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# A POSTERIORI ERROR ESTIMATES FOR APPROXIMATE SOLUTIONS OF VARIATIONAL PROBLEMS WITH POWER GROWTN FUNCTIONALS

**Dedicated to the jubilee of dear**
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**Abstract.** This paper is concerned with deriving upper estimates of the difference $\|v-u\|$ where $u$ is a minimizer of a variational problem and $v$ is some element of the corresponding functional space. By using methods of the duality theory we derive a majorant functional which explicitly depends only on $v$ and on the data of the problem. The advantage of this majorant is that it does not contain unknown constants and can be directly computed by simple numerical methods.

## §1. Introduction

In this paper we consider methods of a posteriori error estimation for conforming approximations to the following variational problem which is hereafter called **Primal problem**. Find $u \in V_0 + u_0$ such that

$$I_\alpha(u) = \inf_{w \in V_0+u_0} I_\alpha(w), \quad I_\alpha(w) = \int\limits_\Omega \left( \frac{1}{\alpha} \mid \nabla w \mid^\alpha - fw \right) dx. \qquad (1)$$

Here $\Omega$ is an open, bounded domain in $\mathbb{R}^n$ with Lipschitz continuous boundary, $\alpha \geq 2$, $f \in \mathrm{L}_2(\Omega)$, $u_0 \in V := \mathrm{W}^1_\alpha(\Omega)$ and

$$V_0 := \{ v \in V \ \ \| \ v = 0 \text{ on } \partial\Omega \},$$

where the symbol := means "equal by definition" and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Known results of the calculus of variations guarantee existence of the unique solution (minimizer) $u \in \mathrm{W}^1_\alpha(\Omega)$ to problem (1) (see [1, 2]). If $\Omega$, $f$ and $u_0$ are regular enough then $u$ is a classical solution of the boundary-value problem

$$\operatorname{div}\left( \mid \nabla u \mid^{\alpha-2} \nabla u \right) + f = 0, \quad u = u_0 \text{ on } \partial\Omega. \qquad (2)$$

Let $v$ be an element of the admissible set $V_0 + u_0$. Our aim is to find a functional $M_\alpha : V_0 + u_0 \rightarrow \mathbb{R}$ with the following properties:

$$\begin{array}{ll} \text{(i)} & \|\nabla(v-u)\|_{\alpha,\Omega} \leq M_\alpha(v)\ \forall v \in V_0 + u_0\,, \\ \text{(ii)} & M_\alpha(v) = 0 \iff v = u, \\ \text{(iii)} & M_\alpha(v_k) \rightarrow 0 \text{ if } \|\nabla(v_k - u)\|_{\alpha,\Omega} \xrightarrow[k\rightarrow+\infty]{} 0, \end{array} \tag{3}$$

where $\|.\|_{\alpha,\Omega}$ denotes the norm in the space $L_\alpha(\Omega)$. It is clear that the functional $M_\alpha$ must depends also on $\Omega$, $f$ and $u_0$. For the sake of simplicity we do not include them as arguments explicitly, but always assume that this dependence takes place. The functional $M_\alpha$ with the above properties we name *a posteriori error majorant.*

The interest for getting such type functionals was stimulated by the necessity of measuring the accuracy of approximate solutions to boundary-value problems. For linear problems a functional $M$ satisfying $3-(i)$ can be easily obtained. For example, if $\alpha = 2$, then (2) is the Dirichlet problem $\Delta u + f = 0$, $u = u_0$, on $\partial\Omega$ and by using the first (energy) inequality (see [1]) we derive the estimate

$$\|\nabla(u-v)\|_{2,\Omega} \leqslant C_\Omega \|\Delta v + f\|_{2,\Omega}, \tag{4}$$

where $C_\Omega$ is the constant in Friedrichs–Poincare inequality and $v$ is an approximation. The above estimate is valid only for $v \in (V_0 + u_0) \cap W_2^2(\Omega)$ and cannot be used for all $v \in V$. However, contemporary numerical methods (e.g., finite element method) usually produce sequences of approximate solutions $\{u_n\} \in V$ which converge to the exact solution only in the energy spece $V$. Therefore, error majorants based on the "strong" norm of the residual $\Delta v + f$ (as in (4)) does not satisfy (3)-iii and, thus, are usually inapplicable. By this reason in the so-called *residual method* (see e.g., [4–7]) the error is estimated via a "weak" norm of the residual. In the capacity of the latter the norm in the Sobolev's space $H^{-1}$ is oftenly used. The difficulties in using these estimates are stipulated by the presence of the unknown constants which, in general, should be found numerically. Besides, they are applicable for the Galerkin approximations only and, thus, require exact solutions of the corresponding finite dimensional problems.

In [10] we have presented a different approach for deriving a posteriori error estimates based on duality theory in convex analysis. This approach can be applied to various boundary value problems for linear and nonlinear differential equations. Moreover, in [10] it was proved that main presently known groups of a posteriori error estimates (which follow from the residual, gradient averaging and equilibrated data methods) can be obtained as particular forms of the introduced *duality error estimates*. An essential advantage of the proposed duality error estimates consists in their applicability to *any* conforming approximations of the exact solutions what opens a way for measuring errors of approximate solutions obtained by various post-processing methods.

In this paper we use duality theory for deriving a posteriori error estimates for variational problems with power growth functionals (1). The first form of the corresponding error majorant $M^1_\alpha$ follows from the inequality (8) and is given by (14). In §2 we prove that the functional $M^1_\alpha$ satisfies the required conditions (3)(i)–(iii). However, computation of this functional is related to an auxiliary minimization problem which can be rather complicated. Therefore, in §3 we get another error majorant named $M^2_\alpha$. The latter is much more convenient from the computational viewpoint because, in addition to (3)(i)–(iii), it possesses the following properties: (a) the computation of $M^2_\alpha(v)$ for any $v \in V_0 + u_0$ is inexpensive and can be implemented by simple numerical methods, (b) the majorant $M^2_\alpha(v)$ provides an upper bound of the error even if the corresponding auxiliary problem is solved approximately.

## §2. Duality a posteriori error estimate

The basic principle of the duality theory is that the *primal* problem (1) is considered together with a new

**Dual Problem.** Find $p^* \in Y^*$ such that

$$I_{\alpha^*}(p^*) = \sup_{y^* \in Y^*} I_{\alpha^*}(y^*) \tag{5}$$

where $\alpha^* = \frac{\alpha}{\alpha - 1}$,

$$I_{\alpha^*}(y^*) = \begin{cases} \int\limits_\Omega \left(\nabla u_0 \cdot y^* - \dfrac{1}{\alpha^*} \mid y^* \mid^{\alpha^*} - f u_0\right) dx & \text{if } y^* \in Q^*_f \\ +\infty & \text{if } y^* \notin Q^*_f \end{cases},$$

$$Q^*_f = \left\{ y^* \in Y^* \;\; \| \; \int\limits_\Omega \tau^* \cdot \nabla w \, dx = \int\limits_\Omega f w \, dx \; \forall v \in V_0 \right\}$$

and

$$Y^* = \mathrm{L}_{\alpha^*}(\Omega, \mathbb{R}^n) :=$$
$$:= \Big\{ y = (y_1, y_2, \ldots, y_n) \Big| \, \|y\|_{\alpha^*} := \Big( \int\limits_\Omega \mid y \mid^{\alpha^*} \, dx \Big)^{1/\alpha^*} \Big) \leqslant +\infty \Big\}.$$

**Theorem 1.** *There exist unique solutions $u$ and $p^*$ to the problems* (1) *and* (4)*, respectively. They satisfy the following relations*

$$I_\alpha(u) = I_{\alpha^*}(p^*), \tag{6}$$

$$\nabla u = \mid p^* \mid^{\alpha^*-2} p^*, \quad p^* = \mid \nabla u \mid^{\alpha-2} \nabla u. \tag{7}$$

The proof of this theorem follows from the known results of the duality theory in convex analysis (see, e.g., [3]).

**Theorem 2.** *For any $v \in V_0 + u_0$ the following estimate holds:*

$$\|\nabla(v-u)\|^\alpha_{\alpha,\Omega} \le C_\alpha \left( I_\alpha(v) - I_{\alpha^*}(q^*) \right) \; \forall q^* \in Q^*_f, \tag{8}$$

*where $C_\alpha = \alpha 2^{\alpha-1}$ and $\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1$.*

**Proof.** We use the inequality

$$\int\limits_\Omega \left( \left| \frac{w_1 + w_2}{2} \right|^\alpha + \left| \frac{w_1 - w_2}{2} \right|^\alpha \right) dx \le$$
$$\le \frac{1}{2} \int\limits_\Omega \mid w_1 \mid^\alpha \, dx + \frac{1}{2} \int\limits_\Omega \mid w_2 \mid^\alpha \, dx$$

which is valid for all $w_1$, $w_2 \in \mathrm{L}_\alpha(\Omega)$ (see [9, 11]). By setting $w_1 = \nabla u$ and $w_2 = \nabla v$, we obtain

$$\begin{aligned}
\int\limits_\Omega \mid \nabla(v-u) \mid^\alpha \, dx \le 2^{\alpha-1} \Big[ \int\limits_\Omega \mid \nabla v \mid^\alpha \, dx + \int\limits_\Omega \mid \nabla u \mid^\alpha \, dx - \\
- 2 \int\limits_\Omega \mid \nabla \left( \frac{u+v}{2} \right) \mid^\alpha \, dx \Big] \le \\
\le \alpha 2^{\alpha-1} \Big[ I_\alpha(v) + I_\alpha(u) - 2 I_\alpha \left( \frac{u+v}{2} \right) \Big] \le \\
\le \alpha 2^{\alpha-1} \Big[ I_\alpha(v) - I_\alpha(u) \Big].
\end{aligned} \tag{9}$$

The relation (6) yields the inequality

$$I_\alpha(u) \geq I_{\alpha^*}(q^*) \;\; \forall\, q^* \in Q^*_f,$$

which together with (9) results in the desired (7). $\square$

Now we can use (8) in order to derive the first form of the majorant $M_\alpha$. Let $\tau^*$ be an arbitrary element of the space $Y^*$. Then the difference in the right-hand side of (8) can be represented as follows

$$I_\alpha(v) - I_{\alpha^*}(q^*) =$$
$$= I_\alpha(v) - I_{\alpha^*}(\tau^*) + \int\limits_\Omega \left\{ \nabla u_0 \cdot (\tau^* - q^*) + \frac{1}{\alpha^*}\Big( \mid q^* \mid^{\alpha^*} - \mid \tau^* \mid^{\alpha^*} \Big) \right\} dx.$$

Since the function $\xi \mapsto \mid \xi \mid^{\alpha^*}$ is convex for $\alpha^* \geq 1$ and

$$I_\alpha(v) - I_{\alpha^*}(\tau^*) =$$
$$= \int\limits_\Omega \left\{ \frac{1}{\alpha} \mid \nabla v \mid^\alpha + \frac{1}{\alpha^*} \mid \tau^* \mid^{\alpha^*} - \nabla v \cdot \tau^* + \nabla (v - u_0) \cdot (\tau^* - q^*) \right\} dx$$

we get the estimate

$$I_\alpha(v) - I_{\alpha^*}(q^*) \leq$$
$$\leq D(v, \tau^*) + \int\limits_\Omega \Big( \nabla v - \mid q^* \mid^{\alpha^* - 2} q^* \Big) \cdot (\tau^* - q^*)\, dx \qquad (10)$$

where

$$D(v, \tau^*) = \int\limits_\Omega \left[ \frac{1}{\alpha} \mid \nabla v \mid^\alpha + \frac{1}{\alpha^*} \mid \tau^* \mid^{\alpha^*} - \nabla v \cdot \tau^* \right] dx \geq 0.$$

We note that $D(v, \tau^*) = 0$ if and only if $v$ and $\tau^*$ satisfy the duality relations (7).

The estimate (10) can be also represented in the form

$$I_\alpha(v) - I_{\alpha^*}(q^*) \leq$$
$$\leq D(v, \tau^*) + \int\limits_\Omega \Big[ \Big( \nabla v - \mid \tau^* \mid^{\alpha^* - 2} \tau^* \Big) \cdot (\tau^* - q^*) +$$
$$+ \Big( \mid \tau^* \mid^{\alpha^* - 2} \tau^* - \mid q^* \mid^{\alpha^* - 2} q^* \Big) \cdot (\tau^* - q^*) \Big]\, dx.$$

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a convex nonnegative function such that $h(0) = 0$ and let $h^*$ be the function conjugate to $h$, i.e.,

$$h^*(y) = \sup_{x \in \mathbb{R}^n} (y \cdot x - h(x)).$$

It is easy to verify that $h^*$ is also nonnegative and equal to zero if and only if $y = 0$. Then, in virtue of Young–Fenchel inequality, we transform the above inequality into the following one

$$I_\alpha(v) - I_{\alpha^*}(q^*) \le J_1(v, \tau^*) + J_2(\tau^*) \tag{11}$$

where

$$J_1(v, \tau^*) = D(v, \tau^*) + \int\limits_\Omega h\Big(\nabla v - \mid \tau^* \mid^{\alpha^*-2} \tau^*\Big)\, dx,$$

and

$$J_2(\tau^*) = \inf_{q^* \in Q^*_f} \int\limits_\Omega h^*(\tau^* - q^*) + \\ + \Big( \mid \tau^* \mid^{\alpha^*-2} \tau^* - \mid q^* \mid^{\alpha^*-2} q^* \Big) \cdot (\tau^* - q^*)\, dx.$$

In particular, one can take $h(x) = \frac{\beta}{\alpha} \mid x \mid^\alpha$ with some $\beta > 0$ and $h^*(y) = \beta^{1-\alpha^*} \frac{1}{\alpha^*} \mid y \mid^{\alpha^*}$. Then the functionals $J_1$ and $J_2$ above have the form

$$J_1^\beta(v, \tau^*) = D(v, \tau^*) + \frac{\beta}{\alpha} \Big\| \nabla v - \mid \tau^* \mid^{\alpha^*-2} \tau^* \Big\|^\alpha_{\alpha,\Omega}, \tag{12$a$}$$

and

$$J_2^\beta(\tau^*) = \inf_{q^* \in Q^*_f} \Big[ \frac{\beta^{1-\alpha^*}}{\alpha^*} \|\tau^* - q^*\|^{\alpha^*}_{\alpha^*,\Omega} + \\ + \int\limits_\Omega \Big( \mid \tau^* \mid^{\alpha^*-2} \tau^* - \mid q^* \mid^{\alpha^*-2} q^* \Big) \cdot (\tau^* - q^*)\, dx \Big]. \tag{12b}$$

The superscript $\beta$ in (12a,b) emphasizes that the functionals also depend on this value. From (7) and (11) we deduce

$$\|\nabla(v - u)\|^\alpha_{\alpha,\Omega} \le C_\alpha \left( J_1^\beta(v, \tau^*) + J_2^\beta(\tau^*) \right) \quad \forall \tau^* \in Y^*. \tag{13}$$

**Remark 1.** The right-hand side of (13) depends on $v$ and $\tau$ which can be considered as approximate solutions to the primal and dual problems, receptively. It is a sum of two parts which represent errors in the duality relations ($J_1^\beta$) and in the differential equations for the dual variable ($J_2^\beta$).

In order to obtain an error majorant depending only on the function $v$ we can use the duality relations (7) and set $\tau^* = \mid \nabla v \mid^{\alpha-2} \nabla v$. Then we arrive at the following statement.

**Theorem 3.** *The any $\beta > 0$ the functional*

$$M_\alpha^1(v) = \left[C_\alpha\left(J_1^\beta(v, \mid \nabla v \mid^{\alpha-2} \nabla v) + J_2^\beta(\mid \nabla v \mid^{\alpha-2} \nabla v)\right)\right]^{\frac{1}{\alpha}} \tag{14}$$

*satisfies* (3) (i)–(iii).

**Proof.** The inequality (3)(i) is a consequence of (13).

Let us suppose that $v = u$. Then $\mid \nabla u \mid^{\alpha-2} \nabla u = p^* \in Q_f^*$ and, hence, both terms $J_1^\beta$ and $J_2^\beta$ are equal to zero, so that $M_\alpha^1(v, \beta) = 0$. If it is known that $M_\alpha^1(v, \beta) = 0$ then $\tau^* = \mid \nabla v \mid^{\alpha-2} \nabla v$ satisfies the equation $\mathrm{div}\tau^* + f = 0$ what means that $v$ is a solution to the problem (1). Thus, (3)(ii) is proved.

In order to show the validity of (3)(iii) we consider a sequence $\{v_k\} \in V_0 + u_0$ such that $v_k \to u$ in $V$ as $k \to +\infty$. In this case $\tau_k^* = \mid \nabla v_k \mid^{\alpha-2} \nabla v_k \to p^*$ in $Y^*$ and, consequently, $J_1^\beta(v_k, \tau_k^*) \to 0$. Since

$$J_2^\beta(\tau_k^*) \leq \frac{\beta^{1-\alpha^*}}{\alpha^*} \|\tau_k^* - p^*\|_{\alpha^*,\Omega}^{\alpha^*} + \|\nabla(v_k - u)\|_{\alpha,\Omega} \|\tau_k^* - p^*\|_{\alpha^*,\Omega} \tag{15}$$

we see that the second term also tends to zero. Hence, $M_\alpha^1(v_k) \to 0$ as $k \to +\infty$. □

**Corollary 1.** *From the above theorem it follows that the functional*

$$M_\alpha^1(v) := \inf_{\beta > 0} M_\alpha^1(v, \beta)$$

*provide the error estimate which is in a sense optimal.*

## §3. Another form of the duality error estimate

The value of $J_1^\beta(v, \tau^*)$ in (13) can be easily computed for any $v$ and $\tau^*$. Finding the value of $J_2^\beta(\tau^*)$ is related to a minimization of a convex functional over the set $Q_l^*$ (see (12b)). Since this set consists of functions $q^*$ that *exactly* satisfy the equation $\mathrm{div}\, q^* + f = 0$ this problem is, in fact, as complicated as the dual problem $\mathcal{P}^*$. By this reason the estimate (13) has mainly theoretical significance (it decomposes the error into two meaningful independent parts) and is not effective in immediate computations. Below we derive another form of the duality error majorant named

$M_\alpha^2(v)$. The latter necessitates solving much simpler auxiliary problem. Moreover, this majorant provides an upper bound of the error even if the auxiliary problem is solved approximately.

We also prove that $M_\alpha^2(v_k) \to 0$ if $v_k$ tends to the exact solution $u$.

Let $\tau^*$ be some element of the space $Y^*$. We consider the following auxiliary problem:

**Problem $\mathcal{P}_0^*$.** Find $q_\tau^* \in Q_f^*$ such that

$$-\frac{1}{\alpha^*}\int\limits_\Omega \mid \tau^* - q_\tau^* \mid^{\alpha^*} \, dx =$$

$$= \sup_{q^* \in Q_f^*}\left\{-\frac{1}{\alpha^*}\int\limits_\Omega \mid \tau^* - q^* \mid^{\alpha^*} \, dx\right\} := \sup \mathcal{P}_0^*. \qquad (16)$$

It is not difficult to prove that this problem has a solution for any $\tau^* \in Y^*$ and that

$$\sup \mathcal{P}_0^* = \sup_{y^* \in Y^*} \inf_{w \in V_0} \mathcal{L}(w, y^*),$$

where

$$\mathcal{L}(w, y^*) = \int\limits_\Omega \left(-\frac{1}{\alpha^*} \mid \tau^* - y^* \mid^{\alpha^*} + y^* \cdot \nabla w - fw\right) \, dx.$$

The function $w \mapsto \mathcal{L}(w, y^*)$ is convex and continuous in the topology of $V$ for any $y^* \in Y^*$. The function $y^* \mapsto \mathcal{L}(w, y^*)$ is concave and continuous in the topology of $Y^*$ for any $w \in V_0$. Besides, $\mathcal{L}(0, y_k^*) \to -\infty$ if $\|y_k^*\|_{\alpha^*,\Omega} \to +\infty$ as $k \to +\infty$. These properties of $\mathcal{L}$ implies the following equality

$$\sup \mathcal{P}_0 = \inf_{w \in V_0} \sup_{y^* \in Y^*} \mathcal{L}(w, y^*) =$$

$$= \inf_{w \in V_0} \int\limits_\Omega \left(\frac{1}{\alpha} \mid \nabla w \mid^\alpha + \tau^* \cdot \nabla w - fw\right) \, dx. \qquad (17)$$

Since

$$\|\tau^* - q_\tau^*\|_{\alpha^*,\Omega}^{\alpha^*} = \alpha^* \mid \sup \mathcal{P}_0^* \mid \qquad (18)$$

we can use (10) with $q^* = q_\tau^*$ if some upper bound for $\mid \mathcal{P}_0^* \mid$ is known. Therefore, our aim is to obtain easily computable lower bound for the right-hand side of (17).

Let $\eta^* \in Q^* := \{y^* \in Y^* \ \| \ \operatorname{div} y^* \in L_2(\Omega)\}$. Then we have

$$\sup \mathcal{P}_0^* = \inf_{w \in V_0} \int\limits_\Omega \left( \frac{1}{\alpha} \mid \nabla w \mid^\alpha - (\operatorname{div} \eta^* + f) w + (y^* - \eta^*) \cdot \nabla w \right) dx \geq$$

$$\geq \inf_{w \in V_0} \left[ \frac{1}{\alpha} \|\nabla w\|^\alpha_{\alpha,\Omega} - \|\operatorname{div} \eta^* + f\|_{2,\Omega} \|w\|_{2,\Omega} - \|\tau^* - \eta^*\|_{\alpha^*,\Omega} \|\nabla w\|_{\alpha,\Omega} \right].$$

Making use of the Friedrichs inequality

$$\|w\|_{2,\Omega} \leq C_\Omega \|\nabla w\|_{2,\Omega} \ \ \forall w \in V_0$$

we get

$$\begin{aligned} 0 \geq \sup \mathcal{P}_0^* \geq \inf_{w \in V_0} \Big[ \frac{1}{\alpha} \|\nabla w\|^\alpha_{\alpha,\Omega} - \\ - \Big( C_\Omega \mid \Omega \mid^\mu \|\operatorname{div} \eta^* + f\|_{2,\Omega} + \|\tau^* - \eta^*\|_{\alpha^*,\Omega} \Big) \|\nabla w\|_{\alpha,\Omega} \Big] \geq \\ \geq -\frac{1}{\alpha^*} \Big[ C_\Omega \mid \Omega \mid^\mu \|\operatorname{div} \eta^* + f\|_{2,\Omega} + \|\tau^* - \eta^*\|_{\alpha^*,\Omega} \Big]^{\alpha^*}. \end{aligned} \tag{19}$$

Here $\mu = \frac{\alpha - 2}{2\alpha}$ and $\mid \Omega \mid$ denotes the Lebesque measure of the domain $\Omega$. Hence,

$$\|\tau^* - q^*_\tau\|_{\alpha^*,\Omega} \leq F(\tau^*, \eta^*) \quad \forall \eta^* \in Q^* \tag{20}$$

where

$$F(\tau^*, \eta^*) := C_\Omega \mid \Omega \mid^\mu \|\operatorname{div} \eta^* + f\|_{2,\Omega} + \|\tau^* - \eta^*\|_{\alpha^*,\Omega}.$$

From (7) and (10) it follows that

$$\begin{aligned} &\|\nabla(v - u)\|^\alpha_{\alpha,\Omega} \leq \\ &\leq C_\alpha \Big[ D(v, \tau^*) + \Big( \|\nabla v\|_{\alpha,\Omega} + \|q^*\|^{\alpha^* - 1}_{\alpha^*,\Omega} \Big) \|\tau^* - q^*\|_{\alpha^*,\Omega} \Big] \quad \forall q^* \in Q^*_f. \end{aligned}$$

By setting here $q^* = q^*_\tau$ and using (20) we arrive at

**Theorem 4.** *For any $v \in V_0 + u_0$ the estimate*

$$\begin{aligned} \|\nabla(v - u)\|^\alpha_{\alpha,\Omega} \leq C_\alpha \Big[ D(v, \tau^*) + \Big( \|\nabla v\|_{\alpha,\Omega} + \\ + (\|\tau^*\|_{\alpha^*,\Omega} + F(\tau^*, \eta^*))^{\alpha^* - 1} \Big) F(\tau^*, \eta^*) \Big] \end{aligned} \tag{21}$$

*is valid for any $\tau^* \in Y^*$ and any $\eta^* \in Q^*$.*

The inequality (21) gives various forms of the error majorant $M^2_\alpha(v)$.

Let us set in (21) $\tau^* = | \nabla v |^{\alpha-2} \nabla v$. In this case we get

$$\|\nabla(v-u)\|^{\alpha}_{\alpha,\Omega} \leq M^2_{\alpha}(v) \tag{22}$$

where

$$M^2_{\alpha}(v) = C_{\alpha}\Big[\|\nabla v\|_{\alpha,\Omega} + \Big((\|\nabla v\|^{\alpha-1}_{\alpha,\Omega} + \mathfrak{M}_F(v)\Big)^{\frac{1}{\alpha-1}}\Big]\mathfrak{M}_F(v) \tag{23}$$

and

$$\mathfrak{M}_F(v) = \inf_{\eta^* \in Q^*} F(|\nabla v|^{\alpha-2} \nabla v,\, \eta^*). \tag{24}$$

The estimate (22) gives the upper bound of the error as well as the estimate (13). However, there is a considerable difference between them. Finding the value of the right-hand side in (13) is related to the variational problem on the set $Q^*_f$ formed by vector-valued functions *exactly* satisfying the equation $\mathrm{div} q^* + f = 0$. By this reason it is very difficult to use this estimate in practice. On the contrary, estimate (22) requires solving an auxiliary variational problem (24) for functions defined on the set $Q^*_f$ which does not involve any differential equations and, therefore, can be approximated by standard methods (e.g., by piecewise affine finite element approximations).

**Remark 2.** It is easy to see that estimate (22) also holds if in (23) $\mathfrak{M}_F(v)$ is replaced by any $\overline{\mathfrak{M}}(v)$ such that $\overline{\mathfrak{M}}(v) \geq \mathfrak{M}_F(v)$ for any $v \in V_0 + u_0$. For this purpose we can take

$$\overline{\mathfrak{M}}(v) = C_{\Omega} \mid \Omega \mid^{\mu} \sqrt{\mathfrak{M}_G(v)} + \sqrt[\alpha^*]{\mathfrak{M}_G(v)} \tag{25}$$

where

$$\mathfrak{M}_G(v) = \inf_{\eta^* \in Q^*} G(|\nabla v|^{\alpha-2} \nabla v,\, \eta^*) \tag{26}$$

and

$$G(\tau^*, \eta^*) :=$$
$$:= \int\limits_{\Omega} \Big( \mid \mathrm{div}\eta^* + f \mid^2 + \mid \tau^* - \eta^* \mid^{\alpha^*} \Big)\, dx \quad \forall\, \tau^* \in Y^*,\ \eta^* \in Q^*.$$

Thus, we see that in this case the auxiliary problem (26) is a minimization problem for a strictly convex functional with quadratic main part. This problem can be easily solved by standard methods. Moreover, approximate solutions of this problem give an upper bound for $\mathfrak{M}_G(v)$, so that the corresponding value of $M^2_\alpha(v)$ also provides upper bound for the approximation error.

If $\tau^* \in L_2(\Omega, \mathbb{R}^n)$, then

$$F(\tau^*, \eta^*) \leq \mid \Omega \mid^\mu [C_\Omega \|\operatorname{div} \eta^* + f\|_{2,\Omega} + \|\tau^* - \eta^*\|_{2,\Omega}]$$

and finding the value of a majorant $\overline{\mathfrak{M}}$ can be reduced to a *linear problem*. In this case we have instead of (25)–26):

$$\overline{\mathfrak{M}}(v) = \inf_{\eta^* \in Q^* \cap L_2(\Omega, \mathbb{R}^n)} \mid \Omega|^\mu \Big(2 \int\limits_\Omega (C^2_\Omega (\operatorname{div} \eta^* + f)^2 + \mid \tau^8 - \eta^* \mid^2) dx\Big)^{\frac{1}{2}}.$$

Finally, we show that the majorant $M^2_\alpha$ possesses the same properties as $M^1_\alpha$ (cf. Theorem 3).

**Theorem 5.** *The functional $M^2_\alpha$ satisfies the conditions* (3) (i)–(ii).

**Proof.** First of all, we note that the inequality (3)(i) is a consequence of (22). Since the equality $M^2_\alpha(u) = 0$ is evident, it remains to prove the implication

$$M^2_\alpha(v) = 0 \Rightarrow v = u$$

in order to establish (3)(ii).

Suppose that $M^2_\alpha(v) = 0$. This means that $\mathfrak{M}_F(v) = 0$ and, thus,

$$\operatorname{div} \big(\mid \nabla v \mid^{\alpha-2} \nabla v\big) + f = 0.$$

Hence, we arrive at the conclusion that $v = u$.

Let $\{v_k\} \in V_0 + u_0$ be a sequence converging to $u$ in $V$. Then

$$\tau^*(v_k) := \mid \nabla v_k \mid^{\alpha-2} \nabla v_k \to p^* \quad \text{in} \quad Y^*$$

and

$$\mathfrak{M}_F(v_k) \leq F(|\nabla v_k|^{\alpha-2} \nabla v_k, p^*) \leq \|\tau^*(v_k) - p^*\|_{\alpha^*,\Omega} \to 0$$

$$\mathfrak{M}_G(v_k) \leq G(|\nabla v_k|^{\alpha-2} \nabla v_k, p^*) \leq \|\tau^*(v_k) - p^*\|^{\alpha^*}_{\alpha^*,\Omega} \to 0.$$

We see that $\mathfrak{M}_F(v_k)$ and its upper estimate $\overline{\mathfrak{M}}(v_k)$ tend to zero what implies

$$M^2_\alpha(v_k) \to 0.$$

□

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