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FUSION OF q–TENSOR OPERATORS: QUASI–HOPF–ALGEBRAIC POINT OF VIEW

ABSTRACT. We discuss fusion of tensor operators for $U_q(\mathcal{J})$ using R-matrix approach. The problem is reduced to construction of the twisting element $\mathcal{F}$ which appears in Drinfeld's description of quasi-Hopf algebras. The discussion is illustrated by explicit calculations for the case $U_q(sl(2))$.

1. INTRODUCTION

§1.1 MOTIVATIONS AND NOTATIONS

The theory of tensor operators arose originally as a result of group-theoretical treatment to quantum mechanics [1]. And conversely, further development of the representation theory had been inspired by the physical interpretation of its mathematical content. Relatively recent appearance of the theory of quantum groups [2] led to development of the theory of q-deformed tensor operators [3]. The latter turned out to be not purely mathematical construction; it is employed, in particular, in description of quantum WZW model [4, 5].

In the present paper we discuss some aspects of the *fusion* procedure for (deformed) tensor operators in its R-matrix formulation [6]. We consider the special case of the fusion scheme – to construct a set of basic tensor operators for given irrep ρ^K if we are given those for two other irreps ρ^I and ρ^J (and ρ^K appears in decomposition of $\rho^I \otimes \rho^J$). This problem is closely related to Drinfel's construction of quasi-Hopf algebras [7]. Our aim is to obtain exact prescriptions applicable in practice, but to formulate the problem precisely we need to given first rather detailed introduction to the subject.

We suppose that the reader is familiar with the notion of Hopf algebra. The latter is an associative algebra $\mathcal{G}$ equipped with unit $\epsilon \in \mathcal{G}$, a homomorphism $\Delta : \mathcal{G} \mapsto \mathcal{G} \otimes \mathcal{G}$ (the *co-product*), an anti-automorphism $S : \mathcal{G} \mapsto \mathcal{G}$ (the *antipode*) and a one-dimensional representation $\epsilon : \mathcal{G} \mapsto \mathbb{C}$ (the *co-unit*) which obey a certain set of axioms [8]. A quasitriangular Hopf algebra [2] possesses in addition an invertible element $R \in \mathcal{G} \otimes \mathcal{G}$

(the universal R-matrix) obeying certain relations which, in particular, imply the Yang-Baxter equation. Throughout the paper we use so-called R-matrix formalish [9, 10]. Let us recall that its main ingredients are operator-valued matrices (L-operators)

$$L_+^I = (\rho^I \otimes id)R_+, \quad L_-^I = (\rho^I \otimes id)R_- \tag{1}$$

and numerical matrices (R-matrices)

$$R_+^{IJ} = (\rho^I \otimes \rho^J)R_+, \quad R_-^{IJ}(\rho^I \otimes \rho^J)R_-, \tag{2}$$

with ρ^I being an irrep of $\mathcal{G}$ and $R_+ = R$, $R_- = (R'_+)^{-1}$. Here and further on $'$ stands for permutation in $\mathcal{G} \otimes \mathcal{G}$.

Our consideration is restricted to the case of $\mathcal{G} = U_q(\mathcal{J})$ with $|q| = 1$ and with $\mathcal{J}$ being semi-simple Lie algebra (the way of generalization to $\mathcal{J}$ being arbitrary semi-simle modular Hopf algebra is described in [11]). Fir simplicity we assume also that q is not a root of unity.

We perform all explicit computations only in the case of $U_q(sl(2))$, but they be certainly repeated for, at least, $U_q(sl(n))$. Let us underline also that, although we deal with deformed tensor oparators and keep index q in some formulate, the classical (i.e., non-deformed) theory is recovered in the limit $q = 1$ and, therefore, it does not need special comments.

§1.2 (Deformed) tensor operators, generating matrices

Let given quasi-triangular Hopf algebra $\mathcal{G}$ be a symmetry algebra of some physical model. This means that the oparators corresponding to physical variables in this model are classified with respect to their transformation properties under the adjoint action of $\mathcal{G}$. Recall that if $\mathcal{H}$ is a certain Hilbert space such that $\mathcal{G} \subset \mathrm{End}\,\mathcal{H}$, then ($q$-deformed) adjoint action of an element $\xi \in \mathcal{G}$ on some element $\eta \in \mathrm{End}\,\mathcal{H}$ is defined as follows [3]

$$(\mathrm{ad}_q \xi)\eta = \sum_k \xi_k^1 \eta S(\xi_k^2), \tag{3}$$

where ξ_k^a are the components of the co-product $\Delta\xi = \sum\limits_k \xi_k^1 \otimes \xi_k^2 \in \mathcal{G} \otimes \mathcal{G}$ and $S(\xi) \in \mathcal{G}$ stands for the antipode of ξ.

From the physical point of view, the space $\mathcal{H}$ in (3) is the Hilbert space of the model in question. For $\mathcal{G}$ being a (quantum) Lie algebra one often chooses $\mathcal{H}$ as the corresponding model space. The latter is defined as a sum of all irreducible representations taken with multiplicity one: $\mathcal{M} = \oplus_I \mathcal{H}_I$ (I runs over all highest weights).

Let $\rho^J : \mathcal{G} \mapsto \mathrm{End}\, V^J$ be a highest weight J irrep of $\mathcal{G}$ with the carrier vector space V^J. The set of operators $\{T^J_m\}_{m=1}^{\dim \rho^J_\rho}$ acting on the Hilbert space $\mathcal{H}$ is called a tensor operator (of highest weight J) if

$$(\mathrm{ad}_q \xi) T^J_m = \sum_n T^J_n (\rho^J(\xi))_{nm} \quad \text{for all} \quad \xi \in \mathcal{G}. \tag{4}$$

If $\{T^I_m\}_{m=1}^{\dim {}^I_\rho}$ and $\{T^J_{m'}\}_{m'=1}^{\dim {}^J_\rho}$ are tensor operators acting on the same Hilbert space, then, using corresponding (deformed) Clebsch–Gordan coefficients, we can construct tensor operator of weight K' as follows [3]

$$T^K_{m''} = \sum_{m,m'} \begin{Bmatrix} I & J & K \\ m & m' & m'' \end{Bmatrix}_q T^I_m T^J_{m'}. \tag{5}$$

This formula describes *fusion* of tensor operators.

In the case of $\mathcal{G} = U_q(sl(2))$ the tensor operators are labeled by spin J and the definition (4) acquires the form:

$$\begin{aligned} X^{\pm} T^J_m q^H - q^{H \mp 1} T^J_m X^{\pm} &= \sqrt{[J \mp m][J \pm m + 1]}\, T^J_{m \pm 1}, \\ q^H T^J_m q^{-H} &= q^m T^J_m, \end{aligned} \tag{6}$$

where $[x] = (q^x - q^{-x})/(q - q^{-1})$ denotes q-number and $X_\pm$ and H are the generators with the relations

$$[H, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = [2H].$$

An example of (deformed) tensor operator (of spin 1) is provided by the following set of combinations of the generators

$$T^1_1 = q^{-H} X_+, \quad T^1_0 = (q^{-1} X_- X_+ - q X_+ X_-)/\sqrt{[2]}, \quad T^1_{-1} = -q^{-H} X_-. \tag{7}$$

But this is rather special case because, generally speaking, components of tensor operators act on the model space as shifts between different subspaces $\mathcal{H}^I$, whereas for components in (7) $\mathcal{H}$ are invariant subspaces.

Let us remark that along with tensor operator of covariant type introduced in (4) one can define contravariant tensor operator as the set of operators $\{\bar{T}^J_m\}_{m=1}^{\dim {}^J_\rho}$ obeying the relations

$$(\mathrm{ad}_q \xi) \bar{T}^J_m = \sum_n (\rho^J(S(\xi)))_{mn} \bar{T}^J_n \quad \text{for all} \quad \xi \in \mathcal{G}. \tag{8}$$

Further we consider only the covariant case since the theory and computations for the contravariant case are quite analogous.

In the case of quasi-triangular Hopf algebra we can describe tensor operators using R-matrix language. Let ρ^J denotes highest weight J irrep of $\mathcal{G}$ with the carrier space V^J. Consider the matrix $U^J \in \mathrm{End}\, V^J \otimes \mathrm{End}\, \mathcal{H}$ obeying following R-matrix relations

$$\overset{1}{L}{}^I_\pm \overset{2}{U}{}^J = \overset{2}{U}{}^J R^{IJ}_\pm \overset{1}{L}{}^I_\pm , \tag{9}$$

where $L^J_\pm$ and $R^{IJ}_\pm$ are defined as in (1)–(2). Equations (9) are equivalent [6] to the statement that each row of U^J satisfies (4); that is, all rows of U^J are tensor operators of weight J. We shall refer to U^J as *generating matrices* because, according to the Wigner–Eckart theorem matrix elements of entries of U^J (evaluated, say, on the corresponding model space) give (q-deformed) Clebsch–Gordan coefficients.

Note that U^J in (9) may have arbitrary number of rows and they are not necessarily linearly independent tensor operators. But from now on we shall work with the case of square generating matrices. Moreover, for $\mathcal{G} = U_q(sl(n))$ (and very likely even for $U_q(\mathcal{J})$) for any semi=simple CJ) one can construct a square generating matrices with all the rows being linearly independent tensor operators [4, 5, 12]. Notice also that, if U^J obeus (9) and M is a matrix with entries commuting with all the elements of $\mathcal{G}$, then

$$\widetilde{U}^J = M U^J \tag{10}$$

also obeys (9), i.e., $\widetilde{U}^J$ also is a generating matrix.

Now let U^I, U^J obey (9). The analogue of the fusion formula (5) for generating matrices reads [6]:

$$U^{IJ}_K = P^{IJ}_K F^{IJ} \overset{2}{U}{}^J \overset{1}{U}{}^I P^{IJ}_K \in V^I \otimes V^J \otimes \mathrm{End}\, \mathcal{H}, \tag{11}$$

where l.h.s. is a new generating matrix of weight K written in the basis of $V^I \otimes V^J$. Here $F^{IJ} \in V^I \otimes V^J$ stands for arbitrary matrix whose entries commute with all elements of $\mathcal{G}$ and P^{IJ}_K denotes a projector (i.e., $(P^{IJ}_K)^2 = P^{IJ}_K$) onto the subspace in $V^I \otimes V^J$ corresponding to the representation ρ^K.

One can rewrite (11) in the standard basis of the space V^K:

$$U^K_{mn} = \epsilon^t_m U^{IJ}_K \, \epsilon_n , \quad m, n = 1, \ldots , \dim \rho^K . \tag{12}$$

Here $\{\epsilon_n\}$ is an orthonormal set of the eigenvectors of the projector P^{IJ}_K, that is, $P^{IJ}_K = \sum \epsilon_n \otimes \epsilon^t_n$ and $\epsilon^t_m \epsilon_n = \delta_{mn}$.

Formula (11) resembles the fusion formula for R-matrices [10]:

$$\overset{1,32}{R}{}^{LK}_{\pm} = \overset{23}{P}{}^{IJ}_{K}\,\overset{13}{R}{}^{LJ}_{\pm}\,\overset{12}{R}{}^{LI}_{\pm}\,\overset{23}{P}{}^{IJ}_{K}\,, \tag{13}$$

where the l.h.s. stands for $R^{LK}_{\pm}$ written in the basis of $V^L \otimes V^I \otimes V^J$ and we use notations of [10]. Of course, the origin of both (11) and (13) is the Hopf structure of $\mathcal{G}$.

The fusion formulae given above are of direct practical use since they allow to construct corresponding objects (generating matrices and R-matrices) for higher representations starting with those for the fundamental irreps. For latter use we rewrite (12)–(13) as follows

$$U^K = C[IJK]F^{IJ}\overset{2}{U}{}^J\overset{1}{U}{}^I C'[IJK], \tag{14}$$

$$R^{LK}_{\pm} = \overset{23}{C}[IJK]\overset{13}{R}{}^{LJ}_{\pm}\,\overset{12}{R}{}^{LI}_{\pm}\,\overset{23}{C}[IJK]. \tag{15}$$

Here we used so called Clebsch Gordan maps, $C[IJK] : V^I \otimes V^J \mapsto V^K$ and $C'[IJK] : V^K \mapsto V^I \otimes V^J$, which are constructed according to the rules

$$C[IJK] = \sum_{n=1}^{\dim \rho^K} \widehat{\epsilon}_n \otimes \epsilon'_n\,, \quad C'[IJK] = \sum_{n=1}^{\dim \rho^K} \epsilon_n \otimes \widehat{\epsilon}^{\,t}_n\,, \tag{16}$$

where $\widehat{\epsilon}_n$ stands for the vector ϵ_n rewritten in the basis of the space V^K. The main properties of the CG maps are:

$$C[IJK](\rho^I \otimes \rho^J)\Delta(\xi)C'[IJK] = \rho^K(\xi) \quad \text{for any} \quad \xi \in \mathcal{G}, \tag{17}$$

$$\sum_K C'[IJK]C[LMK] = \delta_{IL}\delta_{JM}\,, \quad CD[IJK]C[IJL] = \delta_{KL}\,. \tag{18}$$

§1.3 Exact generating matrices

Let $\mathcal{C}$ denote a commutative $*$-algebra of functions on the weight space of $U_q(\mathcal{J})$. It is convenient to parametrize the coordinate on $\mathcal{C}$ by the vector $\vec{p} = 2J + \rho$, where J runs over all highest weights and ρ is the sum of the simple roots of CJ. Thus, $\mathcal{C}$ is an algebra of functions depending on the "variables" p_i – components of $\vec{p}$.

Let $\mathcal{J}$ be a Lie algebra of rank n and let $|q| = 1$. Generating matrices for $\mathcal{G} = U_q(\mathcal{J})$ can be constructed using following scheme [4, 5, 11, 12]. Define $D = q^{2\vec{H}\otimes\vec{p}} \in \mathcal{G} \otimes \mathcal{C}$ and $\Omega = q^{4\vec{H}\otimes\vec{H}} \in \mathcal{G} \otimes \mathcal{G}$, where $\vec{A} \otimes \vec{B}$ is understood as $\sum_{i=1}^{n} A_i \otimes B_i$, and H_i are the basic generators of the Cartan subalgebra of $\mathcal{G}$. Next, intriduce the map

$$\sigma : \mathcal{C} \mapsto \otimes\mathcal{C}, \quad \sigma(\vec{p}) = \epsilon \otimes \vec{p} + 2\vec{H} \otimes \epsilon. \tag{19}$$

Let $\mathcal{R}_\pm(\vec{p}) \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{C}$ be solutions (related as usual by $\mathcal{R}_-(\vec{p}) = (\mathcal{R}'_+(\vec{p}))^{-1}$) of the equations

$$\overset{12}{\mathcal{R}}_\pm(\vec{p})\overset{13}{R}_\pm(\vec{p}_2)\overset{23}{\mathcal{R}}_\pm(\vec{p}) = \overset{23}{\mathcal{R}}_\pm(\vec{p}_1)\overset{13}{\mathcal{R}}_\pm(\vec{p})\overset{12}{\mathcal{R}}_\pm(\vec{p}_3), \tag{20}$$

$$[\mathcal{R}_\pm(\vec{p}), q^{H_i} \otimes q^{H_i}] = 0 \quad \text{for all} \quad i, \tag{21}$$

$$\mathcal{R}_-(\vec{p})\overset{2}{D} = \Omega\overset{2}{D}\mathcal{R}_+(\vec{p}), \tag{22}$$

$$\mathcal{R}^*_\pm(\vec{p}) = \mathcal{R}^{-1}_\pm(\vec{p}). \tag{23}$$

The subscripts in (20) denote shifts of the corresponding arguments[1] according to (19). It is easy to verify that the property of unutary[2] (23) of $\mathcal{R}_\pm(\vec{p})$ is consistent with (20)–(22) for $|q| = 1$.

In general, for given $\mathcal{J}$ there exists a family of solutions of Eq. (20). But in the case of $\mathcal{J} = sl(n)$ the additional conditions (21)–(23) fix the solution uniquely [5, 13], and it is very likely that the same picture holds for any (deformed) semi-simple Lie algebra $\mathcal{J}$. Moreover, such a solution has the remarkable property [4, 5, 20] – entries of $\mathcal{R}^{IJ}_\pm(\vec{p})$ are nothing but corresponding (deformed) $6j$-symbols (involving weights I, J and all K allowable by the triangle inequality).

Now consider the element $U \in \mathcal{G} \otimes \operatorname{End}\mathcal{H}$ which obeys the equations

$$\mathcal{R}_\pm(\vec{p})\overset{2}{U}\overset{1}{U} = \overset{1}{U}\overset{2}{U}R_\pm, \tag{24}$$

$$\overset{1}{U}\overset{2}{D} = \Omega\overset{2}{D}\overset{1}{U}, \tag{25}$$

$$U^{-1}DU = q^{2C(\vec{p})}L_+L_-^{-1}, \tag{26}$$

where $C(\vec{p}) = \frac{1}{4}(\vec{p}+\rho)(\vec{p}-\rho)$ is the Casimir element. It can be shown [4, 5, 11, 12] that such U (if it exists) is a generating matrix[3] for $\mathcal{G}$. Note

[1] Notice that (20) takes values in $\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$. The subscript $i = 1, 2, 3$ means that $\vec{H}$-term of (19) appears in the i^{th} tensor component.

[2] The conjugation of an object belonging to n-fold tensor product $\mathcal{G}^{\otimes n}$ is understood as follows: $(\xi_1 \otimes \xi_2 \ldots \otimes \xi_n)^* = \xi_1^* \otimes \xi_2^* \ldots \otimes \xi_n^*$.

[3] To be more precise, U and $\mathcal{R}_\pm(\vec{p})$ are not matrices but so called *universal* objects. If we fix representations of their $\mathcal{G}$-parts: $U^J = (\rho^J \otimes id)U$, $\mathcal{R}^{IJ}_\pm(\vec{p}) = (\rho^I \otimes \rho^J)\mathcal{R}_\pm(\vec{p})$, we obtain a generating matrix and $\mathcal{C}$-valued counterparts of standard R-matrices.

that (20)–(21) are nothing but consistency conditions for (24), (25); (21) are the same symmetry conditions which are known[4] for the standard R-matrices of $U_q(\mathcal{J})$. Relation (25) is a matrix form of the equation

$$U\vec{p} = \sigma(\vec{p})U. \tag{27}$$

Equation (26) is a kind of normalization condition.

Notice that from the group of transformations (10) only following one

$$U \mapsto D^\alpha U \quad \text{for any} \quad \alpha \in \mathbb{C} \tag{28}$$

survives for the solution U of Eqs. (24)–(26). A validity of (28) can be easily checked with the help of (21) and (22). Additionally, it is easy to verify that the rescaling

$$U \mapsto (\epsilon \otimes f(\vec{p}))U(\epsilon \otimes f^{-1}(\vec{p})) \tag{29}$$

with arbitrary function $f(\vec{p}) \in \mathcal{C}$ is allowed.

Let us explain why the generating matrix obeying (24)–(27) is of especial interest from the point of view of the theory of tensor operators. Notice that the property (27) ensures that if such U exists, then its rows are linearly independent tensor operators. In other words, if for given irrep ρ^I and given vector $|I,m\rangle$ in the model space $\mathcal{M}$ of $\mathcal{G}$ we consider a set of vectors $U^J_{ij}|I,m\rangle$ $i,j = 1,\ldots,\dim\rho^J$, then all non-vanishing vectors in this set are pairwise linearly independent. In particular, if J is a fundamental irrep, then entries of U^J provide a set of basic shifts on $\mathcal{M}$. Thus, the solution of (24)–(26) presents very special but, in fact, the most interesting case of generating matrix. We shall refer to it as exact generating matrix.

It is worth to mention that matrix elements $\langle K,m''|U^J_{ij}|I,m'\rangle$ coincide with the Clebsch–Gordan coefficients $\left\{ \begin{smallmatrix} I & J & K \\ m & m' & m'' \end{smallmatrix} \right\}_q$ (up to some p_i-dependent factors which are allowed according to (28), (29)) which appear in decomposition of the tensor product $\rho^I \otimes \rho^J$ (with the weights I, J, K restricted by the triangle inequality). This property of exact generating matrices make them especially important from practical point of view.

That concerning physical content of relations given above, Eq. (20) has appeared in various forms in studies of quantum versions of Liouville [14, 14], Toda [16], and Calogero–Moser [17] models. In these models

[4] Recall that quantization does not deform the co-multiplication for elements of the Cartan subalgebra.

$\mathcal{R}(\vec{p})$ is interpreted as a dynamical R-matrix. From the point of view of the theory of tensor operators relations (19)–(27) most closely connected with quantization of WZW model [4, 5, 11]. Here $\mathcal{R}9\vec{p})$ plays a role of the brading matrix and Eqs. (24)–(26) with appropriate dependence on the spatial coordinate (or its discretized version) describe vertex operators. Let us mention that in WZW theory the quantum-group parameter of $\mathcal{G} = U_q(\mathcal{J})$ is given by $q = \epsilon^{i\gamma\hbar}$, where $\hbar > 0$ is the Planck constant and the deformation parameter $\gamma > 0$ is interpreted as a coupling constant. That is why we consider the case of $|q| = 1$.

II. Fusion of exact generating matrices

§2.1 Formulation of the problem

Suppose we are given generating matrix U^I, U^J for some irreps of $\mathcal{G}$. Then by formula (11) we can build up generating matrices U^K for every irrep ρ^K which appears in the decomposition of $\rho^J \otimes \rho^J$. For the sake of shortness we shall call them *descendant* matrices. But, as we explained before, it is natural to deal not with all possible generating metrices but only with exact ones, i.e., with those obeying the additional Eqs. (24)–(27) with $\mathcal{R}_{\pm}(\vec{p})$, D and Ω introduced above. Here we face the problem – to find such a matrix F^{IJ} that the descendant matrix U^K obtained by the formula (11) would be also an exact generating matrix.

Let us underline that this problem would not arise if (24) contained in l.h.s. the standard R-matrix instead of $\mathcal{R}(\vec{p})$. Indeed, for operator-valued matrix $g^J \in \mathrm{End}\, V^J \otimes \mathcal{G}$ (it may be regarded as L-operator type object[5]) which obeys usual quadratic relations[6]

$$R^{IJ} \overset{2}{g}{}^J \overset{1}{g}{}^I = \overset{1}{g}{}^I \overset{2}{g}{}^J R^{IJ}, \tag{30}$$

the fusion formula is well known (Eq. (13) is its particular realization):

$$(g^K)_{mn} = \epsilon^t_m \overset{2}{g}{}^J \overset{1}{g}{}^I \epsilon_n, \tag{31}$$

where ϵ_n, $n = 1, \ldots, \dim \rho^K$ are the eigenvectors of the projector P^{IJ}_K as before. For example, in the case of $\mathcal{G} = U_q(sl(2))$ starting with $g^{\frac{1}{2}}$ and

[5] For $\mathcal{G}$ replaced by its dual G' the matrix g^J is regarded as the quantum group-like element. The fusion formulate (31)–(32) are also valid in this case.

[6] We prefer this order of auxiliary spaces since it is the same as in (24).

applying (31) sufficiently many times one can matrix g^J for any spin J:

$$g^0 = 1,\; g^{\frac{1}{2}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
$$g^1 = \begin{pmatrix} a^2 & q^{-\frac{1}{2}}\sqrt{[2]}ab & b^2 \\ q^{-\frac{1}{2}}\sqrt{[2]}ac & ad + q^{-1}bc & q^{-\frac{1}{2}}\sqrt{[2]}bd \\ c^2 & q^{-\frac{1}{2}}\sqrt{[2]}cd & d^2 \end{pmatrix}, \dots . \tag{32}$$

For generating matrices the fusion problem is more complicated because $\mathcal{R}(\vec{p})$ in Eqs. (20) and (24) is an atttribute not of Hopf but of quasi-Hopf algebra. In this section we discuss some general aspects of the fusion problem in quasi-Hopf case. In the next section we shall consider an example – the case of $U_q(sl(2))$.

It also should be underlined that the fusion problem (in the form formulated above) does not appear if one uses the language of universal objects (see, e.g., [19, 11]) instead of the language of operator-valued matrices. For example, instead of the set of matrices $g^J \in \mathrm{End}\, V^J \otimes \mathcal{G}$ obeying (30) we could introduce the element $g \in \mathcal{G} \otimes \mathcal{G}$ and fix its *functoriality* relation as follows:

$$(\Delta \otimes id)(g) = \overset{2}{g}\overset{1}{g}. \tag{33}$$

Then both quadratic relations (30) and fusion formula (31) can be obtained from (33) with the help of the axioms of quasi-triangular Hopf algebra. In fact, in this approach we even do not need the fusion formula because each g^J can be obtained simply by evaluation of g in fixed representation: $g^J = (\rho^J \otimes id)g$.

Similarly, we could introduce the universal object $U \in \mathcal{G} \otimes \mathrm{End}\,\mathcal{H}$ with the functoriality relation [11]:

$$(\Delta \otimes id)(U) = \mathcal{F}\overset{2}{U}\overset{1}{U}, \tag{34}$$

where $\mathcal{F}$ obeys a certain set of axioms. Then quadratic relations (24) (with $\mathcal{R}(\vec{p})$ constructed from $\mathcal{F}$ and R according to (41)) would be consequences of (34). Again, fixing representation of $\mathcal{G}$-part of the universal element U, we obtain a generating matrix $U^J = (\rho^J \otimes id)U$ and, therefore, we do not need the fusion formula.

Although the language universal objects is more convenient in abstract theoretical constructions, in practice we usually do not have explicit formulae for involved universal objects (for instance, the universal R-matrices are known only for $U_q(sl(2))$ and $U_q(sl(3))$. Therefore, in the present paper we intentionally adopted the matrix language to discuss how to construct from exact generating matrices of given representations those for other representations if explicit universal formulae are unknown.

§2.2 Quasi-Hopf features

Let us remind that an associative algebra $\mathcal{G}$ is said to be quasi-Hopf algebra [7] if its co-multiplication is "quasi-coassociative," that is, for all $\xi \in \mathcal{G}$ we have

$$\begin{gathered}((id \otimes \Delta)\Delta(\xi))\Phi = \Phi((\Delta \otimes id)\Delta(\xi)),\\ (\epsilon \otimes id)\Delta(\xi) = (id \otimes \epsilon) = (id \otimes \epsilon)\Delta(\xi) = \xi.\end{gathered} \tag{35}$$

Here $\Phi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$ is some invertible element (co-associator) which must satisfy certain equations. For quasi-triangular quasi-Hopf algebra one postulates in addition existence of an invertible $\mathcal{R} \in \mathcal{G} \otimes \mathcal{G}$ (twisted R-matrix) such that

$$\mathcal{R}\Delta(\xi) = \Delta'(\xi)\mathcal{R} \quad \text{for all} \quad \xi \in \mathcal{G}, \tag{36}$$

$$(\Delta \otimes id)\mathcal{R} = \Phi_{312}\overset{13}{\mathcal{R}}\Phi^{-1}_{132}\overset{23}{\mathcal{R}}\Phi_{123}, \quad (id \otimes \Delta)\mathcal{R} = \Phi^{-1}_{231}\overset{13}{\mathcal{R}}\Phi_{213}\overset{12}{\mathcal{R}}\Phi^{-1}_{123}, \tag{37}$$

$$(\epsilon \otimes id)\mathcal{R} = (id \otimes \epsilon)\mathcal{R} = \epsilon. \tag{38}$$

The analogue of Yang-Baxter equation for $\mathcal{R}$ follows from (36) and (37) and looks like

$$\overset{12}{\mathcal{R}}\Phi_{312}\overset{13}{\mathcal{R}}\Phi^{-1}_{132}\overset{23}{\mathcal{R}}\Phi_{123} = \Phi_{321}\overset{23}{\mathcal{R}}\Phi^{-1}_{231}\overset{13}{\mathcal{R}}\Phi_{213}\overset{12}{\mathcal{R}}. \tag{39}$$

A crucial observation [5, 11, 18] is that construction used in §2, 3 for description of exact generating matrices involves the quasi-Hopf algebra (with $\mathcal{R}_+(\vec{p})$ playing the role of the element $\mathcal{R}$) which is obtained as a twist of the quasi-triangular Hopf algebra $\mathcal{G}$. More precisely, there exists an invertible element $\mathcal{F}(\vec{p}) \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$ such that with its help one can construct from standard co-multiplication and R-matrices (which obey axioms of Hopf-algebra) the following objects

$$\Delta_{\mathcal{F}}(\cdot) = \mathcal{F}^{-1}(\vec{p})\Delta(\cdot)\mathcal{F}(\vec{p}), \tag{40}$$

$$\mathcal{R}_\pm(\vec{p}) = (\mathcal{F}'(\vec{p}))^{-1}R_\pm\mathcal{F}(\vec{p}), \tag{41}$$

$$\Phi(\vec{p})_{123} = \overset{12}{\mathcal{F}}{}^{-1}(\vec{p}_3)\overset{12}{\mathcal{F}}(\vec{p}) \tag{42}$$

which obey all the axioms of quasi-Hopf algebra[7]. In particular, (20) provides a realization of the abstract form (39) of the twisted Yang-Baxter equation.

The fact that $\mathcal{R}_\pm(\vec{p})$ introduced in (20)–(22) admit decomposition of type (41) is very important in the context of the fusion problem for exact generating matrices. Indeed, let we are given two exact generating matrices, U^I and U^J, which obey (24)–(27) with certain $\mathcal{R}^{IJ}_\pm(\vec{p})$. Applying the formula (11) with some matrix $F^{IJ}(\vec{p})$ (it may be $\mathcal{C}$-valued) to these U^I and U^J we get new matrix U^{IJ}_K. It automatically obeys (9). Moreover, it is easy to verify that exchange relations between U^{IJ}_K and any exact generating matrix U^L are again of from (24) but with new R-matrix in l.h.s. which in the basis of $V^L \otimes V^I \otimes V^J$ looks like

$$\overset{1,32}{\mathcal{R}}{}^{LK}_{\pm}(\vec{p}) = \overset{23}{P}{}^{IJ}(\vec{p}_1)\overset{13}{\mathcal{R}}{}^{LJ}_{\pm}(\vec{p})\overset{12}{\mathcal{R}}{}^{LI}_{\pm}(\vec{p}_3)(\overset{23}{F}{}^{IJ}(\vec{p}))^{-1}\overset{23}{P}{}^{IJ}_{K}. \tag{43}$$

This is an analogue of the fusion formula (13) for standard R-matrices. The demand that the new generating matrix U^{IJ}_K is exact, i.e., in particular, it must obey (24), implies that expression (43) being rewritten in the basis of $V^I \otimes V^J$ must coincide with $\mathcal{R}^{LK}_\pm(\vec{p})$. Taking into account that $\mathcal{R}_\pm(\vec{p})$ satisfies (41) some matrix $\mathcal{F}$, we get the equality

$$\begin{aligned}\mathcal{R}^{IJ}_\pm(\vec{p}) = \overset{23}{C}[IJK]\overset{23}{F}{}^{IJ}(\vec{p}_1)\overset{13}{\mathcal{R}}{}^{LJ}_{\pm}(\vec{p})(\overset{21}{\mathcal{F}}{}^{IL})^{-1}(\vec{p}_3)\overset{21}{\mathcal{F}}{}^{IL}(\vec{p})\times \\ \times\, \overset{12}{\mathcal{R}}{}^{LI}_{\pm}(\vec{p})(\overset{12}{\mathcal{F}}{}^{LI})^{-1}\overset{12}{\mathcal{F}}{}^{LI}(\vec{p}_3)(\overset{23}{F}{}^{IJ})^{-1}(\vec{p})\overset{23}{C'}[IJK]. \end{aligned} \tag{44}$$

The later is equivalent (due to (17) and (40)) to the identity

$$\begin{aligned}(\rho^L \otimes \rho^I \otimes \rho^J)(id \otimes \Delta_{\mathcal{F}})\mathcal{R} = (\overset{23}{F}{}^{IJ}(\vec{p}))^{-1}\overset{23}{F}{}^{IJ}(\vec{p}_1)\overset{13}{\mathcal{R}}{}^{LJ}_{\pm}(\vec{p})\times \\ \times\, (\overset{21}{\mathcal{F}}{}^{IL})^{-1}(\vec{p}_3)\overset{21}{\mathcal{F}}{}^{IL}(\vec{p})\overset{12}{\mathcal{R}}{}^{LI}_{\pm}(\vec{p})(\overset{12}{\mathcal{F}}{}^{LI})^{-1}(\vec{p})\overset{12}{\mathcal{F}}{}^{LI}(\vec{p}_3),\end{aligned}$$

which, as we see from (37) and (42), takes place only if $F^{IJ} = \mathcal{F}^{IJ}(\vec{p})$.

§2.3 Properties of the twisting element

The practical resume of the previous paragraph is: if we are given exact generating matrices U^I and U^J (and, hence, we known $\mathcal{R}^{IJ}_\pm(\vec{p})$), then to construct new exact matrix U^K we must substitute in the fusion formula

[7] In fact, here we deal here with some generalization of the Drinfeld's scheme since $\mathcal{F}(\vec{p})$, $\mathcal{R}_\pm(\vec{p})$ and $\Phi(\vec{p})$ possess additional $\mathcal{C}$-valued tensor component. But all Hopf-algebra operations are applied only to $\mathcal{G}$-parts of these objects.

(11) the matrix $F^{IJ} = (\rho^I \otimes \rho^J)\mathcal{F}(\vec{p})$ where $\mathcal{F}(\vec{p})$ is the twisting element of the quasi-Hopf algebra introduced above. An obstacle to application of this prescription is that usually an explicit universal expression for $\mathcal{F}(\vec{p})$ is unknown. But, assuming that such $\mathcal{F}(\vec{p})$ exists, we can look for $\mathcal{F}^{IJ}(\vec{p})$ as a matrix satisfying the list of conditions:

1. $\mathcal{F}^{IJ}(\vec{p})$ is a solution of the Eq. (41) for given $\mathcal{R}^{IJ}_{\pm}(\vec{p})$, that can be rewritten in the following more convenient in practice form

$$R^{IJ}_{\pm}\mathcal{F}^{IJ}(\vec{p}) = (\mathcal{F}^{IJ}(\vec{p}))'\mathcal{R}^{IJ}_{\pm}(\vec{p}); \tag{45}$$

2. $\mathcal{F}^{IJ}(\vec{p})$ obeys the symmetry condition

$$[\mathcal{F}^{IJ}(\vec{p}), q^{H^I_i} \otimes q^{H^J_i}] = 0 \quad \text{for all} \quad i; \tag{46}$$

3. $\mathcal{F}^{IJ(\vec{p})}$ is sucg that all entries of the matrix[8]

$$U^{IJ}_0 = P^{IJ}_0 \mathcal{F}^{IJ}(\vec{p})\overset{2}{U}{}^J \overset{1}{U}{}^I P^{IJ}_0, \tag{47}$$

or, equivalently, of the matrix

$$U^0 = C[IJ0]\mathcal{F}^{IJ}(\vec{p})\overset{2}{U}{}^J \overset{1}{U}{}^I C'[IJ0], \tag{48}$$

commute with all entries of generating matrix U^M for any weight M;

4. $\mathcal{F}^{IJ}(\vec{p})(\mathcal{F}^{IJ}(\vec{p}))^*$ is a p-independent object, or, in other words $\mathcal{F}^{IJ}(\vec{p})(\mathcal{F}^{IJ}(\vec{p})^* = \chi$ with χ being element of $\mathcal{G} \otimes \mathcal{G}$.

Let us comment these conditions. Necessity of the first of them was explained in the previous paragraph. But it is not a sufficient condition since, in general, (45) possesses a family of solutions. In principle, we could separate the right solution in this family verifying whether a substitution of this solution in fusion formulae (43) or (44) yields matrices $\mathcal{R}(\vec{p})$ obeying (20)–(23). But such a verification would be quite cumbrous in practice.

The second condition ensures that the descendant matrix U^K obeys (27), as a consequence, (25). This can be easily checked applying (27) to (11). Notice also that (46) implies that for this specific quasi-Hopf algebra the co-multiplication on the Cartan subalgebra is not deformed and, hence, it is the same as for $U_q(\mathcal{J})$ and CJ.

[8] Formulae (47), (48) can be regarded [6] as a generalization of the formula for quantum determinant [9, 10].

Condition 3 originates from (38) and the same property $(\epsilon \otimes id)R = (id \otimes \epsilon)R = \epsilon$ known for standard R-matrices (recall that ϵ stands for the trivial one-dimensional representation of $\mathcal{G}$). Indeed, applying $(\epsilon \otimes id)$ or $(id \otimes \epsilon)$ to (24), we conclude that $U^0 = (\epsilon \otimes id)U$ commutes with all entries of U^J for any J. Therefore, if the trivial representation $\rho^0 \equiv \epsilon$ appears in the decomposition of the product $\rho^I \otimes \rho^J$ (e.g., in the case when both irreps coincide with fundamental one), then l.h.s of Eqs. (47), (48) do not vanish and represent a tensor operator of zero weight (i.e., scalar with respect to the adjoint action of $\mathcal{G}$. In this case condition 3 is non-trivial because a tensor operator of zero weight may be p-dependent and, hence, in general, it does not commute with other tensor operators.

To clarify condition 4, we recall first that for $|q| = 1$ the standard comultiplication has the following property with respect to the conjugation in $\mathcal{G} : \Delta^*(\xi) = \Delta'(\xi^*)$. On the other hand, relations (23) imply that (see also [11])

$$(\Delta_{\mathcal{F}}(\xi))^* = \Delta_{\mathcal{F}}(\xi^*). \tag{49}$$

Hence, the self-conjugated element $\chi = \mathcal{F}(\vec{p})\mathcal{F}^*(\vec{p})$ obeys the relation $\chi\Delta = \Delta'\chi$. Additionally, we derive from (23) that

$$R_{\pm}\chi(\vec{p}) = (\chi(\vec{p}))' R_{\mp}^{-1}. \tag{50}$$

Next, we notice that from (23), (37), and (49) the unitariry of the coassociator follows:

$$\Phi^*(\vec{p})_{123} = \Phi^{-1}(\vec{p})_{123}. \tag{51}$$

According to (42), the latter equation leads to the condition $\overset{12}{\chi}(\vec{p}) = \overset{12}{\chi}(\vec{p}_3)$, which, due to the possibility to apply (27) to (51) arbitrary many times, implies p-independence of χ. Let us mention that the universal expression for the element $\chi \in \mathcal{G} \otimes \mathcal{G}$ was found in [11].

III. $U_q(sl(2))$ CASE

In this section we are going to illustrate the preceding discussion by some explicit calculations. Although solutions for the twisted Yang-Baxter Eq. (20) are known [5, 13, 14, 16] for fundamental representations of $\mathcal{G} = U_q(sl(n))$, we shall consider here only the case of $U_q(sl(2))$. But let us stress, that in the more general case of $U_q(sl(n))$ the computations would be essentially the same.

§3.1 $\mathcal{R}(\vec{p})$ AND U IN FUNDAMENTAL REPRESENTATION

As we stressed above, the twisted Yang-Baxter equation possesses a family of solutions. It takes place even in the simplest case of the fundamental representation of $U_q(sl(2))$. But imposing the additional conditions (21)–(23), we get the unique solution [5, 13] which depends on the single variable $p = 2J + 1$ with J being the spin. In particular, the fundamental matrices $\mathcal{R}_{\pm}(p)$ (that is with both auxiliary spaces being of spin 1/2) are given by (all non-specified entries are zeros)

$$\mathcal{R}_{+}^{\frac{1}{2}\frac{1}{2}} = P(\mathcal{R}_{-}^{\frac{1}{2}\frac{1}{2}})^{-1}P = q^{-\frac{1}{2}} \begin{pmatrix} q & & & \\ & \frac{\sqrt{[p+1][p-1]}}{[p]} & \frac{q^{p}}{[p]} & \\ & -\frac{q^{-p}}{[p]} & \frac{\sqrt{[p+1][p-1]}}{[p]} & \\ & & & q \end{pmatrix}, \quad (52)$$

where $[x] = (q^{x} - q^{-1})/(q - q^{-1})$ stands, as usually, for q-number. Entries of matrices (52) coincide with values of certain $6j$-symbols for $U_q(sl(2))$ [20].

Let us notice that the following asymptotics

$$\mathcal{R}_{\pm}^{\frac{1}{2}\frac{1}{2}}(p) \to R_{\pm}, \quad \mathcal{R}_{\pm}^{\frac{1}{2}\frac{1}{2}}(p) \to R_{\mp}^{-1} \quad (53)$$

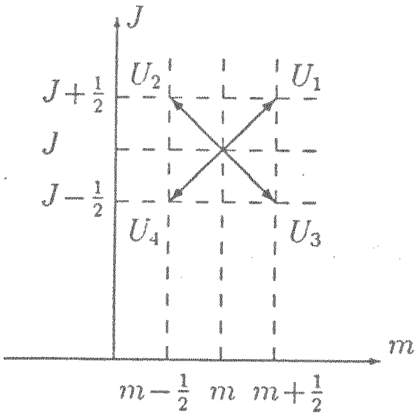

Fig. 1. Action of the operators U_i on the model space.

hold in the formal limits $q^p \to +\infty$ and $q^{-p} \to +\infty$, respectively; that is, here we return to the case of Hopf algebra, in particular, the co-associator becomes trivial: $\Phi_{123} = \epsilon \otimes \epsilon \otimes \epsilon$. Moreover, (53) together with (41), (50) allows to add the following condition to the list given in §2.3.

5. (Asymptotic behaviour)

$$\mathcal{F}(q^p \to +\infty) = \epsilon \otimes \epsilon, \quad \mathcal{F}(q^{-p} \to +\infty) = \chi, \tag{54}$$

where $\chi \in \mathcal{G} \otimes \mathcal{G}$ was described in the end of §2.3. Let us stress that we derived this additional condition only for $\mathcal{J} = sl(2)$. It would be interesting to find its generalization for $\mathcal{J} = sl(n)$ when we have the vector $\vec{p}$ instead of single variable.

Now let us turn to a solution of Eqs. (24)–(26) for $\mathcal{R}_\pm^{\frac{1}{2}\,\frac{1}{2}}(p)$ given by (52). It had been considered in different contexts in [4, 5, 12] and was shown to be unique up to the transformations (28), (29). In particular, it can be written [12] in terms of operators of multiplication and shift (difference derivative) of two complex variables:

$$\begin{aligned} U^{\frac{1}{2}} &= \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} = \\ &= \begin{pmatrix} z_1 q^{\frac{1}{2} z_2 \partial_2} & z_2 q^{-\frac{1}{2} z_1 \partial_1} \\ -z_2^{-1} [z_2 \partial_2] q^{-\frac{1}{2}(z_1 \partial_1 + 1)} & z_1^{-1} [z_1 \partial_1] q^{\frac{1}{2}(z_2 \partial_2 + 1)} \end{pmatrix} \frac{1}{\sqrt{[p]}}. \end{aligned} \tag{55}$$

Here $p = z_1 \partial_1 + z_2 \partial_2 + 1$ and r.h.s. is a realization of the exact generating matrix of spin 1/2. This means, in agreement with the general description of §1.3, that the entries of $U^{\frac{1}{2}}$ act on the model space $\mathcal{M} = \otimes_{J=0}^{\infty} \mathcal{H}_J$ as the basic shifts (see Fig. 1). This can be directly verified [12] by means of realization of the model space as $D_q(z_1, z_2)$ – the space of holomorphic functions of two complex variables with such a scalar product (a deformation of the standard one) that the monomials $|J, m\rangle = \frac{z_1^{J+m} z_2^{J-m}}{[J+m]![J-m]!}$ form an orthonormal basis. Moreover, for the concrete realization (55) the matrix elements $\langle J', m | U_i | J'', m'' \rangle$ (evaluated on $D_q(z_1, z_2)$) exactly coincide with the Clebsch–Gordan coefficients $\left\{ \begin{smallmatrix} J' & 1/2 & J'' \\ m' & \pm 1/2 & m'' \end{smallmatrix} \right\}_q$ (four of them are non-vanishing); let us call this property "preciseness."

Now we encounter the simplest version of the fusion problem – to build up the exact generating matrix of spin 1 from $U^{\frac{1}{2}}$. For this purpose we have to find an explicit form of the corresponding twisting element $\mathcal{F}$ in the fundamental representation.

Before going into the computations let us mention that in [15] a universal formula (i.e., applicable for representations of any spin) for solution $\widetilde{\mathcal{R}}(p)$ of Eq. (20) was found and a universal expression for $\widetilde{\mathcal{F}}(p)$ obeyin (41) with given $\widetilde{\mathcal{R}}(p)$ was obtained. But this solution $\widetilde{\mathcal{R}}(p)$ does not satisfy (23) and, therefore, taken, say, in the fundamental representation it differs from (52). Thus, solutions $\widetilde{U}$ of (24) with such $\widetilde{\mathcal{R}}(p)$ would not be exact generating matrices in our sense. In particular, the solution for spin 1/2 would differ from one given by (55) and, therefore, would not have the remarable properties spelt out above.

Let us stress although that these new matrices $\widetilde{U}$ still would generating matrices in the sense of definition (9). Therefore, one could examine whether they can be converted into exact generating matrices by means of the transformation $\widetilde{U} = M(p)U$ with $M(p) \in \mathcal{G} \otimes \mathcal{C}$. If such $M(p)$ exists, then the following relations hold:

$$\widetilde{\mathcal{F}}(p) = (\Delta \otimes id) M(p) \mathcal{F} (\overset{1}{M}(p_2) \overset{2}{M}(p))^{-1},$$
$$\widetilde{\mathcal{R}}_{\pm}(p) = \overset{2}{M}(p_1) \overset{1}{M}(p) \mathcal{R}_{\pm}(p) (\overset{1}{M}(p_2) \overset{2}{M}(p))^{-1},$$

and we could construct our $\mathcal{F}(p)$ from $\widetilde{\mathcal{F}}(p)$ and $M(p)$. But, bearing in mind possible extensions to the cases where no universal formulae for $\mathcal{R}(p)$ are known, we prefer to give more direct computations of $\mathcal{F}(p)$ instead of the quest of such $M(p)$.

§3.2 Computation of $\mathcal{F}^{\frac{1}{2}\frac{1}{2}}(p)$ and U^1

The matrix $\mathcal{F}^{\frac{1}{2}\frac{1}{2}}$ must satisfy the list of conditions given in §3.2. First of all, it must be a solution of the equation (45), where $\mathcal{R}_{\pm}(p)$ in r.h.s. are given by (52) and the standard R-matrices in l.h.s. are

$$\begin{aligned} \mathcal{R}_{+}^{\frac{1}{2}\frac{1}{2}} &= q^{-1/2} \begin{pmatrix} q & & & \\ & 1 & \omega & \\ & & 1 & \\ & & & q \end{pmatrix}, \\ \mathcal{R}_{-}^{\frac{1}{2}\frac{1}{2}} &= q^{1/2} \begin{pmatrix} q^{-1} & & & \\ & 1 & & \\ & -\omega & 1 & \\ & & & q^{-1} \end{pmatrix}, \end{aligned} \tag{56}$$

with $\omega = q - q^{-1}$. The symmetry condition 2 dictates to look for the solution of Eq. (45) in the following form

$$\mathcal{F}(p)=\begin{pmatrix}1&&&\\&\alpha(p)&\beta(p)&\\&\gamma(p)&\delta(p)&\\&&&1\end{pmatrix}. \qquad (57)$$

The straightforward check shows that only two of functions $\alpha(p)$, $\beta(p)$, $\gamma(p)$, $\delta(p)$ are independent and we can express, say entries of the third line in (57) via those of the second. The result reads

$$\begin{aligned}\gamma(p)&=\frac{q^{-p}}{[p]}\,\alpha(p)+\frac{\sqrt{[p+1][p-1]}}{[p]}\,\beta(p),\\ \delta(p)&=\frac{\sqrt{[p+1][p-1]}}{[p]}\,\alpha(p)+\frac{q^{p}}{[p]}\,\beta(p).\end{aligned} \qquad (58)$$

Now we should employ condition 3. To this end we use the following formulae for the fundamental R-matrices of $U_q(sl(n))$ (see [9] for details):

$$P_{\pm}=\frac{q^{\frac{1}{n}\pm1}\widehat{R}_{+}-q^{-\frac{1}{n}\mp1}\widehat{R}_{-}}{q^{2}-q^{-2}}, \qquad (59)$$

where $\widehat{R}_{\pm}=PR_{\pm}$ and P_{+}, P_{-} are the projectors in $\mathbb{C}^n\otimes\mathbb{C}^n$ (q-symmetrizer and q-antisymmetrizer) of ranks $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$, respectively. In the case of $U_q(sl(2))$ these projectors are

$$P_{\pm}=\begin{pmatrix}1&&&\\&q^{\pm1}\lambda&\pm\lambda&\\&\pm\lambda&q^{\pm1}\lambda&\\&&&1\end{pmatrix}, \qquad (60)$$

where $\lambda=\frac{1}{[2]}=(q+q^{-1})^{-1}$. It is easy to find their eigenvectors $\vec{x}_i$ such that $\vec{x}_i^t\vec{x}_j=\delta_{ij}$:

$$\begin{aligned}&P_{+}=\sum_{i=1}^{3}\vec{x}_i\otimes\vec{x}_i^t,\quad \vec{x}_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\\ &\vec{x}_2=\sqrt{\lambda}\begin{pmatrix}0\\q^{-1/2}\\q^{1/2}\\0\end{pmatrix},\quad \vec{x}_3=\begin{pmatrix}0\\0\\0\\1\end{pmatrix};\end{aligned} \qquad (61)$$

$$P_- = \vec{x}_0 \otimes \vec{x}_0^t, \quad \vec{x}_0 = \sqrt{\lambda} \begin{pmatrix} 0 \\ q^{1/2} \\ -q^{-1/2} \\ 0 \end{pmatrix}, \tag{62}$$

According to (16) we can construct from these vectors the following CG maps

$$\begin{aligned} C[\frac{1}{2}\,\frac{1}{2}0] &= \sqrt{\lambda}(0, q^{1/2}, -q^{-1/2}, 0), \\ C[\frac{1}{2}\,\frac{1}{2}1] &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\lambda}\, q^{-1/2} & \sqrt{\lambda}\, q^{1/2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{63}$$

Now, substituting (60) into (48) we can compute U^0, which in this simplest case is not a matrix but a single operator. To be able to use condition 3 we have to compare U^0 with the central element of the algebra $\mathcal{U}$ generated by the entries U_i of the matrix $U^{\frac{1}{2}}$ and the spin operator p. As was shown in [12], the only nontrivial central element of the algebra $\mathcal{U}$ given by the following analogue of determinant of $U^{\frac{1}{2}}$:

$$\mathrm{Det}\, U^{\frac{1}{2}} = (U_1 U_4 - q U_2 U_3) \sqrt{\frac{[p]}{[p-1]}} = (q U_4 U_1 - U_3 U_2) \sqrt{\frac{[p]}{[p+1]}}. \tag{64}$$

Omitting simple calculations, we give the result: U^0 coincides (up to a numerical factor) with (64) only if the following constraint

$$\alpha(p)\sqrt{[p+1]} - \beta(p)\sqrt{[p-1]} = \varepsilon \sqrt{[p]}. \tag{65}$$

holds and, thus, we have only one independent variable. Here a numerical constant ε in r.h.s. can be arbitrary (non-zero). But the additional condition 5 says that $\varepsilon = q^{1/2}$.

Finally, we can use conditions 4 and 5. To apply 4 in practice, we can first consider the non -deformed case ($q = 1$) when entries of $\mathcal{F}(p)$ are self-conjugated and next extend the solution to generic q in such a way that 5 would be satisfied. After simple calculations we get

$$\begin{aligned} \alpha(p) = \delta(p) &= \frac{1}{[2]} \left(q^{\frac{1}{2}} \sqrt{\frac{[p+1]}{[p]}} + q^{-\frac{1}{2}} \sqrt{\frac{[p-1]}{[p]}} \right), \\ \beta(p) = -\gamma(p) &= \frac{1}{[2]} \left(q^{-\frac{1}{2}} \sqrt{\frac{[p+1]}{[p]}} - q^{\frac{1}{2}} \sqrt{\frac{[p-1]}{[p]}} \right). \end{aligned} \tag{66}$$

Thus $\mathcal{F}^{\frac{1}{2}\frac{1}{2}}(p)$ is found. Notice that $\det \mathcal{F}^{\frac{1}{2}\frac{1}{2}}(p) = 1$. It is instructive to derive with help (66) and (54) explicit expression for the element χ (in fundamental representation):

$$\chi = \begin{pmatrix} 1 & & & \\ & 2\lambda & -\omega\lambda & \\ & \omega\lambda & 2\lambda & \\ & & & 1 \end{pmatrix}. \tag{67}$$

In the non-deformed limit $q = 1$ we have $\chi = \epsilon \otimes \epsilon$, as expected.

Finally, substituting (63) into (14) and expllllloting the explicit form of $\mathcal{F}\frac{1}{2}\frac{1}{2}(p)$, we can build up the exact generating matrix of spin 1. It looks like following

$$U^1 = \begin{pmatrix} U_1^2 & q^{-\frac{1}{2}}\sqrt{[2]}U_1U_2 & U_2^2 \\ \sqrt{\frac{[2][p]}{[p+1}}U_1U_3 & \sqrt{\frac{[p]}{[p+1]}}(q^{\frac{1}{2}}U_1U_4 +^{-\frac{1}{2}} U_2U_3) & \sqrt{\frac{[2][p]}{[p+1]}}U_2U_4 \\ U_3^2 & q^{-\frac{1}{2}}\sqrt{[2]}U_3U_4 & U_4^2 \end{pmatrix}. \tag{68}$$

Let us briefly comment this formula. First, as we could expect, in the formal limit $q^p \longrightarrow +\infty$ it coincides with the expression (32) for g^1. Next, it is easy to see that thesecond row of (68) coincides (up to rescaling by $\sqrt{\frac{[p]}{[p+1]}}$) with the spin 1 tensor operator (7) constructed from the generators of $U_q(sl(2))$.

Finally, notice that the elements U^1_{ij}, $i,j = 1,2,3$ act on the model space $\mathcal{M}$ as shifts from state $|J,m\rangle$ to state $|J+(2-i), m+(2-j)\rangle$, which is natural because we applied the fusion scheme to the matrix $U^{\frac{1}{2}}$ whose entries are basic shifts on $\mathcal{M}$. But, furthermore, if we realize in (68) the operators U_i as in the r.h.s. of (55), then U^1 also will have a property of "preciseness." Namely, it can be checked then that matrix elements $\langle J', m'|U^1_{ij} J'', m''\rangle$ evaluated on $D_q(z_1, z_2)$ precisely coincide with the CG coefficients $\begin{Bmatrix} J' & 1 & J'' \\ m' & 2-j & m'' \end{Bmatrix}_q$ (nine of them are non-vanishing). Thus, the fusion procedure preserves the "preciseness" of exact generating matrices. This observation might be useful for practical computations.

§3.3 Another construction for $\mathcal{F}$

Computations of the previous paregraph inspire us to introduce p-dependent counterparts of the projectors $P_\pm$ used above. Indeed, it is obvious from the connection formula $\widehat{\mathcal{R}}_\pm(\vec{p}) = \mathcal{F}^{-1}\widetilde{R}_\pm\mathcal{F}$ that the objects

$\mathcal{P}_+$ and $\mathcal{P}_-$ introduced by the following analogues of the decomposition formulae (59):

$$\mathcal{P}_\pm = \frac{q^{\frac{1}{n}\pm 1}\widehat{\mathcal{R}}_+(\vec{p}) - q^{-\frac{1}{n}\mp 1}\widehat{\mathcal{R}}_-(\vec{p})}{q^2 - q^{-2}}. \tag{69}$$

also are projectors of ranks $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$, respectively. In the case of $U_q(sl(2))$ we find

$$\mathcal{P}_\pm = \begin{pmatrix} 1 & & & \\ & \lambda\frac{[p\mp 1]}{[p]} & \pm\lambda\frac{\sqrt{[p+1][p-1]}}{[p]} & \\ & \pm\lambda\frac{\sqrt{[p+1][p-1]}}{[p]} & \lambda\frac{[p\mp 1]}{[p]} & \\ & & & 1 \end{pmatrix}. \tag{70}$$

Repeating the procedure described in the previous paragraph, we can find the eigenvectors $\vec{x}_i$ such that $\vec{x}_i^t\vec{x}_i = \delta_{ij}$ and $\mathcal{P}_- = \vec{x}_0 \otimes \vec{x}_0^t$, $\mathcal{P}_+ = \sum_{i=1}^{3} \vec{x}_i \otimes \vec{x}_i^t$. Next, using the same formulae (16), we can construct p-dependent counterparts of the CG-maps. They look like following

$$\begin{aligned} C_p[\frac{1}{2}\,\frac{1}{2}0] &= \sqrt{\lambda}\left(0, \sqrt{\frac{[p+1]}{[p]}}, -\sqrt{\frac{[p-1]}{[p]}}, 0\right), \\ C_p[\frac{1}{2}\,\frac{1}{2}1] &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\lambda}\sqrt{\frac{[p-1]}{[p]}} & \sqrt{\lambda}\sqrt{\frac{[p+1]}{[p]}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{71}$$

Now, straightforward check shows that the matrix $\mathcal{F}^{\frac{1}{2}\,\frac{1}{2}}(p)$ found before can be obtained as follows: $\mathcal{F}^{\frac{1}{2}\,\frac{1}{2}}(p) = C'[\frac{1}{2}\,\frac{1}{2}0]C_p[\frac{1}{2}\,\frac{1}{2}0] + C'[\frac{1}{2}\,\frac{1}{2}1] + C'[\frac{1}{2}\,\frac{1}{2}1]C_p[\frac{1}{2}\,\frac{1}{2}1]$. Actually, we can give more general version of this formula:

$$\mathcal{F}^{IJ}(\vec{p}) = \sum_K C'[IJK]C_p[IJK] \tag{72}$$

since very similar expressions have already appeared in [11, 18], where $C[IJK]$ and $C_p[IJK]$ were taken a-priory as the CG coefficients and $6j$-symbols with specific dependence on $\vec{p}$, respectively. With these object being suitable defined, it can be proved [11] that $\mathcal{F}(\vec{p})$ given by (72) obeys all axioms for the twisting element.

Thus, we have another way of computation of the twisting element in practice. Although it is not much simpler than what we used before. Indeed, to apply it we have to know values of CG coefficients and $6j$-symbols a-priory. Here the problem of appropriare (in the sense of compatibility with given matrices $R_\pm$ and $\mathcal{R}_\pm(\vec{p})$ basis and normalization arises. Therefore, our prescription (rather "experimental" as we didn't prove that (72) obeys the list of conditions for $\mathcal{F}(\vec{p})$ given above) for constructing of $C[IJK]$ and $C_p[IJK]$ from eigenvectors of the projectors P_K^{IJ} and $\mathcal{P}_K^{IJ}$ is, probably, quite useful from practical point of view.

Finally, let us make an algebraic sense of (72) more transparent. To this end we can note that, since $P_K^{IJ} = C[IJK]C'[IJK]$, we are able to rewrite the formula for decomposition of R-matrices over projectors (we used its simplest case (59) above) in the following form:

$$\widehat{R}_{\pm}^{IJ} = \sum_K C'[IJK] r_{K,\pm}^{IJ} C[IJK], \tag{73}$$

were $r_{K,\pm}^{IJ}$ are the corresponding eigenvalues (for the fundamental representations of the main series they are given in [9], for arbitrary irrep of $sl(2)$ see [10]). Now, bearing in mind the properties (18) of CG maps, we see that (72) transfers, according to (41), expression (73) into similar one for the twisted R-matrices:

$$\widetilde{\mathcal{R}}_{\pm}^{IJ}(\vec{p}) = \sum_K C_p'[IJK] r_{K,\pm}^{IJ} C_p[IJK]. \tag{74}$$

Thus, Hopf and quasi-Hopf structures turn out to be identical in terms of projectors.

Conclusion

In present paper we demonstrated that the theory (deformed) tensor operators and, in particular, the fusion procedure can be most naturally described making use of the R-matrix approach and revealing the underlying quasi-Hopf-algebraic structure. From ptactical point of view the prescription for constructing of exact generating matrices can, probably, be employed for explicit computations, e.g., for calculations of (deformed) CG coefficients for quantum Lie algebras of higher ranks. On the other hand, the specific quasi-Hopf algebra appearing in this context should certainly be studied in more detail since it provides non-trivial (and presumably somewhat simplified) realization of the abstract general scheme.

Although the present papaer dealt mainly with mathematical side of the theory of tensor operators, we are going to discuss some physical applications in future.

Finally, we would like to note that it would be interesting to extend developed technique to the case of q being a root of unity, which will involve truncated quasi-Hopf algebras.

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