \mathbf{0} \quad \text{for any } \varphi \in L.$$

Let $\mathbf{L}^\perp$ be the orthogonal complement (in the space $\mathbf{H}$) of the subspace $\mathbf{L}$, $\mathbf{L}_\star$ be the orthogonal complement (in the space $\mathbf{H}_\star$) of the subspace $\mathbf{L}^\perp$ and $\mathbf{L}^\star$ be the orthogonal complement (in the space $\mathbf{H}$) of the subspace $\mathbf{L}_\star$. We put $\mathbf{P}_\star = \mathbf{P}(\mathbf{L}, \mathbf{L}^\star)$ It is easily to see that

$$\{\mathbf{P}_\star\}^*\varphi = \begin{cases} \varphi, & \text{if } \varphi \in \mathbf{L}_\star \\ \mathbf{0}, & \text{if } \varphi \in \mathbf{L}^\perp, \end{cases} \tag{14}$$

and

$$\mathbf{P}_\star\varphi = \begin{cases} \varphi, & \text{if } \varphi \in \mathbf{L} \\ \mathbf{0}, & \text{if } \varphi \in \mathbf{L}^\star. \end{cases} \tag{15}$$

We consider an estimator $\mathbf{s}^\star = \mathbf{P}_\star\, y$.

**Proposition 1.** *An estimator* $\mathbf{s}^\star$ *is the best linear unbiased estimator for* $\mathbf{s}$,
*that is*

$$\mathbf{E}\,\mathbf{s}^\star = \mathbf{s} \quad \textit{and} \quad \boldsymbol{\sigma}_T^2(\hat{\mathbf{s}},\ \mathbf{s}) \geq \boldsymbol{\sigma}_T^2(\mathbf{s}^\star,\ \mathbf{s})$$

*for any linear unbiased estimator* $\hat{\mathbf{s}}$.

**Proof.** An estimator $\mathbf{s}^\star$ is an unbiased estimator. We put $\hat{\mathbf{s}} = \mathbf{P}y$, where $\mathbf{P} = \mathbf{P}(\mathbf{L},\, \mathbf{L}^*)$. Let $\{\varphi_j,\ j = 1, 2, ..., m\}$ be an orthonormal basis of the space $\mathbf{L}$ and $\psi_j = \mathbf{p}\varphi_j$ $(j = 1, ..., m)$. So

$$\varphi_j = \psi_j + \varphi^j, \qquad \text{where } \varphi^j \in \mathbf{L}^\perp. \tag{16}$$

In this case (see (12),(14))

$$\boldsymbol{\sigma}_T^2(\hat{\mathbf{s}},\ \mathbf{s}) = \sum_{j=1}^m \mathbf{E}\,|x[\psi_j]|^2 = \sum_{j=1}^m ||\psi_j||_\star^2. \tag{17}$$

Hence,

$$\boldsymbol{\sigma}_T^2(\hat{\mathbf{s}},\ \mathbf{s}) = \sum_{j=1}^m ||\varphi_j - \varphi^j||_\star^2 \geqslant \sum_{j=1}^m ||\mathbf{P}_\star^*\varphi_j||_\star^2,$$

because (see (14)) $\mathbf{P}_\star^*$ is the orthoprojector on the space $\mathbf{H}_\star$ on subspace $\mathbf{L}_\star$, which is orthogonal complement (in the space $\mathbf{H}_\star$) of subspace $\mathbf{L}^\perp$. It is evident that

$$\boldsymbol{\sigma}_T^2(\mathrm{s}^\star,\ \mathrm{s}) \ = \ \sum_{j=1}^m ||\mathbf{P}_\star^*\varphi_j||_\star^2. \tag{18}$$

So we have

$$\boldsymbol{\sigma}_T^2(\hat{\mathrm{s}},\ \mathrm{s}) \geqslant \boldsymbol{\sigma}_T^2(\mathrm{s}^\star,\ \mathrm{s}).$$

## §7. The least-squares method in the estimate problem

The least-squares estimator $\mathbf{s}^*$ is defined by the relation

$$\mathrm{s}^* \ = \ arg \inf_{\mathbf{s} \,\in\, \mathbf{L}} ||y[\cdot] \ - \ s[\cdot]||,$$

$$\text{where} \qquad \mathbf{s} \ = \ \upsilon s \qquad \text{and} \qquad ||g[\cdot]|| \ = \ \sup_{\varphi \in D_T} \frac{|g[\varphi]|}{||\varphi||}$$

It is easily to see that

$$\mathbf{s}^* \ = \ \mathbf{P}(\mathbf{L})\, y, \qquad \text{that is} \qquad \mathbf{P}_{\mathbf{s}^*} \ = \ \mathbf{P}(\mathbf{L}),$$

where $\mathbf{P}(\mathbf{L})$ is the operator of orthogonal projection in space $\mathbf{H}$ on subspace $\mathbf{L}$. The least-squares estimator $\mathbf{s}^*$ is not the best linear unbiased estimator in general case.

**Proposition 2.** *Suppose that subspaces $\mathbf{L}$ and $\mathbf{L}^\perp$ are orthogonal in the metric of Hilbert space $\mathbf{H}_\star$, then the least-squares estimator $\mathbf{s}^*$ is the best linear unbiased estimater.*

**Proof.** In this case (see (14)) we have $\mathbf{P}_{\mathbf{s}^*} \ = \ \mathbf{P}(\mathbf{L}) \ = \ \mathbf{P}_\star \ = \ \mathbf{P}_{\mathbf{s}^\star}$, therefore $\mathbf{s}^* \ = \ \mathbf{s}^\star$.

## §8. The comparison of the accuracy of a linear unbiased estimator with the accuracy of the best linear unbised estimator

We would compare the accuracy of an estimator $\hat{\mathrm{s}} = \mathbf{P}y$, where $\mathbf{P} = \mathbf{P}(\mathbf{L}, \mathbf{L}^*)$, with the accuracy of the best linear unbiased estimator $\mathbf{s}^\star$. It has been stated in §6 (see (17) and (18)) that

$$\boldsymbol{\sigma}_T^2(\hat{\mathrm{s}},\ \mathrm{s}) \ = \ \sum_{j=1}^m ||\mathbf{P}^*\varphi_j||_\star^2, \qquad \boldsymbol{\sigma}_T^2(\mathrm{s}^\star,\ \mathrm{s}) \ = \ \sum_{j=1}^m ||\mathbf{P}_\star^*\varphi_j||_\star^2,$$

where $\{\varphi_j,\ j=1,...,m\}$ is an orthonormal basis of $\mathbf{L}$. Since

$\varphi_j = \psi_j + \varphi^j$, where $\varphi^j \in \mathbf{L}^\perp$ and $\psi_j = \mathbf{P}^*_\star \varphi_j,\ (j=1,...,m)$,

then

$$\mathbf{P}^*\varphi_j \;=\; \mathbf{P}^*[\psi_j + \varphi^j] \;=\; \mathbf{P}^*[\psi_j] \;=\; \mathbf{P}^*[\mathbf{P}^*_\star\varphi_j] \quad (j=1,...,m).$$

So we have

$$\mathbf{P}^*\varphi_j \;=\; \mathbf{P}^*[\mathbf{P}^*_\star\varphi_j] \quad (j=1,...,m).$$

Therefore

$$||\mathbf{P}^*\varphi_j||_\star \;\le\; ||\mathbf{P}^*||_\star \cdot ||\mathbf{P}^*_\star\varphi_j||_\star,$$

where for an operator $A$ we denote by $||A||_\star$ the uniform operator norm in the space $\mathbf{H}\star$. At last we obtain

$$\boldsymbol{\sigma}^2_T(\mathbf{s}^\star,\ \mathbf{s}) \;\le\; \boldsymbol{\sigma}^2_T(\hat{\mathbf{s}},\ \mathbf{s}) \;\le\; ||\mathbf{P}^*||_\star \cdot \boldsymbol{\sigma}^2_T(\mathbf{s}^\star,\ \mathbf{s}). \tag{19}$$

Now we would formulate and proof a simple geometrical result. We assume that $\mathbf{L}$ and $\mathbf{M}$ are subspaces of some Hilbert space $\mathbf{H}$ with inner product $(\cdot,\cdot)$ and norm $||\cdot||$. Suppose that

$$\overline{\mathbf{L}+\mathbf{M}} \;=\; \mathbf{H} \quad \text{and} \quad \mathbf{L}\cap\mathbf{M} \;=\; \mathbf{0}.$$

We denote

$$\cos(\mathbf{L},\mathbf{M};\mathbf{H}) \;=\; \sup_{h_1\in\mathbf{L},h_2\in\mathbf{M}} \frac{|(\mathbf{h}_1,\mathbf{h}_2)|}{||h_1||\cdot||h_2||}.$$

Let operator $\mathbf{P}$ is defined by the relation

$$\mathbf{Ph} \;=\; \begin{cases} \mathbf{h}, & \text{if} \quad \mathbf{h}\in\mathbf{L} \\ \mathbf{0}, & \text{if} \quad \mathbf{h}\in\mathbf{M}. \end{cases}$$

**Lemma.** *The uniform operator norm $||\mathbf{P}||$ of the operator $\mathbf{P}$ in the space $\mathbf{H}$ satisfies to the relation*

$$||\mathbf{P}||^2 \;\le\; \frac{1}{1-cos^2(\mathbf{L},\mathbf{M};\mathbf{H})}. \tag{20}$$

**Proof.** We assume that a linear set $\mathbf{L}+\mathbf{M}$ is dense in $\mathbf{H}$. Therefore

$$||\mathbf{P}|| \;=\; \sup_{\mathbf{h_L}\in\mathbf{L},\mathbf{h_M}\in\mathbf{M}} \frac{||\mathbf{h_L}||}{||\mathbf{h_L}-\mathbf{h_M}||}.$$

It is clear that

$$\inf_{\mathbf{h_M}\in\mathbf{M}} ||\mathbf{h_L} - \mathbf{h_M}|| \;=\; ||\mathbf{h_L} - \mathbf{P}(\mathbf{L})\mathbf{h_L}|| \quad \text{and}$$

$$\inf_{\mathbf{h}\in\mathbf{L}} ||\mathbf{h} - \mathbf{P}(\mathbf{L})\mathbf{h}||^2 \;=\; (1-\cos^2(\mathbf{L},\mathbf{M};\mathbf{H}))||\mathbf{h}||^2.$$

So we obtain (20).

**Proposition 3.** *For any linear unbiased estimator* $\hat{\mathrm{s}} \;=\; \mathbf{P}(\mathbf{L},\mathbf{L}^*)\,y$

$$\boldsymbol{\sigma}^2_T(\mathrm{s}^\star,\ \mathrm{s}) \;\le\; \boldsymbol{\sigma}^2_T(\hat{\mathrm{s}},\ \mathrm{s}) \;\le\; \frac{1}{1-cos^2(\mathbf{L}_*,\mathbf{L}^\perp;\mathbf{H}_\star)}\cdot\boldsymbol{\sigma}^2_T(\mathrm{s}^\star,\ \mathrm{s}). \qquad (21)$$

**Proof.** The statement of the *Proposition* follows from (19) and (21).

## §9. The spectral representation

### The reproducing kernel $G^f_t(u,\,v)$ of the space $\mathbf{H}_T(f)$

Now we consider the spectral representation of process $x$ (see in details [1]):

$$x[\varphi] \;=\; \int\limits_{-\infty}^{\infty} \hat{\varphi}(u)\; Z(du),$$

where $Z(\cdot)$ is such an orthogonal measure that

$$\mathbf{E}\,Z(A)\cdot\overline{Z(B)} \;=\; \int\limits_{A\cap B} f(u)\;du, \quad \mathbf{E}\,Z(A) = 0.$$

We denote by $\mathbf{X}^b_a$ such subspace of the space $L^2(dP)$, that

$$\mathbf{X}^b_a \;=\; \overline{sp}\{x[\varphi]:\ supp\,\varphi\subset[a,b]\}$$

(where we denote by $\overline{sp}\{M\}$ the closure of the linear manifold of $M$) and put $\mathbf{X} \;=\; \mathbf{X}^{\infty}_{-\infty}$. It is well known that the relation $U\,x[\varphi] \;=\; \hat{\varphi}$ defines an isometry $U\ :\mathbf{X}\rightarrow L^2_f$.

Let $\mathbf{H}_T(f)$ be a linear set of all entire function of power less than $T$ which belongs to the space $L^2_f$. The M.Krein alternative states that if $\overline{\mathbf{H}_T(f)} \;\ne\; L^2_f$, than $\mathbf{H}_T(f)$ is a closed subspace of the space $L^2_f$ and the functional $l_z\,:\,l_z(\varphi) \;=\; \varphi(z)$, is bounded on $\mathbf{H}_T(f)$. It must be mentioned that

$$U\,\mathbf{X}^T_{-T} \;=\; \mathbf{H}_T(f).$$

We denote by $G_T^f(u,v)$ the reproducing kernel of the space $\mathbf{H}_T(f)$:

$$\big(\,\varphi(\cdot),\,G_T^f(z,\cdot)\big) \;=\; \varphi(z).$$

It is clear that

$$G_T^f(u,v) \;=\; \sum_{j=1}^{\infty} \overline{\varphi_j(u)}\cdot\varphi_j(v), \qquad ||G_T^f(v,\cdot)||_f^2 \;=\; G_T^f(v,\,v)$$

where $\{\varphi_j,\; j=1,2,...\}$ is an orthonormal basis of the space $\mathbf{H}_T(f)$, and

$$|h(v)|^2 \;\leqslant\; G_T^f(v,\,v)\cdot||h(\cdot)||_f^2 \qquad \text{for any function} \quad h\in\mathbf{H}_T(f). \tag{22}$$

For example if function $f(t)\equiv 1$ we have

$$G_T^{\mathbf{1}}(u,v) \;=\; \frac{sin\,T(u-v)}{\pi(u-v)}, \qquad ||G_T^{\mathbf{1}}(u,\,\cdot)||_{\mathbf{1}}^2 \;=\; G_T^{\mathbf{1}}(u,\,u) \;=\; \frac{T}{\pi}, \tag{23}$$

where $\mathbf{1}(t)\;\equiv\;1$. We denote by $\Phi_T(v,u)$ –Fejer's kernel

$$\Phi_T(v,u) \;=\; \frac{sin^2\,T(u-v)}{\pi T\,(u-v)^2}.$$

**Proposition 4.** *Suppose that* $f, \frac{1}{f}\in L^2_{\frac{1}{1+u^2}}$. *Then*

$$\frac{T}{\pi}\cdot\Big(\int_{-\infty}^{\infty}\Phi_T(v,u)\cdot f(u)\,du\Big)^{-1} \;\leqslant\; G_T^f(v,v) \;\leqslant\; \frac{t}{\pi}\cdot\int_{-\infty}^{\infty}\Phi_T(v,u)\cdot\frac{1}{f(u)}\,du. \tag{24}$$

**Proof.** The function $G_T^{\mathbf{1}}(v,\,\cdot)$ is the reproducing kernel of the space $\mathbf{H}_T(\mathbf{1})$, therefore

$$\Big(\varphi(\cdot),\,g(\cdot)\Big)_f \;=\; \varphi(v),\;(\varphi\in\mathbf{H}_T(f)\cap\mathbf{H}_T(\mathbf{1})\,)$$

$$\text{for function} \qquad g(u) \;=\; \frac{G_T^{\mathbf{1}}(v,\,u)}{f(u)}.$$

Because of the linear set $\mathbf{H}_T(f)\cap\mathbf{H}_T(\mathbf{1})$ is dence in $\mathbf{H}_T(f)$, we have

$$G_T^f(v,\,\cdot) \;=\; \mathbf{P}_T(f)\,g, \quad where \quad \mathbf{P}_T(f)$$

is the orthoprojector in $L_f^2$ on $\mathbf{H}_T(f)$. So we conclude

$$G_T^f(v,\,v) \;=\; \|G_T^f(v,\,\cdot)\|_f^2 \;\leqslant\; \|\mathbf{P}_T(f)g\|_f^2 \;=\; \frac{T}{\pi}\int\limits_{-\infty}^{\infty} \Phi_T(v,u)\cdot\frac{1}{f(u)}\,du. \tag{25}$$

Now we apply (22) to the function $h(u) = \frac{1}{\sqrt{T\pi}}\frac{\sin T(v-u)}{(v-u)}$, which belongs to $\mathbf{H}_T(f)$ and therefore

$$\frac{T}{\pi} \;=\; |h(v)|^2 \;\leqslant\; G_T^f(v,\,v)\cdot\|h(\cdot)\|_f^2 \;=\; G_T^f(v,\,v)\cdot\int\limits_{-\infty}^{\infty} \Phi_T(v,u)\cdot f(u)\,du. \tag{26}$$

The relation (24) follows from (25) and (26).

We put

$$\mu_f(v) \;=\; \sup_{T\geqslant 0}\int\limits_{-\infty}^{\infty} \Phi_T(v,u)f(u)\,du\cdot\int\limits_{\infty}^{\infty} \Phi_T(v,u)\frac{1}{f(u)}\,du. \tag{27}$$

It is clear that $\mu_f(v)\geqslant 1$. From Proposition 4. we conclude:

**Proposition 5.** *Suppose that* $f, \frac{1}{f}\in L^2_{\frac{1}{1+u^2}}$. *Then*

$$\frac{T}{\pi\mu_f(v)}\cdot\int\limits_{-\infty}^{\infty} \Phi_T(v,u)\frac{1}{f(u)}\,du \;\leqslant\; G_T^f(v,\,v) \;\leqslant\; \frac{T}{\pi}\cdot\int\limits_{-\infty}^{\infty} \Phi_T(v,u)\frac{1}{f(u)}\,du \tag{28}$$

*and*

$$\frac{T}{\pi}\cdot\Big(\int\limits_{-\infty}^{\infty} \Phi_T(v,u)f(u)\,du\Big)^{-1} \;\leqslant\; G_T^f(v,\,v) \;\leqslant \tag{29}$$

$$\frac{T\mu_f(v)}{\pi}\cdot\Big(\int\limits_{-\infty}^{\infty} \Phi_T(v,u)f(u)\,du\Big)^{-1}.$$

**Proposition 6.** *Suppose that* $f, \frac{1}{f}\in L^2_{\frac{1}{1+u^2}}$. *Then*

$$\frac{T}{\pi\mu_f(v)} \;\leqslant\; G_T^f(v,\,v)\cdot G_t^{\frac{1}{f}}(v,\,v) \;\leqslant\; \frac{T\mu_f(v)}{\pi}. \tag{30}$$

## §10. The estimate problem for a signal which Fourier transform is a finite measure

Now we consider a case, when unknown signal $\mathbf{s}(t)$ may be represented in the form

$$\mathbf{s}(t) = \sum_{j=1}^{m} \theta_j \cdot exp\{itu_j\}, \tag{31}$$

where $\Lambda = \{u_j \quad (j = 1, 2, ..., m)\}$ we suppose that $\Lambda$ is a known finite subset of the real line. We denote by $\mathbf{L}$ the linear set of all functions $s(t)$ which may be represented in the form (31) and put

$$\mathbf{L}_T = \mathbf{1}_{[-T,T]}(t) \cdot \mathbf{L}.$$

We suppose that $\mathbf{H}$ is the $L^2$ space on interval $[-T, T]$ and put

$$s[\varphi] = (\varphi, \mathbf{s}).$$

Suppose that we observe random variables

$$y[\varphi] = s[\varphi] + x[\varphi], \qquad \varphi \in D_T.$$

**Theorem.** *Suppose that the spectral density $f$ of process $x$ satysfies to the condition*

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} \Phi_T(v, u) f(u)\, du \cdot \int_{\infty}^{\infty} \Phi_T(v, u) \frac{1}{f(u)}\, du = 1 \quad \textit{for} \quad v \in \Lambda.$$

*Then*

$$\frac{\boldsymbol{\sigma}_T^2(\mathbf{s}^*,\ \mathbf{s})}{\boldsymbol{\sigma}_T^2(\mathbf{s}^\star,\ \mathbf{s})} \to 1 \quad \textit{when} \quad T \to \infty.$$

**Proof.** It is not so difficult to observe that the statement of the Theorem follows from the Propositions 3–5 (see in detail [4]).

## Aknowlegements

The author would like to thank Prof. V. Bagdonavicius and M. S. Nikulin for helpful discussion.

## References

1. Yu. A. Rozanov, *Processus aleatoires stationnaires (en russe)*. MIR, Moscou, 1963.
2. I. A. Ibragimov, Yu. A. Rozanov, *Processus aleatoires gaussiens*. MIR, Moscou, 1974.
3. U. Grenander, M. Rozenblatt, *Statistical analyses of stationary time series*. New-Jercy 1956.
4. V. M. Solev, *Accuracy of the least squares method in the estimate problems on stationary noise*. Zap. Nauchn. Semin. POMI **232** (1996).
5. A. S. Holevo, N. Pasulov, *The regression problem for the processes with continuous time*. Teor. veroyatn. i ee primenen **23**, No. 4 (1978), 762–771.

С.-Петербургское отделение
Математического института
им. В. А. Стеклова РАН

Universite Bordeaux-2 – France

Поступило 18 декабря 1997