A. A. Arkhipova

ON THE PARTIAL REGULARITY UP TO THE BOUNDARY OF WEAK SOLUTIONS TO QUASILINEAR PARABOLIC SYSTEMS WITH QUADRATIC GROWTH

ABSTRACT. We consider quasilinear nondiagonal parabolic systems with quadratic growth on the gradient in a parabolic cylinder Q. Under Dirichlet and Neumann boundary conditions partial Hölder continuity up to the lateral surface of Q of solutions $u \in W_2^{1,1}(Q) \cap L^\infty(Q)$ is proved. Hausdorff dimension of a singular set is estimated. In the proof we get of the maximum principle theorem for corresponding model linear problems.

Dedicated to Olga A. Ladyzhenskaya

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with sufficiently smooth boundary $\partial\Omega$, $Q = \Omega \times (0,T)$. Let $u : Q \to \mathbb{R}^N$, $N > 1$, be a solution of quasilinear parabolic system

$$u_t^k - \left(A_{kl}^{\alpha\beta}(z,u)\, u_{x_\beta}^l\right)_{x_\alpha} + b^k(z,u,u_x) = 0,$$
$$z = (x,t) \in Q, \quad k = 1,\dots,N. \tag{1}$$

We assume that

I. $A_{kl}^{\alpha\beta}$ are uniformly continuous functions on $\overline{Q} \times \mathbb{R}^N$ and

$$A_{kl}^{\alpha\beta}(z,u)\,\zeta_\beta^l\,\zeta_\alpha^k \geqslant \nu|\zeta|^2, \quad \forall\,\zeta \in \mathbb{R}^{nN}, \quad \left|A_{kl}^{\alpha\beta}(z,u)\right| \leqslant L; \tag{2}$$

II. functions $b^k(z,u,u_x)$ are mesurable in Q for all $u \in U(Q) \equiv W_2^{1,1}(Q) \cap L^\infty(Q)$*) and

$$|\, b(z,u,p)\,| \leqslant a|p|^2 + L, \quad (z,u,p) \in Q \times \mathbb{R}^N \times \mathbb{R}^{nN}. \tag{3}$$

In conditions (2) and (3) ν, $L = \mathrm{const} > 0$, $a = \mathrm{const} \geqslant 0$.

---

*)We follow the notations of spaces in the monograph [1] and write $v \in \mathcal{B}(Q)$ instead of $v \in \mathcal{B}(Q;\mathbb{R}^N)$ for brevity.

For solutions $u \in W_2^{1,0}(Q) \cap L^\infty(Q)$ of the system (1) under the smallness condition

$$2a\|u\|_{\infty,Q} < \nu \tag{4}$$

the regularity problem was studied in [2–7]. To prove partial Hölder continuity of weak solutions inside Q the so-called direct method was applied. In the frame of this method there arises the following difficulty: a model linear problem has "bad" initial-boundary conditions. More exactly, a solution of the model problem in a cylinder $Q_R \subset Q$ coincides on the parabolic boundary $\partial' Q_R$ of the Q_R with the solution u of (1) which is not sufficiently smooth in "t". Poor information on the smoothness of u in "t" does not allow 1) to use well-known results on the solvability of the model problem, 2) to estimate $L^\infty(Q_R)$-norm of the solution of the model problem in terms of the known norms of u. Some integral continuity conditions of u in t were formulated in [3–7] to state partial Hölder continuity of u inside Q.

We mention also papers [8, 9] where the regularity problem for *nonlinear* parabolic systems with quadratic growth in the gradient was studied. In [8] partial Hölder continuity of u_{x_α}, $\alpha - 1, \dots, n$, inside Q was proved for $u \in W_2^{2,0}(Q) \cap C^\gamma(\overline{Q})$, $\gamma > 0$. It was stated in [9] that weak solutions from $W_2^{1,0}(Q) \cap C^\gamma(\overline{Q})$, $\gamma > 0$, are the functions from the class $W_{2,\mathrm{loc}}^{2,1}(Q)$.

There are different ways of defining the concept of a generalized solution to an initial-boundary problem for system (1) under condition (3). At the same time, the solvability question for such problem in any class of functions is open up to now. An attempt to answer this question is author's work [10] where for two-dimensional domain Ω the global in time solvability of Cauchy–Dirichlet problem is proved for system (1) of variational structure (without smallness condition like (4)). In [10] we modified M. Struwe's idea [11] to prolong a local solution from some interval $[0, t_0)$ to $[0, t_0]$ if it has uniformly small "local energies" on $[0, t_0]$. Constructed solution u is regular on $\overline{Q}$ with exception of at most finite number of points and at the same time it is a distribution solution with $u_t \in L_2(Q)$, $\sup\limits_{[0,T]} \|u_x\|_{2,\Omega} < +\infty$.

Let $u \in U(Q) = W_2^{1,1}(Q) \cap L^\infty(Q)$ be a solution of (1) under Dirichlet condition

$$u\big|_\Gamma = \psi(z), \quad \Gamma = \partial\Omega \times (0, T), \tag{5}$$

or Neumann condition

$$\frac{\partial u}{\partial n_A}\bigg|_{\Gamma} = \psi(z), \quad \frac{\partial u^{(k)}}{\partial n_A} = \left\{\frac{\partial u^{(k)}}{\partial n_A}\right\}^{k\leqslant N}, \tag{6}$$

$\frac{\partial u^{(k)}}{\partial n_A} = A^{\alpha\beta}_{kl}(z,u)u^l_{x_\beta}\cos(\mathbf{n}, x_\alpha)$, $\mathbf{n} = \mathbf{n}(x)$ is the outer to Ω normal vector at the point $x \in \partial\Omega$.

The aim of the work is to prove partial Hölder continuity up to the lateral surface Γ for a solution of (1), (5) and (1), (6) from the class $U(Q)$. The existence of $u_t \in L^2(Q)$ removes the solvability question for the model problem inside Q and near the lateral surface. As for the L^∞-estimate of the solution v of the model problem in a borderline cylinder Q_R (the base of Q_R is a half ball B^+_R) we don't know any result of such type. Moreover, in the case of the Neumann problem (1), (6) there appear mixed boundary conditions for v. We demonstrate here how to get rid of the "maximum principle" estimate of v in Q_R proving partial regularity of $u \in U(Q)$. Actually, we extend to the parabolic case partial regularity results obtained earlier by the author for elliptic systems with quadratic growth [12, 13]. The main idea of the method we use here goes back to works [14, 15].

Remark 1. It is known that problem (1), (5) or (1), (6) with smooth data (under initial condition $u\big|_{t=0} = \varphi$) has a unique regular solution on some time interval $[0, T_0)$ [16]. On the other hand there exists an example of quasilinear system (1) ($b^k \equiv 0$) with a solution of Cauchy–Dirichlet problem which is regular for $t \in [0, t_0)$ and it has a singular point inside cylinder Q at the moment $t = t_0$ [17]. It means one can expect only partial regularity of a solution in $\overline{\Omega} \times (0, T)$ where T is an arbitrary positive number.

In this paper we demonstrate the idea of the proof considering the Neumann problem (1), (6).

First step is to prove L^p-integrability ($p > 2$) of the gradient of $u \in U(Q)$.

Let ψ in (6) be a function from anisotropic space $L^{m_1+\delta, m_2+\delta}(\Gamma)$ and

$$\left.\begin{aligned} &\frac{2}{m_2} + \frac{n-1}{m_1} = 1 + \frac{n}{2},\\ &\text{where } \; m_1 \in \left[\frac{2(n-1)}{n}, \frac{2(n-1)}{n-2}\right], \quad m_2 \in [1,2]\\ &\text{for } \; n > 2 \; \text{ and } \; m_1 \in (1, \infty], \; m_2 \in [1, 2) \; \text{ for } \; n = 2; \end{aligned}\right\} \tag{7}$$

$\delta > 0$ is fixed arbitrarily.

Theorem 1. *Suppose that 1) Ω is a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with C^2 – smooth boundary, 2) conditions (2), (3) hold, 3) ψ satisfies assumption (7) with $m_1 \leqslant m_2$, 4) $u \in U(Q) = W_2^{1,1}(Q) \cap L^\infty(Q)$ is a solution of the problem (1), (6) and condition (4) holds. Then there exist constants $p > 2$ and $c_0 > 0$ such that for an arbitrarily fixed $\tau > 0$ $u_x \in L^p(Q')$, $Q' = \Omega \times (\tau, T)$ and*

$$\|u_x\|_{p,Q'} \leqslant c_0 \left\{ \|u_x\|_{2,Q} + \|\psi\|_{\widetilde{m}_1, \widetilde{m}_2; \Gamma} \right\}, \quad \widetilde{m}_i = m_i + p - 2, \tag{8}$$

with constants p and c_0 depending on $(\nu - 2a\|u\|_{\infty,Q})^{-1}$, L, m_1, m_2, δ. In addition, c_0 depends on the value τ^{-1}.

The validity of Theorem 1 follows from more general result proved by the author in [18, Theorem 3] (see also [19]).

Now we need some standard notation:

$$B_R(x^0) = \left\{ x \in \mathbb{R}^n : |x - x^0| < R \right\}, \quad \Lambda_R(t^0) = (t^0 - R^2, t^0),$$

$$Q_R(z^0) = \left(B_R(x^0) \times \Lambda_R(t^0)\right) \cap Q, \quad \Gamma_R(z^0) = \left(B_R(x^0) \times \Lambda_R(t^0)\right) \cap \Gamma,$$

we write B_R, $Q_R \dots$ instead of $B_R(0)$, $Q_R(0) \dots$

$$\fint_{Q_R(z^0)} v\, dz = \frac{1}{|Q_R|} \int_{Q_R^{(z^0)}} v\, dz, \quad |Q_R| = \mathrm{meas}_{n+1} Q_R = \mathcal{O}(R^{n+2}),$$

$\mathcal{H}_k(D; \delta)$ is the k-dimensional Hausdorff measure with respect to the parabolic metric δ,

$$\delta(z^1, z^2) = \max\left\{ |x^1 - x^2|, \, |t^1 - t^2|^{\frac{1}{2}} \right\}.$$

Remark 2. Performing a similarity transformation one can deduce from (8) a local estimate of the L^p-norm of u_x in Q_R ([19], Remark 6).

There exists a constant c_0 such that for every $z^0 \in Q' = \overline{\Omega} \times (\tau, T)$, $R < \frac{\sqrt{\tau}}{2}$ we have the estimate:

$$\left(\fint_{Q_R(z^0)} |u_x|^p\, dz \right)^{\frac{2}{p}} \leqslant c_0 \left\{ \fint_{Q_{2R}(z^0)} |u_x|^2\, dz + R^{-2\delta} \|\psi\|^2_{\widetilde{m}_1, \widetilde{m}_2; \Gamma_{2R}(z^0)} \right\}, \tag{9}$$

$\delta = \frac{n-1}{\widetilde{m}_1} + \frac{2}{\widetilde{m}_2}$, $\widetilde{m}_i = m_i + p - 2$.

From now on we assume that $\psi \in L^{s_1, s_2}(\Gamma)$ where

$$\frac{2}{s_2} + \frac{n-1}{s_1} < 1, \quad s_1 \in (n-1, \infty], \quad s_2 \in (2, \infty]. \tag{10}$$

For such functions ψ it follows from (9) that

$$\left(\fint_{Q_R(z^0)} |u_x|^p \, dz \right)^{\frac{2}{p}} \leqslant c_0 \left\{ \fint_{Q_{2R}(z^0)} |u_x|^2 \, dz + R^{-2\alpha} \|\psi\|^2_{s_1, s_2; \Gamma_{2R}(z^0)} \right\}, \quad (11)$$

with $\alpha = \frac{n-1}{s_1} + \frac{2}{s_2}$, $\alpha \in [0, 1)$.

Now we formulate the main result of the paper.

Theorem 2. *Suppose that assumptions* 1), 2) *and* 4) *of Theorem* 1) *are fulfilled and function* ψ *satisfies condition* (10)*. Then there exists a set* $Q' \subset \widehat{Q} = \overline{\Omega} \times (0, T)$ *open with respect to* $\widehat{Q}$ *such that* $u \in C^{\gamma}(Q'; \delta)$ *and* $\mathcal{H}_{n-\varepsilon}\,(\widehat{Q} \setminus Q'; \delta) = 0$ *for some* $\varepsilon > 0$*. Exponent* $\gamma = 1 - \alpha$*, if number* α *from ineqiality* (11) *is positive, and for* $\alpha = 0$ γ *is an arbitrary number from the interval* $(0, 1)$*.*

Proof of Theorem 2. Let the set Ω be covered by $\bigcup\limits_{j=0}^{M} V^{(j)}$, where $V^{(0)} \subset\subset \Omega$ and for any fixed $j = 1, \dots, M$ there exists a C^2-diffeomorphism $y^{(j)} = y^{(j)}(x)$ transforming the set $V^{(j)} \cap \Omega$ into a halfball $B^+_{d_j} = B_{d_j} \cap \{y^{(j)}_n > 0\}$. Moreover, $y^{(j)}(V^{(j)} \cap \partial\Omega) = B_{d_j} \cap \{y^{(j)}_n = 0\} = \gamma_{d_j}$. For fixed $j = 1, \dots, M$ we put $y = y^{(j)}(x)$ and in the coordinate system $\zeta = (y_1, \dots, y_n, t)$ define function $\widetilde{u}(y, t) = u(x, t)$. Function $\widetilde{u}(y, t)$ is a local solution of the transformed problem (1), (6):

$$\widetilde{u}^k_t - \left(\widetilde{A^{\alpha\beta}_{kl}}(\zeta, \widetilde{u}) \, \widetilde{u}^l_{y_\beta} \right)_{y_\alpha} + \widetilde{b}^k(\zeta, \widetilde{u}, \widetilde{u}_y) = 0,$$

$$\zeta = (y, t) \in Q^+_d = B^+_d \times (0, T),$$

$$\left. \frac{\partial u}{\partial n_{\widetilde{A}}} \right|_{\Gamma_d} = \widetilde{\psi}(\zeta), \quad \Gamma_d = \gamma_d \times (0, T),$$

where the matrix $\widetilde{A}(\zeta, \widetilde{u})$ satisfies condition (2) with new positive constants $\widetilde{\nu}$, $\widetilde{L}$, function $\widetilde{\psi} \in L^{s_1, s_2}(\Gamma_d)$, s_1, s_2 being defined in (10). For function $\widetilde{b}(\zeta, \widetilde{u}, q)$ we have the following estimate:

$$\left| \widetilde{b}\,(\zeta, \widetilde{u}, q) \right| \leqslant \widetilde{a}\,|q|^2 + L_1 |q| + \widetilde{L}. \quad (12)$$

All new constants $\widetilde{\nu}$, $\widetilde{L}$, L_1, $\widetilde{a}$ depend on the parameters ν, L, a and C^2-norms of $y(x)$ and the inverse transformation $x(y)$. Note, that

$\|\,\widetilde{u}\,\|_{\infty,Q_d^+} \leqslant M = \|u\|_{\infty,Q}$. Recalculating of the parameters gives inequality $2\,\widetilde{a}\,M < \widetilde{\nu}$ if the number $d \leqslant d_j$ is small enough. Now we fix $\varepsilon_0 > 0$ in such a way that for $a_1 = \widetilde{a} + \varepsilon_0$ the inequality $2a_1 M < \widetilde{\nu}$ holds. Applying Cauchy inequality with the parameter ε_0 to (12) we derive

$$|\,\widetilde{b}\,(\zeta, \widetilde{u}, q)\,| \leqslant a_1 |q|^2 + c(\widetilde{L}, L_1, \varepsilon_0^{-1}).$$

It means that in new coordinates condition (3) with restriction (4) is fulfilled. We may choose now $d \leqslant \min\limits_{j=1,\dots,M} d_j$ and assume later on that the parameters depend on the C^2-characteristic of $\partial\Omega$ but not on the fixed map $y^{(j)}$. It allows to save the old notations of the variables and the functions and to consider the local problem

$$\left.\begin{aligned} &u_t^k - \Big(A_{kl}^{\alpha\beta}(z,u)u_{x_\beta}^l\Big)_{x_\alpha} + b^k(z,u,u_x) = 0, \qquad k = 1,\dots,N,\\ &z \in Q_d^+ = B_d^+ \times (0,T),\\ &\frac{\partial u}{\partial n_A}\bigg|_{\Gamma_d} = \psi, \end{aligned}\right\} \tag{13}$$

under conditions (2)–(4) in $\overline{Q_d^+}$ and (10) on $\Gamma_d = \gamma_d \times (0,T)$.

The solution $u \in W_2^{1,1}(Q_d^+) \cap L^\infty(Q_d^+)$ satisfies the identity

$$\left.\begin{aligned} &\int\limits_{Q_d^+} \Big(u_t^k\,\eta^k + A_{kl}^{\alpha\beta}\,u_{x_\beta}^l\,\eta_{x_\alpha}^k + b^k\,\eta^k\Big)\,dz = \int\limits_{\Gamma_d} \psi^k\,\eta^k\,d\Gamma,\\ &\forall\,\eta \in V(Q_d^+) \cap L^\infty(Q_d^+), \quad \eta\big|_{S_d^+\times(0,T)} = 0, \quad S_d^+ = \partial B_d^+ \setminus \gamma_d. \end{aligned}\right\} \tag{14}$$

The space $V(Q_d^+)$ is defined by functions with a finite norm

$$|\,\eta\,|_{Q_d^+} = \left(\operatorname*{ess\,sup}_{(0,T)} \|\eta\|_{2,B_d^+}^2 + \|\eta_x\|_{2,Q_d^+}^2\right)^{\frac{1}{2}}. \tag{15}$$

Due to the imbedding theorem for functions from $V(Q_d^+)$ (estimate (3.11), Chap. II, [1]) and condition (7) it follows:

$$\left.\begin{aligned} &\bigg|\int\limits_{\Gamma_d} \psi\eta\,d\Gamma\bigg| \leqslant \|\psi\|_{m_1,m_2;\Gamma_d}\|\eta\|_{m_1',m_2';\Gamma_d} \leqslant\\ &\qquad \leqslant c\|\psi\|_{m_1,m_2;\Gamma_d}\,|\,\eta\,|_{Q_d^+} < +\infty,\\ &\qquad (m_1', m_2' \text{ are the conjugate exponents to } m_1,\ m_2). \end{aligned}\right\} \tag{16}$$

Let us fix $z^0 = (x^0, t^0)$ where $x^0 \in \overline{B^+_{d/2}}$, $t^0 \in (\frac{d^2}{4}, T)$ and $R \leqslant \frac{d}{4}$. We write $Q_R(z^0) = \Omega_R(x^0) \times \Lambda_R(t^0)$ for $\Omega_R(x^0) = B^+_d \cap B_R(x^0)$ and $\overset{0}{A}{}^{\alpha\beta}_{kl} = A^{\alpha\beta}_{kl}(z^0, u^0)$ for $u^0 = \fint_{Q_R(z^0)} u\,dz$.

Let v be a solution of the model problem

$$\left.\begin{aligned} v^k_t - \overset{0}{A}{}^{\alpha\beta}_{kl}\, v^l_{x_\beta x_\alpha} = 0 \quad \text{in} \quad Q_R(z^0),\\ \frac{\partial v}{\partial n_{\overset{0}{A}}}\Big|_{\Gamma_R(z^0)} = 0, \quad v\big|_{\mathbb{P}_R(z^0)} = u, \end{aligned}\right\} \tag{17}$$

where $\Gamma_R(z^0) = \gamma_R(x^0) \times \Lambda_R(t^0)$, $\gamma_R(x^0) = \gamma_d \cap B_R(x^0)$ and $\mathbb{P}_R(z^0) = \partial' Q_R(z^0) \setminus \Gamma_R(z^0)$, $\partial' Q_R$ being the parabolic boundary of the cylinder Q_R.

From (17) for $w = u - v \in V(Q_R(z^0))$ it follows:

$$\left.\begin{aligned} \int\limits_{Q_R(z^0)} \left(-w^k\,\eta^k_t + \overset{0}{A}{}^{\alpha\beta}_{kl}\, w^l_{x_\beta}\,\eta^k_{x_\alpha}\right) dz = \int\limits_{Q_R(z^0)} \left(F^k_\alpha\,\eta^k_{x_\alpha} + f^k\,\eta^k\right) dz,\\ \forall\,\eta \in W^{1,1}_2(Q_R(z^0)), \quad \eta\big|_{t=t^0} = 0, \quad \eta\big|_{S'_R(x^0)\times\Lambda_R(t^0)} = 0, \end{aligned}\right\} \tag{18}$$

where

$$F^k_\alpha = \overset{0}{A}{}^{\alpha\beta}_{kl}\, u^l_{x_\beta} \in L^2(Q_R), \ f^k = u^k_t \in L^2(Q_R), \ S'_R(x^0) = \partial\Omega_R(x^0) \setminus \gamma_R(x^0).$$

According to the linear theory (§4, 5, Chap. III, [1]) there exists a unique solution $w \in W^{1,\frac{1}{2}}_2(Q_R)$, $w\big|_{\mathbb{P}_R} = 0$, of problem (18). It means that problem (17) has a unique solution $v \in W^{1,\frac{1}{2}}_2(Q_R)$. It is known that $v \in C^\infty(Q_R(z^0) \cup \Gamma_R(z^0))$ and the following Campanato type estimates hold:

$$\fint_{Q_\rho(\widehat{z})} |v|^2 dz \leqslant c \fint_{Q_r(\widehat{z})} |v|^2 dz, \quad \forall \widehat{z} \in \overline{Q_{\frac{R}{2}}(z^0)}, \quad r \leqslant \frac{R}{2}; \tag{19}$$

$$\fint_{Q_\rho(\widehat{z})} |v_x|^2 dz \leqslant c \fint_{Q_r(\widehat{z})} |v_x|^2 dz, \quad \forall \widehat{z} \in \overline{Q_{\frac{R}{2}}(z^0)}, \quad r \leqslant \frac{R}{2}, \tag{20}$$

the constants $c > 0$ depend on ν, L, n.

For cylinder $Q_r(\widehat{z}) \subset\subset Q_R(z^0)$ estimates (19), (20) were derived in [20] and for borderline cylinders $Q_r(\widehat{z})$ under zero Neumann condition on the straightened part of the lateral surface they were deduced in [16].

We also need the energy type estimate for w, which easily follows from (18) (§2, Chap. III, [1]):

$$|\,w\,|^2_{Q_R(z^0)} \leqslant c(\nu, L) \int\limits_{Q_R(z^0)} \left(|u_x|^2 + R^2|u_t|^2\right) dz. \tag{21}$$

Taking into account (19) for $\widehat{z} \in \overline{Q_{R/2}(z^0)}$, $r = R/2$, and (21) we get the relations

$$\fint\limits_{Q_\rho(\widehat{z})} |v|^2 dz \leqslant c \fint\limits_{Q_{\frac{R}{2}}(\widehat{z})} |v|^2 dz \leqslant c_1 \fint\limits_{Q_R(z^0)} \left(|w|^2 + |u|^2\right) dz \leqslant$$

$$\leqslant c_2\, R^2 \fint\limits_{Q_R(z^0)} |w_x|^2 dz + c_3 M^2 \leqslant$$

$$\leqslant c_4 \left\{ R^{-n} \int\limits_{Q_R(z^0)} |u_x|^2 dz + R^{-(n-2)} \int\limits_{Q_R(z^0)} |u_t|^2 dz + M^2 \right\},\ \ c_i = c_i(\nu, L, n).$$

Let $\rho \to 0$ at the left-hand side of the last relations then we have that

$$\sup_{\widehat{z} \in \overline{Q_{\frac{R}{2}}(z^0)}} |v(\widehat{z})|^2 \leqslant c \left\{ M^2 + \fint\limits_{Q_R(z^0)} |u_x|^2 dz + R^{-(n-2)} \int\limits_{Q_R(z^0)} |u_t|^2 dz \right\}. \tag{22}$$

Here and in what follows we denote the term $R^{-n} \int_{Q_R} |u_x|^2 dz$ by $\fint\limits_{Q_R} |u_x|^2 dz$. To estimate it we test the identity (14) with $\eta = u\zeta^2$ where ζ is a cut-off function for $Q_{2R}(z^0)$, $\zeta \equiv 1$ in $Q_R(z^0)$ and find that

$$\fint\limits_{Q_R(z^0)} |u_x|^2 dz \leqslant c \left\{ M^2 + R^{2(1-\alpha)} \|\psi\|^2_{s_1, s_2; \Gamma_{2R}(z^0)} \right\}, \tag{23}$$

$\alpha \in [0, 1)$ is defined in relation (11), s_1, s_2 are parameters from (10), $c > 0$ depends on $(\nu - Ma)^{-1}$, L, n.

From inequalities (22) and (23) we deduce

$$\sup_{z \in \overline{Q_{\frac{R}{2}}(z^0)}} |v(z)|^2 \leqslant c \left\{ M^2 + R^{2(1-\alpha)} \|\psi\|^2_{s_1, s_2; \Gamma(z^0)} + D_R(z^0) \right\}, \tag{24}$$

$$D_R(z^0) = R^{-(n-2)} \int\limits_{Q_R(z^0)} |u_t|^2 dz.$$

If we denote the expression on the right-hand side of inequality (24) as $m_R^2(z^0)$ then

$$\|w\|_{\infty, Q_{\frac{R}{2}}(z^0)} \leqslant m_R(z^0) + M = M_R(z^0). \tag{25}$$

For an arbitrarily fixed $s \in (0,1)$ we choose $K > 2M_R$ $(M_R = M_R(z^0))$ from the condition

$$K^s - (2M_R)^s = \frac{K^s}{2}, \tag{26}$$

it follows that $K = 2^{1+1/s} M_R$, $K = K(s) \to +\infty$ for $s \to 0$ and fixed R and z^0. Moreover, we get the estimate

$$K \leqslant c\left((\nu - aM)^{-1}, \nu, L\right) 2^{1+\frac{1}{s}} \left(M + \|\psi\|_{s_1, s_2; \Gamma_d} + D_R^{\frac{1}{2}}(z^0)\right), \tag{27}$$

where $D_R(z^0) = R^{-(n-2)} \int_{Q_R(z^0)} |u_t|^2 dz$.

The next step is to derive the main inequality for the norm $|\, w\,|_{Q_{R/2}}(z^0)$ (for the definition of $|\cdot|$ see (15)).

Lemma. *For the solution w of* (18) *and any $s \in (0,1)$ the following inequality holds:*

$$\left.\begin{aligned} |\, w\,|^2_{Q_{\frac{R}{2}}(z^0)} \leqslant c_0 \Big\{ s |\, w\,|^2_{Q_R(z^0)} + \int\limits_{Q_R(z^0)} |\Delta A|^2 \, |u_x|^2 \Big\} dz + \\ + a \int\limits_{Q_R(z^0)} |u_x|^2 \min\{|w|, K\}\, dz + \frac{1}{s} \|\psi\|^2_{m_1, m_2; \Gamma_R(z^0)} + K R^{n+2}, \end{aligned}\right\} \tag{28}$$

where $c_0 = c_0(\nu, L, n)$, $\Delta A = A(z^0, u^0) - A(z, u)$, parameter $a > 0$ is defined in (3), *m_1, m_2 are the exponents from* (7), *the constant $K = K(s)$ is fixed by equality* (26).

Proof. In identity (18) we express the integral $\int_{Q_R(z^0)} f^k \, \eta^k \, dz$ with the help of relation (14) and derive:

$$\left.\begin{aligned}
&\int\limits_{Q_R}\Bigl(-w^k\,\eta^k_t+\overset{0}{A}{}^{\alpha\beta}_{kl}\,w^l_{x_\beta}\,\eta^k_{x_\alpha}\Bigr)\,dz=\\
&=\int\limits_{Q_R}\Bigl(\Delta A^{\alpha\beta}_{kl}\,u^l_{x_\beta}\,\eta^k_{x_\alpha}-b^k(z,u,u_x)\eta^k\Bigr)\,dz+\int\limits_{\Gamma_R}\psi^k\,\eta^k\,d\Gamma,\\
&\forall\,\eta\in W^{1,1}_2(Q_R)\cap L^\infty(Q_R),\quad \eta\big|_{S'_R\times\Lambda_R}=0,\quad \eta=\big|_{t=t_0}=0.
\end{aligned}\right\}\tag{29}$$

Here and later on we write Q_R, Γ_R instead of $Q_R(z^0)$, $\Gamma_R(z^0)$.

Now we fix τ, $t_1\in\Lambda_{R/2}=(t^0-\frac{R^2}{4},t^0)$, $\tau\leqslant t_1$.

For $\omega_{[h]}(x,t)=\frac{1}{h}\int_t^{t+h}\omega(x,t)dt$ from (29) it follows (§2, Chap. III, [1]):

$$\left.\begin{aligned}
&\int\limits_{Q^\tau_R}\Bigl(w^k_{[h]t}\,\xi^k+\overset{0}{A}{}^{\alpha\beta}_{kl}\,w^l_{[h]x_\beta}\,\xi^k_{x_\alpha}\Bigr)\,dz=\\
&=\int\limits_{Q^\tau_R}\Bigl((\Phi^k_\alpha)_{[h]}\,\xi^k_{x_\alpha}+\varphi^k_{[h]}\,\xi^k\Bigr)\,fz=\int\limits_{\Gamma^\tau_R}\psi^k_{[h]}\,\xi^k\,d\Gamma,\\
&\forall\xi\in W^{1,1}_2(Q_R)\cap L^\infty(Q_R),\quad \xi\big|_{S'_R\times\Lambda_R}=0,
\end{aligned}\right\}\tag{30}$$

where

$$Q^\tau_R=\Omega_R\times\Lambda^\tau_R,\ \Gamma^\tau_R=\gamma_R\times\Lambda^\tau_R,\ \Lambda^\tau_R=(t^0-R^2,\tau);$$

$$\Phi^k_\alpha=\Delta A^{\alpha\beta}_{kl}\,u^l_{x_\beta}\in L^2(Q_R),\ \varphi^k=-b^k(z,u,u_x)\in L^1(Q_R),\ h\leqslant t^0-t_1.$$

Moreover, identity (30) holds with any continuous in $L^2(\Omega_R)$-norm function ξ from $V(Q_R)$.

Now we put in (30) $\xi=w_{[h]}(K^s-(M_R+|w_{[h]}|)^s)_+$ with $s\in(0,1)$ and K fixed by condition (26), and find that

$$\left.\begin{aligned}
&\frac12\int\limits_{Q^\tau_R}\bigl(|w_{[h]}|^2\bigr)'_t\,\bigl(K^s-(M_R+|w_{[h]}|)^s\bigr)_+\,dz+\frac{\nu}{2}\int\limits_{Q^\tau_R}|w_{[h]x}|^2(K^s-\\
&-(M_R+|w_{[h]}|)^s)_+dz\leqslant cK^s\Biggl[s\int\limits_{Q^\tau_R}|w_{[h]x}|^2\,dz+\int\limits_{Q^\tau_R}|\Phi_{[h]}|^2\,dz+\\
&+\int\limits_{Q^\tau_R}|\varphi_{[h]}|\min\{|w_{[h]}|,K\}dz+\int\limits_{\Gamma^\tau_R}|w_{[h]}|\,|w_{[h]}|d\Gamma\Biggr],\ c=c(\nu,L,n).
\end{aligned}\right\}\tag{31}$$

Taking into account that $\|w_{[h]}\|_{\infty,Q^{t_1}_{R/2}} \leqslant M_R$ and for $\tau \in \Lambda^{t_1}_{R/2}$ $\|w_{[h]}(\ ,\tau)\|_{\infty,\Omega_{R/2}} \leqslant M_R$ we get the estimates for the integrals at the left-hand side of (31):

$$\text{a)} \qquad \int\limits_{Q^\tau_R} |w_{[h]x}|^2 \left(K^s - (M_R + |w_{[h]}|)^s\right)_+ dz \geqslant \frac{K^s}{2} \int\limits_{Q^\tau_{\frac{R}{2}}} |w_{[h]x}|^2\, dz;$$

$$\text{b)} \qquad \frac{1}{2}\int\limits_{Q^\tau_R} \left(|w_{[h]}|^2\right)'_t \left(K^s - (M_R + |w_{[h]}|)^s\right)_+ dz =$$

$$= \int\limits_{\Omega_R} \int\limits_0^{|w_{[h]}(x,t)|} \zeta\,(K^s - (M_R+\zeta)^s)_+\, d\zeta\, dx \Bigg|_{t=t^0-R^2}^{t=\tau} \geqslant$$

$$\geqslant \frac{K^s}{4} \int\limits_{\Omega_{\frac{R}{2}}} |w_{[h]}(x,\tau)|^2 dx - \int\limits_{\Omega_R} \int\limits_0^{|w_{[h]}(x,t^0-R^2)|} \cdot$$

$$\cdot\, \zeta\,(K^s - (M_R+\zeta)^s)_+\, d\zeta\, dx.$$

Estimating the left-hand side of inequality (31) with the help of the relations a) and b) and dividing the received inequality by K^s we deduce that

$$\left.\begin{aligned} &\int\limits_{\Omega_{\frac{R}{2}}} |w_{[h]}(x,\tau)|^2 dx + \int\limits_{Q^\tau_{\frac{R}{2}}} |w_{[h]x}|^2 dz \leqslant \\ &\leqslant c\Bigg[s\int\limits_{Q_R} |w_{[h]x}|^2\, dz + \int\limits_{Q_R} |\Phi_{[h]}|^2 dz + \int\limits_{Q_R} |\varphi_{[h]}|\, g^{(h)}\, dz + \\ &\quad + \int\limits_{\Omega_R} |w_{[h]}(x,t^0-R^2)|^2\, dx + \int\limits_{\Gamma_R} |\psi_{[h]}|\,|w_{[h]}| d\Gamma\Bigg], \end{aligned}\right\} \qquad (32)$$

where $g^{(h)} = \min\{|w_{[h]}|, K\}$ is the bounded in Q_R function.

We estimate the integral over Γ in (32) using the imbedding theorem and Cauchy inequality:

$$\int\limits_{\Gamma_R} |\psi_{[h]}|\,|w_{[h]}|\, d\Gamma \leqslant c\|\psi_{[h]}\|_{m_1,m_2;\Gamma_R} |\, w_{[h]}\, |_{Q_R} \leqslant$$

$$\leqslant s|\, w_{[h]}\,|^2_{Q_R} + \frac{c}{s}\|\psi_{[h]}\|^2_{m_1,m_2;\Gamma_R}.$$

Taking the supremum over $\tau \in \Lambda^{\tau_1}_{R/2}$ in the left-hand side of (32) we get the inequality

$$\left.\begin{aligned} |\, w_{[h]}\,|^2_{Q^{t_1}_{\frac{R}{2}}} \leqslant c\Big\{ s|\, w_{[h]}\,|^2_{Q_R} + \int\limits_{Q_R} |\Phi_{[h]}|^2 dz + \int\limits_{Q_R} |\varphi_{[h]}|\, g^{(h)}\, dz + \\ + \int\limits_{\Omega_R} |w_{[h]}(x, t^0 - R^2)|^2\, dx + \frac{1}{s}\|\psi_{[h]}\|^2_{m_1,m_2;\Gamma_R}\Big\}. \end{aligned}\right\} \qquad (33)$$

It is known that $w_{[h]} \to w$ in the energy norm, $\Phi_{[h]} \to \Phi$ in $L^2(Q_R)$, $\varphi_{[h]} \to \varphi$ in $L^1(Q_R)$, $\psi_{[h]} \to \psi$ in $L^{m_1,m_2}(\Gamma_R)$ for $h \to 0$ (§4, Chap. II, [1]). We also assert that $J_h = \int_{Q_R} \varphi_{[h]}|\, g^{(h)}\, dz \underset{h\to 0}{\longrightarrow} J = \int_{Q_R} |\varphi| g\, dz$. Indeed, let us rewrite the difference $J_h - J$ in the way:

$$J_h - J = \int\limits_{Q_R} \left(|\,\varphi_{[h]}\,| - |\varphi|\right) g^{(h)}\, dz + \int\limits_{Q_R} |\varphi| \left(g^{(h)} - g\right) dz \equiv j_1(h) + j_2(h).$$

The sequence $j_1(h)$ goes to zero for $h \to 0$ because $\varphi_{[h]} \to \varphi$ in $L^1(Q_R)$ and $\sup\limits_{Q_R} |g^{(h)}(z)| \leqslant K$, $\forall\, h > 0$. Note, that $g^{(h)} \to g$ almost everywhere in Q_R for some subsequence $h \to 0$. Using Lebesque theorem we get that $j_2(h) \to 0$ for the subsequence $h \to 0$. Now it is easy to pass to the limit for $h \to 0$ in inequality (33) and to get the relation:

$$\left.\begin{aligned} |\, w\,|^2_{Q^{t_1}_{\frac{R}{2}}} \leqslant c\Big[s|\, w\,|^2_{Q_R} + \int\limits_{Q_R} |\,\Delta A\,|^2\, |u_x|^2\, dz + \\ + \int\limits_{Q_R} |b(z, u, u_x| \min\{|w|, K\}\, dz + \frac{1}{s}\|\psi\|^2_{m_1,m_2;\Gamma_R}\Big]. \end{aligned}\right\} \qquad (34)$$

To complete the proof of the lemma we take the supremum over $t_1 \in \Lambda_{R/2}$ in the left-hand side of (34) and use condition (3). ■

The next step of proving Theorem 2 is to estimate integrals $I_1 = \int_{Q_R} |\Delta A|^2 |u_x|^2 dz$ and $I_2 = \int_{Q_R} |u_x|^2 \min\{|w|, K\} dz$ in (28) with the help of the L^p-estimate (11).

As functions $A^{\alpha\beta}_{kl}$ are uniformly continuous in $\overline{Q_d} \times \mathbb{R}^N$

$$\|A(z, u) - A(z^0, u^0)\| \leqslant \omega\left(|z - z^0|,\, |u - u_0|\right),$$

where $\omega(\lambda,\mu)$ is continuous, bounded, monotone in λ, $\mu \geqslant 0$ and convex in $\mu \geqslant 0$ function, $\omega(0,0)=0$. Integral I_1 is estimated in a standard way:

$$I_1 \leqslant \int\limits_{Q_R} \omega^2\left(R^2, |u-u^0|^2\right)\, |u_x|^2\,dz \leqslant$$

$$\leqslant \Bigl(\fint\limits_{Q_R} \omega^{\frac{2p}{p-2}}\,dz\Bigr)^{\frac{p-2}{p}} \Bigl(\fint\limits_{Q_R} |u_x|^p\,dz\Bigr)^{\frac{2}{p}} |Q_R| \leqslant$$

$$\leqslant c\Bigl(\fint\limits_{Q_R} \omega\,dz\Bigr)^{\frac{p-2}{p}} \Bigl\{\fint\limits_{Q_{2R}} |u_x|^2\,dz + R^{-2\alpha}\|\psi\|^2_{s_1,s_2;\Gamma_{2R}}\Bigr\}|Q_R| \leqslant$$

$$\leqslant c\omega^{\frac{p-2}{p}}\Bigl(R^2, \fint\limits_{Q_R}|u-u^0|^2\,dz\Bigr)\Bigl\{\int\limits_{Q_{2R}} |u_x|^2\,dz + R^{n+2(1-\alpha)}\Bigr\}.$$

From identity (14) it is easy to get the estimate:

$$R^{-2}\int\limits_{Q_R(z^0)} |u-u^0|^2\,dz \leqslant c_1 \int\limits_{Q_{2R}} |u_x|^2\,dz + c_2R^{n+2\gamma}, \tag{35}$$

where c_1, c_2 depend on the parameters of the problem and c_2 also depend on M, $\|\psi\|_{s_1,s_2;\Gamma_d}$. In (35) and below $\gamma = 1-\alpha$ for $\alpha\in[0,1)$ from (11).

Using estimate (35) we find that

$$I_1 \leqslant \omega_0\Bigl(2R, \fint\limits_{Q_{2R}(z^0)} |u_x|^2\,dz\Bigr)\Bigl\{\int\limits_{Q_{2R}(z^0)} |u_x|^2\,dz + (2R)^{n+2\gamma}\Bigr\}, \tag{36}$$

$\omega_0(r,\xi)\to 0$ for r, $\xi\to 0$.

Now we recall that $g=\min\{|w|,K\}$ and estimate I_2 with the help of (11):

$$I_2 \leqslant \Bigl(\fint\limits_{Q_R} |u_x|^p\,dz\Bigr)\Bigl(\fint\limits_{Q_R} g^{\frac{p}{p-2}}\,dz\Bigr)^{\frac{p-2}{p}} |Q_R| \leqslant$$

$$\leqslant c_0\Bigl\{\int\limits_{Q_{2R}} |u_x|^2\,dz + R^{n+2\gamma}\Bigr\}\Bigl(\fint\limits_{Q_R} g^{\frac{p}{p-2}}\,dz\Bigr)^{\frac{p-2}{p}} \leqslant$$

$$\leqslant cK^{\frac{2}{p}}\Big(\fint_{Q_R}|w|dz\Big)^{\frac{p-2}{p}}\int_{Q_{2R}}|u_x|^2dz+c_0KR^{n+2\gamma}\leqslant$$

$$\leqslant cK^{\frac{2}{p}}\Big(\fint_{Q_R}|w_x|^2\,dz\Big)^{\frac{p-2}{2p}}\int_{Q_{2R}}|u_x|^2dz+c_0KR^{n+2\gamma}\leqslant$$

$$\overset{*}{\leqslant} cK^{\frac{2}{p}}\Big\{\fint_{Q_R}|u_x|^2\,dz+R^{-(n-2)}\int_{Q_R}|u_t|^2\,dz\Big\}^{\frac{p-2}{2p}}\int_{Q_{2R}}|u_x|^2\,dz+c_0KR^{n+2\gamma},$$

the estimate $(*)$ holds due to relation (21).

Denote

$$\chi(R,z^0)=\Big(\fint_{Q_R(z^0)}|u_x|^2\,dz+R^{-(n-2)}\int_{Q_R(z^0)}|u_t|^2\,dz\Big)^{\frac{p-2}{2-p}},$$

then

$$I_2\leqslant cK^{\frac{2}{p}}\chi(2R,z^0)\int_{Q_{2R}(z^0)}|u_x|^2\,dz+cK(2R)^{n+2\gamma}. \tag{37}$$

Taking into account estimates (36), (37) and the inequality

$$\|\psi\|^2_{m_1,m_2;\Gamma_R}\leqslant cR^{n+2(1-\alpha)}\|\psi\|^2_{s_1,s_2;\Gamma_R},$$

we deduce from (28) that

$$\left.\begin{aligned}&|\,w\,|^2_{Q_{\frac{R}{2}}}\leqslant c\Big\{s|\,w\,|^2_{Q_R}+\Big(\omega_0(\dots)+K^{\frac{2}{p}}\chi(\dots)\Big)\cdot\\&\cdot\int_{Q_{2R}}|u_x|^2\,dz+\Big(K+\frac{c_\psi}{s}\Big)(2R)^{n+2\gamma}\Big\},\end{aligned}\right\} \tag{38}$$

where the functions ω_0 and χ have the same arguments as in estimates (36) and (37), constant c depend on ν, L, a, M, $c_\psi=\|\psi\|_{s_1,s_2;\Gamma_d}$.

Using (21) to estimate $|\,w\,|^2_{Q_R}$ in (38) we find that

$$\left.\begin{aligned}&|\,w\,|^2_{Q_{\frac{R}{2}}}\leqslant c\Big\{\Big(s+\omega_0(\dots)+K^{\frac{2}{p}}\chi(\dots)\Big)\int_{Q_{2R}}|u_x|^2\,dz+\\&+(2R)^2\int_{Q_{2R}}|u_t|^2\,dz+\Big(K+\frac{c_\psi}{s}\Big)(2R)^{n+2\gamma}\Big\}.\end{aligned}\right\} \tag{39}$$

We put $\widehat{z}=z^0$, $r=\frac{R}{2}$ in (20) then for $\rho\leqslant\frac{R}{2}$

$$\left.\int\limits_{Q_\rho(z^0)}|u_x|^2\,dz\leqslant c\Big(\frac{\rho}{R}\Big)^{n+2}\int\limits_{Q_{\frac{R}{2}}(z^0)}|u_x|^2\,dz+c\int\limits_{Q_{\frac{R}{2}}(z^0)}|w_x|^2\,dz.\right\}\tag{40}$$

From (39) and (40) it follows that

$$\left.\begin{aligned}&\int\limits_{Q_\rho(z^0)}|u_x|^2\,dz\leqslant c\bigg\{\bigg[\Big(\frac{\rho}{2R}\Big)^{n+2}+s+\omega_0(\dots)+K^{\frac{2}{p}}\chi(\dots)\bigg]\cdot\\&\cdot\int\limits_{Q_{2R}(z^0)}|u_x|^2\,dz+(2R)^2\int\limits_{Q_{2R}(z^0)}|u_t|^2\,dz+\Big(K+\frac{c_\psi}{s}\Big)(2R)^{n+2\gamma}\bigg\},\\&\rho\leqslant\frac{R}{2}.\end{aligned}\right\}\tag{41}$$

Obviously, inequality (41) holds also for $\rho\in(\frac{R}{2},2R]$.

Renaming $2R$ by R in (41) we derive for $G(\rho,z^0)=\fint\limits_{Q_\rho(z^0)}|u|^2\,dz$ the inequality

$$\left.\begin{aligned}&G(\rho,z^0)\leqslant c_0\bigg\{\Big(\frac{\rho}{R}\Big)^2+\Big(\frac{R}{\rho}\Big)^n\big[s+\omega_0(\dots)+K^{\frac{2}{p}}\chi(\dots)\big]\bigg\}\cdot\\&\cdot G(R,z^0)+\frac{R^2}{\rho^n}\int\limits_{Q_R(z^0)}|u_t|^2\,dz+\Big(K+\frac{c_\psi}{s}\Big)\Big(\frac{R}{\rho}\Big)^nR^{2\gamma}\bigg\},\end{aligned}\right\}\tag{42}$$

where c_0 depends on parameters ν, L, a, $M=\|u\|_{\infty,Q}$ only;

$$\omega_0=\omega_0\big(R,G(R,z^0)\big),\quad \omega_0(\lambda,\mu)\to 0\ \text{ for }\ \lambda,\,\mu\to 0;$$

$$\chi=\Big(G(R,z^0)+R^{-(n-2)}\int\limits_{Q_R(z^0)}|u_t|^2\,dz\Big)^{\frac{p-2}{2p}}.$$

For $\tau\in(0,1)$ (42) can be rewritten as

$$\left.\begin{aligned}&G(\tau R,z^0)\leqslant c_0\big\{\tau^2+\tau^{-n}s+\omega_0(\dots)+K^{\frac{2}{p}}\chi(\dots)\big]\big\}G(R,z^0)+\\&+c_0\tau^{-n}R^{-(n-2)}\int\limits_{Q_R(z^0)}|u_t|^2\,dz+c_0\Big(K+\frac{c_\psi}{s}\Big)\tau^{-n}R^{2\gamma}.\end{aligned}\right\}\tag{43}$$

If parameter $\alpha \in (0,1)$ in (11) then for $\gamma = 1-\alpha$ we put $\gamma_1 \in (\gamma, 1)$ and for the case $\alpha = 0$ we fix $\gamma_1 \in (0,1)$ arbitrarily. Let us choose now $\tau \in (0,1)$ from the inequality

$$c_0 \tau^2 \leqslant \frac{\tau^{2\gamma_1}}{3}, \quad \tau = \tau(c_0, \gamma_1). \tag{44}$$

After that we fix $s \in (0,1)$ to get the inequality

$$\frac{s\, c_0}{\tau^n} \leqslant \frac{\tau^{2\gamma_1}}{3}, \quad s = s(c_0, \gamma_1). \tag{45}$$

Now parameters $K = K(s)$ and $K_1 = c_0(K + \frac{c_\psi}{s})\tau^{-n}$ in (43) are also fixed in s and τ. From estimate (27) we have

$$K_1 \leqslant c_1 \left(M + \|\psi\|_{s_1, s_2 : \Gamma_d} + D_R^{\frac{1}{2}}(z^0) \right), \tag{46}$$

where $D_R(z^0) = R^{-(n-2)} \int\limits_{Q_R(z^0)} |u_t|^2 \, dz$.

Note that if $n = 2$ then

$$K + K_1 \leqslant c_1 \left(M + \|\psi\|_{s_1, s_2 : \Gamma_d} + \|u_t\|_{2, Q_d} \right).$$

For any $\varepsilon > 0$ we put

$$\gamma_2 = \begin{cases} \gamma, & \text{if} \quad \gamma \neq 1 \\ \gamma_1 - \varepsilon, & \text{if} \quad \gamma = 1. \end{cases}$$

In any case $\gamma_1 > \gamma_2$.

From now on we assume that

$$\ell(z^0) \equiv \sup_{\rho \leqslant \frac{d}{2}} \frac{1}{\rho^{n-2+2\gamma_2}} \int\limits_{Q_\rho(z^0)} |u_t|^2 \, dz < +\infty. \tag{47}$$

This assumption ensures that for any $\widehat{R} \leqslant \frac{d}{2}$ there exists a neighborhood $V(z^0)$ such that

$$\begin{gathered} \frac{1}{\widehat{R}^{n-2+2\gamma_2}} \int\limits_{Q_{\widehat{R}}(z)} |u_t|^2 \, dz \leqslant \ell(z^0) + 1 \\ \textit{for } z \in V(z^0) \cap (Q_d^+ \cup \Gamma_d). \end{gathered} \tag{48}$$

Taking into account estimates (44)–(46) and the inequality $D_R(z^0) \leqslant \ell(z^0)\, R^{2\gamma_2}$ we derive from (43):

$$G(\tau R, z^0) \leqslant$$
$$\leqslant \left\{ \frac{2}{3}\tau^{2\gamma_1} + c_1\omega_0\big(R, G(R,z^0)\big) + c_2\big(G(R,z^0) + R^{2\gamma_2}\big)^{\frac{p-2}{2p}} \right\} \cdot$$
$$\cdot\, G(R,z^0) + HR^{2\gamma_2}, \quad (49)$$

where c_1, c_2, H depend on ν, L, a, M, n, γ_1 and c_2, H depend also on the value $\ell(z^0)$ from condition (47).

Function $\theta(\rho,\lambda) = c_1\omega_0(\rho,\lambda) + c_2(\lambda + \rho^{2\gamma_2})^{\frac{p-2}{2p}}$ goes to zero for ρ, $\lambda \to 0$. There exists $R_0 > 0$ and $\lambda_0 > 0$ such that for all $R \leqslant R_0$ and $\lambda \leqslant \lambda_0$ the inequality $\theta(R,\lambda) \leqslant \frac{\tau^{2\gamma_1}}{3}$ holds. Hence, for $R \leqslant R_0$ and $G(R,z^0) \leqslant \lambda_0$

$$\theta\big(R, G(R,z^0)\big) \leqslant \frac{\tau^{2\gamma_1}}{3} \quad (50)$$

and it follows from (49):

$$G(\tau R, z^0) \leqslant \tau^{2\gamma_1} G(R,z^0) + HR^{2\gamma_2}. \quad (51)$$

Now we assume in addition to (47) that

$$\varliminf_{\rho\to 0} G(\rho, z^0) = 0. \quad (52)$$

It results that for some $R \leqslant R_0$ $G(R,z^0) \leqslant \lambda_0$ and ineqiality (51) holds.

Recall that $\gamma_1 > \gamma_2$ and choose $R_1 \leqslant R_0$ to satisfty the inequality

$$\frac{HR_1^{2\gamma_2}}{\tau^{2\gamma_2} - \tau^{2\gamma_1}} < \frac{\lambda_0}{2}.$$

Now we revise the choise of R: let $R \leqslant R_1$ be fixed from the condition $G(R,z^0) < \frac{\lambda_0}{2}$. Then it follows from (51) that $G(\tau R, z^0) < \lambda_0$ and we can address to (51) changing R by τR. Repeating the iteration process we derive

$$G(\tau^k R, z^0) \leqslant \tau^{2k\gamma_1} G(R,z^0) + \frac{HR^{2\gamma_2}\tau^{2\gamma_2(k-1)}}{1 - \tau^{2(\gamma_1-\gamma_2)}}$$

and hence

$$G(\tau^k R, z^0) < \lambda_0\, \tau^{2k\gamma_2}, \quad \forall\, k \in N.$$

From the last inequality the following estimate is derived in a usual way:

$$G(\rho, z^0) \leqslant c\left(\frac{\rho}{R}\right)^{2\gamma_2}, \quad \forall\, \rho \leqslant R. \tag{53}$$

For the fixed number R there exists a neighborhood $U(z^0)$ such that for any $z \in U(z^0) \cap (Q_d^+ \cup \Gamma_d)$ the inequality $G(R, z) < \frac{\lambda_0}{2}$ holds. Moreover, from estimate (48) with $\widehat{R} = R$ we have

$$D_R(z) \leqslant \big(\ell(z^0) + 1\big) R^{2\gamma_2}, \quad \forall\, z \in V(z^0) \cap (Q_d^+ \cup \Gamma_d).$$

It follows that for $\forall\, z \in W(z^0) \cap (Q_d^+ \cup \Gamma_d)$, where $W(z^0) = U(z^0) \cap V(z^0)$, we have

$$G(\rho, z) \leqslant c\big(\frac{\rho}{R}\big)^{2\gamma_2}, \quad \forall\, \rho \leqslant R. \tag{54}$$

Now using the estimate similar to (35) we deduce that

$$\left.\begin{array}{l} \displaystyle\int\limits_{Q_\rho(z)} |\, u - u_{\rho,z}\,|^2 \; dz \leqslant c\rho^{n+2+2\gamma_2}, \quad \forall\, \rho \leqslant R, \\ z \in W(z^0) \cap (Q_d^+ \cup \Gamma_d), \end{array}\right\} \tag{55}$$

where $u_{\rho,z} = \frac{1}{|Q_\rho|} \int\limits_{Q_\rho(z)} u\, dz$.

According to the definition and properties of the Campanato space $\mathcal{L}^{2,n+2+2\gamma}$ (see [20, 21]) we conclude that $u \in \mathbf{C}^{\gamma_2}(W(z^0) \cap (Q_d^+ \cup \Gamma_d))$.

Thus assumptions (47) and (52) quarantee that z^0 is a regular point for the solution u.

It is obvious that a singular set $\Sigma \subset Q_d^+ \cup \Gamma_d$ could be described by the following inclusion: $\Sigma \subset \Sigma_1 \cup \Sigma_2$, where

$$\Sigma_1 = \left\{ z \in Q_d^+ \cup \Gamma_d : \sup_{\rho \leqslant \frac{d}{2}} \frac{1}{\rho^{n-2+2\gamma_2}} \int\limits_{Q_\rho(z)} |u_t|^2 \, dz = +\infty \right\},$$

$$\Sigma_2 = \left\{ z \in Q_d^+ \cup \Gamma_d : \varliminf_{\rho \to 0} \; ⨍\limits_{Q_\rho(z)} |u_x|^2 \, dz > 0 \right\}.$$

From the inequality

$$⨍\limits_{Q_\rho} |u_x|^2 \, dz \leqslant \left(\frac{1}{R^{n+2-p}} \int\limits_{Q_\rho} |u_x|^p \, dz \right)^{\frac{2}{p}}$$

and the definition of Σ_2 it follows that $\mathcal{H}_{n+2-p}(\Sigma_2;\delta)=0$ ([2], Proposition 3.2). For $z^0\in\Sigma_1$ we have

$$\sup_{\rho\leqslant\frac{d}{2}}\frac{1}{\rho^{n-2+2\gamma_2}}\int\limits_{Q_\rho(z^0)}|u_t|^2\,dz=\varlimsup_{\rho\to 0}\frac{1}{\rho^{n-2+2\gamma_2}}\int\limits_{Q_\rho(z^0)}|u_t|^2\,dz=+\infty.$$

It means that

$$\Sigma_1\subset\left\{z\in Q_d^+\cup\Gamma_d:\varlimsup_{\rho\to 0}\frac{1}{\rho^{n-2+2\gamma_2}}\int\limits_{Q_\rho(z)}|u_t|^2\,dz>0\right\}$$

and therefore $\mathcal{H}_{n-2+2\gamma_2}(\Sigma_1;\delta)=0$. Put $\varepsilon_0=\min\{p-2,2(1-\gamma_2)\}$ then $\mathcal{H}_{n-\varepsilon_0}(\Sigma;\delta)$. This concludes the proof of Theorem 2.

The research was partially sponsored by the Russian Foundation for Fundamental Research (RFFI) Grant 96-01-00087.

References

1. O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltseva, *Linear and quasilinear equations of parabolic type.* Transl. of Math. Momographs 23. Providence, Rhode Island, AMS (1968).
2. M. Giaquinta, M. Struwe, *On the partial regularity of weak solutions of nonlinear parabolic systems.* — Math. Z. **179** (1982), 437–451.
3. M. Marino, A. Maugeri, *Partial Hölder continuity of solutions of nonlinear parabolic systems of second order with quadratic growth.* — Bolletino U. M. I. **7, 3-B** (1989), 397–435.
4. M. Marino, A. Maugeri, *Maximum principle for parabolic systems.* — St. Petersburg Math. J. **3, No. 6** (1992), 1351–1358.
5. M. Marino, A. Maugeri, *Regularity for parabolic systems in divergence form.* — Methods of Real Analysis and P. D. E., Quaderno n. 14, Acad. Pontiana, Napoli (1992), 63–82.
6. M. Marino, A. Maugeri, *Boundedness result for parabolic systems.* — Methods of Real Analysis and P. D. E., Quaderno n. 14, Acad. Pontaniana, Napoli (1992), 39–62.
7. M. Marino, A. Maugeri, *$L^{2,\Lambda}$-regularity of the spatial derivatives of the solutions to parabolic systems in divergence form.* — Annali di Matem. Pura Appl. IV **CLXIV** (1993), 275–298.
8. M. Marino, A. Maugeri, *A remark on the Note: Partial Hölder continuity of the spatial derivatives of the solutions to nonlinear parabolic systems with quadratic growth.* — Rend. Sem. Mat., Padova **95** (1996).
9. J. Naumann, J. Wolf, *Interior differentiability of weak solutions to parabolic systems with quadratic growth nonlinearities.* — Rend. Sem. Mat., Padova (to appear) 1997.
10. A. A. Arkhipova, *On the global solvability to nondiagonal parabolic systems of variational structure in the case of two spatial variables.* — Problemy Matem. Analiza (St. Petersburg, SPGU) **16** (1997).

11. M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces.* — Comment. Math. Helvetici **60** (1985), 558–581.
12. A. A. Arkhipova, *On the regularity of solutions of boundary-value problem for quasilinear elliptic systems with quadratic nonlinearity.* — J. of Math. Sci. **80, No. 6** (1996), 2208–2225.
13. A. A. Arkhipova, *On the Neumann problem for nonlinear elliptic systems with quadratic nonlinearity.* — St. Petersburg Math. J. **8, No. 5** (1997), 1–17.
14. M. Giaquinta, E. Giusti, *Partial regularity for the solutions to nonlinear parabolic systems.* — Ann. Mat. Pura Appl. **47** (1973), 253–266.
15. M. Giaquinta, E. Giusti, *Nonlinear elliptic systems with quadratic growth.* — Manuskripta Math. **24** (1978), 323–349.
16. M. Giaquinta, G. Modica, *Local existence for quasilinear parabolic systems under nonlinear boundary condition.* — Ann. Mat. Pura Appl. **149** (1987), 41–59.
17. J. Stara, O. John, *Some (new) counterexamples of parabolic systems.* — Comment. Math. Univ. Carolinae **36, No. 3** (1995), 503–510.
18. A. A. Arkhipova, *L^p-estimates of the gradients of solutions to initial-boundary problems for quasilinear parabolic systems.* — Problemy in Math. Anal., SPGU, St.-Petersburg **13** (1992), 5–18.
19. A. A. Arkhipova, *On the regularity of the solution of the Neumann problem for quasilinear parabolic systems.* — Russian Akad. Sci. Izv. Math. **45, No. 2** (1995), 231–253.
20. S. Campanato, *Equazioni paraboliche del secondo ordine e spazi $\mathcal{L}^{2,\theta}(\Omega,\delta)$.* — Ann. Mat. Pura Appl. **73, No. 4** (1966), 55–102.
21. G. da Prato, *Spazi $\mathcal{L}^{2,\theta}(\Omega,\delta)$ e loro proprietá.* — Ann. Mat. Pura Appl. **69** (1965), 383–392.

С.-Петербургский
государственный университет

Поступило 5 мая 1997 г.