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MINI SURVAY OF THE PRINCIPAL NOTIONS IN THE THEORY OF FULLY NONLINEAR ELLIPTIC SECOND-ORDER DIFFERENTIAL EQUATIONS

Abstract. The paper contains a collection of the principal notions in the theory of fully nonlinear elliptic second-order differential equations and also the reduction of nontotally elliptic equations to totally elliptic.

Dedicated to O. A. Ladyzhenskaya

0. Introduction.

The start of the theory of fully nonlinear second-order differential equations can be connected with the papers [4, 12, 1, 7, 2, 3, 6, 11, 15]. It contains at least three separate branches: totally elliptic equations, nontotally elliptic equations, solvability in viscosity sense. These directions were developed comparetively independently generating own notions and methods and one may get the impression they can not be unified. To the author's opinion it is not so and we suggest here the collection of the principal notions in hope to expose the logic of the whole construction. In particular we describe the reduction of nontotally elliptic equation to the totally elliptic, i.e., the reduction to Bellman equations discovered by N. V. Krylov [14]. The up to day situation in the theory of fully nonlinear second-order differential equations was the subject of V. I. Smirnov's seminars, spring 97, St.-Petersburg. The author is gratefull to the all participants and in particular to O. A. Ladyzenskaya and N. N. Uraltseva.

1. The description of evolving functions. Let function F defined onto $D_0 \subset \mathrm{Sym}\,(n)$, where $\mathrm{Sym}\,(n)$ is the set of nth order symmetric matrices. The matter of primary interest in this paper will be the level sets of function F. For nonlinear F their structure can be very complicated in general. We simplify the situation reducing D_0 to D_1 which is the set of nonnegative monotonicity of F,

$$D_1 = \{S : F(S+\eta) \geq F(S),\, \eta \geq \mathbf{0}\}.$$

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Moreover we want to have the convex level surfaces and restrict F to D_2 which is the set of local concavity of F. Eventually let $\overline{D}$ be convex component of intersection $D_1 \cap D_2$. The domain D could as well be the intersection of the sets of nonpositive monotonicity F and its convexity. It would not change the point we are after the convex level surfaces.

We always assume the matrices $S+\eta$, αS, BSB^T belong to $\overline{D}$ if $S \in \overline{D}$, where $\eta \geq \mathbf{0}$, $\alpha \geq 1$ and B is arbitrary orthogonal matrix. Moreover, $F(s) = F(BSB^T)$.

One more requirement to the pair (F, D) is to be described in terms of the numbers:

$$\underline{f} = \sup_{\partial D} \varlimsup_{S \to S_0} F(S),$$

$$\overline{f} = \inf_{S \in D} \varliminf_{\alpha \to \infty} F(\alpha S).$$

Either of numbers $\underline{f}$, $\overline{f}$ can be infinite but in the case $D \neq \mathrm{Sym}\,(n)$ we always assume

$$\underline{f} < \overline{f}.$$

The above description of the pairs (F, D) accords to the collection of nonlinear operators considered in the theory of fully nonlinear second-order differential equations (see for inst. [1, 16, 17] and here we respeak the relevant facts in terms of the pairs (F, D).

We indicate by $F \in \mathbf{DF}$ there is nonempty D for our F and for the sake of simplicity assume $F \in C^2(D)$ if the opposite was not said. Sometimes $D = \mathrm{Sym}\,(n)$ and then we say F is totally monotone. If it is not so the domain D belongs to the halfspace $\mathrm{tr}\, S > c_0$ with $c_0 = \inf_{\partial D} \mathrm{tr}\, S$ due to the invariance of D under orthogonal transformations. In this case we can always reduce to $c_0 \geq 0$ by translation $S \to S + c_0^- I$, where a^- is the negative part of a and then

$$\mathrm{tr}\, S \geq 0, \quad S \in \overline{D}. \tag{1.1}$$

Denote

$$F^{ij}(S) = \partial F(S)/\partial s_{ij}.$$

The following statement holds true.

Lemma 1. *Assume $F \in \mathbf{DF}$, $S \in D$. Then*

$$F^{ij}(S) s_{ij} \; > 0. \tag{1.2}$$

If in addition $F(S) \leq C < \overline{f}$, then

$$\operatorname{tr}(F^{ij}(S) \geq \nu(C) > 0. \tag{1.3}$$

Indeed, if α is sufficiently large the line

$$0 < F(\alpha S) - F(S) \leq (\alpha - 1)F^{ij}(S)s_{ij}$$

is valid due to concavity of F, i.e. (1.2) holds. Moreover, for large α D contains αI. Then

$$F(\alpha I) - F(S) \leq F^{ij}(S)(\alpha\delta_{ij} - s_{ij}),$$

what together with (1.2) leads to existence of some $\nu(C) > 0$ in (1.3).

The most known nontrivial examples of pairs (F, D) are

$$F(S) = F_{m,l}(S) = (S_{m,l})^{1/(m-l)}, \quad 0 \leq l < m \leq n, \tag{1.4}$$

$$D = K_m = \{S : S_i(S) > 0, \ i = 1, \ldots, m\}, \tag{1.5}$$

where $S_i = \operatorname{tr}_i(S)$, $\operatorname{tr}_i(S)$ is the sum of all i-th order principal minors of the matrix S if $i > 0$ and $S_0 = 1$. For the pairs (1.4), (1.5) $\underline{f} = 0$, $\overline{f} = \infty$, $c_0 = 0$.

The pair (1.4), (1.5) with $l = 0$ has appeared first in context of fully nonlinear equations in the paper [7]. The relevant properties for $l > 0$ were established by N. S. Trudinger [16]. The amount of examples can be increased by the following simple statement.

Lemma 2. *Let $F \in \mathbf{DF}$, $F > 0$ and $q < 1$. Then $F^q \in \mathbf{DF}$, $q > 0$, $\log F \in \mathbf{DF}$, $-F^q \in \mathbf{DF}$, $q < 0$.*

In this course we also can relate to the domain (1.5) functions $\log S_{m,l}$ with $\underline{f} = -\infty$, $\overline{f} = +\infty$, $-S_{l,m}$ with $\underline{f} = -\infty$, $\overline{f} = 0$ and eventually

$$F = -\sum_{k=0}^{m-1} a^k S_{k,m}, \tag{1.6}$$

where the coefficients $a^k \geq 0$, $\sum_{k=0}^{m-1} a^k > 0$ and $\underline{f} = -\infty$, $\overline{f} = 0$. Example (1.6) is taken from [14].

For smooth functions F monotonicity can be expressed as $(F^{ij}) \geq \mathbf{0}$. Due to (1.2) functions $F \in \mathbf{DF}$ are srictly monotone over D but there are of particular interest the functions F which are uniformly monotone over $\overline{D}$ in the following sense: the inequality

$$\nu_0 I \operatorname{tr}(F^{ij}(S)) < (F^{ij}(S)) \tag{1.7}$$

holds true with some $\nu_0 > 0$ for $S \in \overline{D}$. Note that inequality (1.3) prevents degeneration in (1.7).

It was noticed by N. S. Trudinger [16] that any pair (F, D) generates the family $(F_\beta, D_\beta), \beta > 0$ with uniformly monotone F_β over D_β. Namely, denote

$$S^\beta = S + \beta I \mathrm{tr}\, S,\ D_\beta = \{S : S^\beta \in D\}, \quad F^\beta(S) = F(S^\beta). \tag{1.8}$$

The following statement holds true.

Proposition 3. *Let $\beta > 0$, $F \in \mathbf{DF}$ and $\underline{f}$, $\overline{f}$ be relating numbers. Then F_β is uniformly monotone over D_β with $\nu_0 = \beta/(1+\beta)$ in (1.7) and $\underline{f}_\beta = \underline{f}$, $\overline{f}_\beta = \overline{f}$.*

Indeed,

$$\begin{aligned} F_\beta^{ij}(S) &= F^{ij}(S^\beta) + \beta\delta^{ij}\mathrm{tr}\,(F^{ij}(S^\beta)),\\ \mathrm{tr}\,(F_\beta^{ij}(S)) &= (1+\beta)\mathrm{tr}\,(F^{ij}(S^\beta)), \end{aligned} \tag{1.9}$$

what leads to (1.7) with $\nu_0 = \beta/(1+\beta)$.

2. The reduction to the totally monotone functions.

To begin with consider one geometrical construction. Let $D \in \mathrm{Sym}\,(n)$ be a convex domain of the same type as in Sec. 1, $\mathbf{0} \notin D$ and let ∂D be its boundary. Denote by $\{N = (n^{ij})\}$ the set of interior unit normals to ∂D, i.e., the set of symmetric matrices N such that $\mathrm{tr}\, N^2 = 1$ and if $S^0 \in \partial D$ then

$$D \subset \{S \in \mathrm{Sym}\,(n) : n^{ij}(S^0)(s_{ij} - s^0_{ij}) > 0\}. \tag{2.1}$$

The term $n^{ij}(S^0)s^0_{ij}$ is the meaning of support function to ∂D at S^0 and can be expressed the following way:

$$n^{ij}(S^0)s_{ij}(S^0) = \inf_D n^{ij}(S^0)s_{ij} = p(N_0, D) \tag{2.2}$$

and we infer from (2.1), (2.2) that ∂D is the envelope of the set of support planes

$$n^{ij}s_{ij} - p(N, D) = 0, \quad N \in \{N\}.$$

Therefore the line

$$\inf_{\{N\}}(n^{ij}s_{ij} - p(N, D)) = 0 \tag{2.3}$$

reads as the equation of ∂D if $\{N\}, p$ are known.

The inequality in (2.1) and also equalities in (2.2), (2.3) remain true after substitution N to $c(N)N$ whatever $c(N) > 0$ be. Keeping in mind

N. V. Krylov's construction [14] we choose $c(N) = (\operatorname{tr} N)^{-1}$. Due to the symmetry of D and by its convexity we have $N \geq \mathbf{0}$, also $c(N) > 0$ and the relevant set of normals looks as

$$\boldsymbol{\omega}(D) = \{\omega \in \overline{M}^{+}, \quad \operatorname{tr} \omega = 1, \quad p(\omega, D) \geq 0\}, \tag{2.4}$$

where M^{+} is the set of symmetric positive definite matrices and

$$p(\omega, D) = \inf_{D} \omega S. \tag{2.5}$$

Having fixed the domain D and therefore ω by (2.4), (2.5), consider $\widehat{D} \in D$ and assume the function

$$\widehat{h}\,(\omega) = \inf_{\widehat{D}} \omega S \tag{2.6}$$

has sense. Eventually we introduce the function $\widehat{H}$ in $\operatorname{Sym}(n)$ by the line

$$\widehat{H}\,(S) = \inf_{\boldsymbol{\omega}}(\omega S - \widehat{h}\,(\omega)) \tag{2.7}$$

and conclude the above consideration by the statement:

Proposition 4. *Function* (2.7) *is totally monotone and if* $\boldsymbol{\omega}(D) = \boldsymbol{\omega}(\widehat{D}\,)$ *then the equation of* $\partial\widehat{D}$ *can be written as* $\widehat{H}\,(S) = 0$.

Function (2.7) is nonsmooth in general and the notion of uniform monotonicity given by (1.7) is irrelevant here. We modify it saying the function H is uniformly monotone if there exists a number $\nu_0 > 0$ such that

$$H(S + \xi \times \xi) \geq \nu_0 \xi^2, \quad \xi \in R^n. \tag{1.7$'$}$$

It is worth to note that Krylov's requirement $\operatorname{tr} \omega = 1$ carries out the following property of function (2.7).

Proposition 5. *Under the assumptions of Proposition* 4 *the set of levels* $\{S : \widehat{H}\,(S) = nc, c \in R\}$ *coincides with the set of translations* $\{\partial\widehat{D} + cI, c \in R\}$, *respectively.*

Indeed, under circumstances we have the identity

$$\widehat{h}\,(\omega) = (\inf_{\widehat{D}+cI} \omega S) - nc$$

Since $\boldsymbol{\omega}(D) = \boldsymbol{\omega}(D + cI)$ the definitions (2.6), (2.7) stay correct and therefore $\widehat{H}\,(S) = nc$ is equivalent to $S \in \partial\widehat{D} + cI$.

As the consequence the domains $\widehat{D}$, $\mathrm{Sym}\,(n)\backslash\widehat{D}$ can be described the following way.

$$\widehat{D} = \{S : \widehat{H}\,(S) > 0\}, \quad \mathrm{Sym}\,(n)\backslash\widehat{D} = \{S : \widehat{H}\,(S) \le 0\}. \tag{2.8}$$

We of course assume $\boldsymbol{\omega}(D) = \boldsymbol{\omega}(\widehat{D}\,)$ in (2.8).

Consider function (2.7) from the point of view of Sec. 1. Proposition 4 acclaimes the domain $D = \mathrm{Sym}\,(n)$ is to be related to H. The qualification by numbers $\underline{f}, \overline{f}$ is senseless here because $\overline{f} = -\infty$. On the other hand the regularization (see Proposition 3) looks particulary simple for H:

$$H^\beta(S) = H(S) + \beta \mathrm{tr}\, S,$$

and it was introduced in [13].

The essence of Krylov's reduction is presented in the Theorem:

Theorem 6. *Assume $F \in \mathbf{DF}$, $\mathbf{0} \notin D$, $\underline{f} \le c < \overline{f}$ and let*

$$\overline{D^c} = \{S \in \overline{D} : F(S) \ge c\}.$$

Then

$$\partial D^c = \{S \in \mathrm{Sym}\,(n) : H^c(S) = 0\}\,,$$

where $H^c = \widehat{H}$ (see (2.7)) with $\widehat{D} = D^c$.

To prove Theorem 6 we fix some $c \in (\underline{f}, \overline{f})$. Since F is smooth, strictly monotone and concave in D, the surface ∂D^c is smooth and convex with respect to interior normal and

$$\boldsymbol{\omega}(D^c) = \big\{(F^{ij}/\mathrm{tr}\,(F^{ij})) : F(S) = c\big\}$$

Consider the set

$$\underline{S}(c) = \{S \in D\backslash D^c\}.$$

It is nonempty for $c > \underline{f}$ and by assumption for any $S \in \underline{S}(c)$ there exists the unique number $\alpha > 1$ such that $F(\alpha S) = c$. But that means any ray from the origin into D intersects ∂D^c exactly at one point. Therefore $\boldsymbol{\omega}(D) = \boldsymbol{\omega}(D^c)$, $c \in [\underline{f}, \overline{f})$, i.e., the assumptions of Proposition 4 with $\widehat{D} = D^c$ are satisfied what proves our Theorem.

In order to embrace example (1.6) we expand the assertion of Theorem 6 to the case of functions F_λ depending also on the set Λ.

Corollary 7. *Assume $F_\lambda \in \mathbf{DF}$ with $D_\lambda = D$ for all $\lambda \in \Lambda$ and let*

$$D_\lambda^c = \{S \in D : F_\lambda(S) > c\}.$$

Then

$$\partial D_\lambda^c = \left\{S \in \mathrm{Sym}\,(n) : \inf_{\boldsymbol{\omega}}(\omega S - h(\omega, \lambda, c)) = 0\right\},$$

where

$$\boldsymbol{\omega} = \left\{(F_{\lambda_0}^{ij}/\mathrm{tr}\,(F_{\lambda_0}^{ij}))(S),\ S \in D\right\} \tag{2.9}$$

with $\lambda_0 \in \Lambda$ to be arbitrary fixed and

$$h(\omega, \lambda, c) = \inf_{D_\lambda^c} \omega S.$$

Corollary 7 serves function (1.6) with

$$\Lambda = \left\{\lambda \in R^{m-1} : \lambda^i \geq 0,\ i = 1, \dots, m-1,\ |\lambda| > 0\right\}, \quad D = K_m.$$

In order to emphasize the subtle way of including parametrs in F consider the function (see (1.4))

$$F_{m,l}^A(S) = F_{m,l}(S^A), \quad S^A = ASA, \quad A \in M^+,$$

i.e., $\Lambda = M^+$. In accordance with (1.5)

$$D_A = \{S : \mathrm{tr}_i\, S^A > 0,\ i = 1, \dots, m\}.$$

We fix up for instance $A_0 = I$. Then $D_{A_0} = K_m$. But it is obvious $\boldsymbol{\omega}(D_A) \neq \boldsymbol{\omega}(K_m)$ if $A \neq I$ and the reduction of Corollary 7 is of no avail.

In contrast to H we can only be certain function F was defined over D. Nevertheless the space $\mathrm{Sym}\,(n)$ can be separated in two parts similary to (2.8) in terms of F. Namely

$$\overline{D^c} = \{S : F(S + \eta) \geq c,\ \eta \geq \mathbf{0}\}, \tag{2.10}$$

$$\mathrm{Sym}\,(n)\backslash D^c = \{S : \inf_{\eta \geq \mathbf{0}} F(S + \eta) \leq c\} \tag{2.11}$$

for $c \in (\underline{f}, \overline{f})$.

Indeed, the relation (2.11) is well defined since $S + \eta \in D^c$ for sufficiently large matrices η by assumption.

Remark eventually that to construct $\widehat{H}$ it was sufficient to know only the surface $\partial \widehat{D}$, i.e., any description of $\partial \widehat{D}$ would suit the case generally speaking.

3. Differential operators and equations.

Let $u \in C^2(\Omega)$, $\Omega \in R^n$ be a bounded domain. Fix up a positive definite matrix $A(x,u,p)$ and also $W(x,u,p)$ defined in $\{x \in \Omega, u \in R^1, p \in R^n\}$. Denote

$$u_{(xx)} = A^{1/2}(x,u,u_x)u_{xx}A^{1/2}(x,u,u_x),$$

where u_x is the gradient and u_{xx} is the Hesse matrix of u and determine the matrix operator

$$S[u] = u_{(xx)} + W. \tag{3.1}$$

Having given the set of sufficiently smooth parametrs $\Lambda(x,u,p)$ we relate to $F_\lambda \in \mathbf{DF}$ with D independent on Λ the operator

$$F_\lambda[u] = F(S[u], \lambda[u]) \tag{3.2}$$

and eventually the equation

$$F_\lambda[u] = f, \tag{3.3}$$

where f is known smooth function in Ω.

By Theorem 6 (Corollary 7) equation (3.3) can be reduced to

$$\inf_{\boldsymbol{\omega}}(a^{ij}(\omega,x,u,u_x)u_{ij} + b(\omega,x,u.u_x)) = 0, \tag{3.4}$$

where

$$\boldsymbol{\omega} = \left\{\omega = ((F^{ij}_{\lambda_0}/\mathrm{tr}\,(F^{ij}_{\lambda_0}))(S)) : S \in \overline{D}\right\}, \tag{3.5}$$

$$a^{ij}(\omega,x,u,p) = (A^{1/2}(x,u,p)\omega A^{1/2}(x,u,p))^{ij}, \tag{3.6}$$

$u_{ij} = d^2u/dx^idx^j$,

$$b(\omega,x,u,p) = (\mathrm{tr}\,\omega W - \inf_{D^f_\lambda}\omega S)(x,u,p) \tag{3.7}$$

(see (2.6), (2.7), (2.9)). The parametr λ_0 in (3.5) is fixed some way and for the sake of simplicity from now on we omit the indication of dependence on λ.

Equations (3.4) are Bellman equations and the idea by N. V. Krylov is to consider any fully nonlinear equation as a particular case of Bellman equation to be treated by the methods developed in the theory of Bellman equations. It seems rather deterministic. All the more at the moment there are no means in this theory to encompass equations (3.4) with the coefficients (3.6), (3.7) depending on p, i.e., to encompass the equations arising in geometry for instance. On the other hand one can see equation (3.4) as a particular case of (3.3) with operator $F \equiv H$ generated by totally monotone and concave function (2.7), where the set $\boldsymbol{\omega} \subset \overline{M}^+$ and function h are given.

We introspect the principal notions of the theory of fully nonlinear equations suggesting the following collection of definitions.

Definition 1. *Function u is admissible to operator F iff $u \in C^2(\Omega)$ and*

$$S[u](x) \in D, \quad x \in \Omega. \tag{3.8}$$

A solution u is admissible to equation (3.3) *iff $u \in C^2(\Omega)$ and satisfies relation* (3.8).

We denote the set of functions admissible to F by $\operatorname{ad} F$.

It is worth to note that all nonlinear second-order differential operators are separated by Definition 1 into two parts depending on existence or nonexistence of admissible functions. By the way any C^2-function is admissible to the operators generated by totally monotone functions.

The monotonicity of function F can degenerate in $D(\overline{D})$ in general and if so one can not count on existence of admissible solutions to equation (3.3). For functions of Sec. 1 with $D \neq \operatorname{Sym}(n)$ the requirement $f > \underline{f}$ prevents degeneration of F whence for $D = \operatorname{Sym}(n)$ it is a question. For instance if the quadratic form $(a\xi, \xi), \xi \in R^n$ degenerates for some ω, ξ in (3.4) it is hard to circumvent degeneration without preliminary knowlege of the origin of function b. But there is regularization of function F given by Proposition 3. In order to make use of it we inroduce the following qualification of operators.

Definition 2. *The operator F (H) is uniformly nondegenerate if inequality* (1.7), ((1.7′)) *holds true with some $\nu_0 > 0$.*

Any second-order differential operator is elliptic on the set of admissible functions and the notion of uniform ellipticity is close to the notion of uniform nondegeneracy. Namely the operator F is uniformly elliptic if the inequality

$$\begin{aligned} F(u_{(xx)} + (A(x,u,u_x)\xi, \xi), \dots) - F(u_{(xx)}, \dots) \\ \geq \nu \operatorname{tr} A(x,u,u_x)\xi^2, \quad \xi \in R^n \end{aligned} \tag{3.9}$$

with some number $\nu > 0$ holds true for any $u \in \operatorname{ad} F$

The inequality (3.9) fits perfectly Bellman operators and looks particulary nice if $\operatorname{tr} A(x,u,p) \sim c$ with some constant $c > 0$. But we think it is reasonable also indicate as uniformly elliptic the operators generated by smooth function F satisfying the inequality

$$(F^{ij}[u])u_{xx} \geq \nu \mathrm{tr}\, A(x,u,u_x)\mathrm{tr}\,(F^{ij}[u])I, \quad u \in \mathrm{ad}\, F \tag{3.10}$$

with some $\nu > 0$.

Requirements (3.9), (3.10) can only be valid for uniformly nondegenerate operators but additionally contain the restriction on behaviour of the eigenvalues of matrix A, namely A has to be uniformly positive definite.

The following statement is the corollary to Proposition 3.

Proposition 8. *The operator F_β,*

$$F_\beta[u] \equiv F(S^\beta[u]), \quad \beta > 0, \tag{3.11}$$

where S^β defined by (1.8), is uniformly nondegenerate if $\mathrm{tr}\,(F^{ij}[u]) \geq \nu_0 > 0$ *for any $u \in \mathrm{ad}\, F$. If in addition the matrix $A(x,u,p)$ is uniformly positive definite the operator (3.11) is uniformly elliptic.*

As a rule $\mathrm{ad}\, F_\beta \neq \mathrm{ad} F$ and if $\mathbf{0} \notin D$, $W \geq \mathbf{0}$ (see (1.1), (3.1)) then $\mathrm{ad}\, F \subset \mathrm{ad}\, F_\beta$. For totally monotone operators $\mathrm{ad}\, F_\beta = \mathrm{ad}\, F = C^2(\Omega)$ as the exception.

Under normalization $\mathrm{tr}\,\omega = 1$ Bellman operator H brings out particulary impressive regularization for $A = I$:

$$H_\beta[u] = \beta\Delta u + H[u]. \tag{3.12}$$

One can see (3.12) as the homotopy of fully nonlinear operator H to Laplace operator but of course it is possible only for Bellman operator.

As it was mentioned above the set of admissible functions is not sufficient to obtain solutions to the equations with degenerate monotone operators. But the above collection of notions can be amplified by the relevant notions from the theory of solvability in viscosity sense to make available viscosity solutions. To follow this course consider a set $\overline{X} \subseteq \Omega$ as the set, where function $u \in C^2(\Omega)$ has the second upper jet, i.e., for any $\overline{x} \in \overline{X}$ there exist the collection $\overline{D}_2[u](\overline{x}) = \{\overline{p} \in R^n, \overline{r} \in \mathrm{Sym}\,(n)\}$ such that

$$u(x) - u(\overline{x}) \leq (p, x - \overline{x}) +$$
$$+1/2(\overline{r}(x-\overline{x}), (x-\overline{x})) + o((x-\overline{x})^2), \quad x \in \Omega. \tag{3.13}$$

Instead of (1.3) we involve

$$S(x,u,p,r) = A^{1/2}(x,u,p,)rA^{1/2}(x,u,p,) + W(x,u,p)$$

and let

$$S(\overline{D}_2[u])(\overline{x}) = S(\quad, u, \overline{p}, \overline{r})(\overline{x}).$$

Definition 3. *Function $u \in C(\Omega)$ is a subfunction to the operator F iff*

$$S(\overline{D}_2[u])(\overline{x}) \in D, \quad \overline{x} \in \overline{X}$$

Denote the set of subfunctions to F by sb F. The inclusion ad $F \subset$ sb F is obvious and the crucial fact of viscosity theory is that any subfunction to F can be approximated by admissible to F functions locally in $C(\Omega)$, i.e., over any subdomain of Ω. To prove assertions of such type one involves Jensen's approximations [11].

Specification of subfunction as a viscosiity solution to equation (3.3) ((3.4)) can be done in terms of sub- and supersolutions to the equation. To proceed with consider the set $\underline{X} \subseteq \Omega$ as the set, where the lower 2-jet $\underline{D}_2[u] = \{\underline{p}, \underline{r}\}$ exists, i.e., for $\underline{x} \in \underline{X}$, $(\underline{p}, \underline{r})(\underline{x})$ the inequality inverse to (3.13) holds true.

Definition 4. *Function $u \in C(\Omega)$ is a subsolution to equation (3.3) iff it satisfies the inequality*

$$F(S(\quad, \overline{p}, \overline{r})[u](\overline{x}) \geq f(\overline{x}), \quad \overline{x} \in \overline{X}.$$

It is a supersolution iff the relation

$$\inf_{\eta \geq \mathbf{0}} F(S(\quad, \underline{p}, \underline{r} + \eta)[u](\underline{x}) \leq f(\underline{x}), \quad \underline{x} \in \underline{X} \tag{3.14}$$

is valid and eventually u is a viscosity solution iff it is simultaniously sub- and supersolution to equation (3.3).

In fact Definition 4 conveys partition (2.10), (2.11) and if ad $F \neq C^2(\Omega)$ the line (3.14) is the natural way in description of supersolutions what was discovered by N. S. Trudinger [16].

In the case of Bellman equation (3.4) or any equation with totally monotone operator the indication "$\inf_{\eta \geq \mathbf{0}}$" in (3.14) is obviously redundant and only then one can see the symmetry in the notions of sub- and supersolutions.

In conclusion we outline the proceedings concerning existence theorems. The subject to investigate is the Dirichlet problem to equation (3.3) ((3.4)). For equations with uniformly elliptic and even with uniformly nondegenerate operators the existence of admissible solutions can be proved under very general conditions (see for inst. [5, 1]). Not in such generality but it also can be done for equations with nonuniformly nondegenerate operators [1, 14, 17]. In particular it is hard to deal with when the coefficients depend on u_x and in this connection we can only mention some cases of curvature equations [10].

One can see that admissible solutions are certainly viscosity and due to the nice limit passage in inequalities the existence of viscosity solution can be obtained through regularization of Proposition 8 for the equations of the same generality as the relevant uniformly nondegenerate equations [16]. But of course it arises the problems of smoothness and uniqueness of viscosity solutions.

This procedure can be realized also for degenerate equations and as a matter of fact in such approach we approximate the viscosity solution by supersolutions. On the other hand the equation can be regularized in a way by replacement f to $f^{\varepsilon} = f + \varepsilon > \underline{f}$ in (3.3). If so, the viscosity solution is approximated by subsolutions [8, 9].

Eventually the existence of viscosity solution can be obtained directly by Perron process [6].

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