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NON-HERMITIAN INTEGRABLE MODELS

ABSTRACT. The non-Hermitian integrable models with the q-bosons as the dynamical variables are considered. It is shown that the exponential phase operators are the special limit of the q-bosons. The introduced phase-differences model is studied in the connection with the dynamical roughening systems.

1. Q-bosons and exponential phase operators

Let us consider the q-boson algebra [1] defined by three independent operators B, $B^{\dagger}$, and $N = N^{\dagger}$:

$$[N, B^{+}] = B^{+}\ , [N, B] = -B, \qquad (1)$$

$$[B, B^{+}] = q^{-2N}.$$

The c-number q is taken to be $q = e^{\gamma}$. We take real $\gamma > 0$. The $q-$boson operators (1) can be expressed in terms of the ordinary canonical bosons $b^{\dagger}$, b,

$$[b, b^{\dagger}] = 1, \qquad (2)$$

such that

$$B = (B^{\dagger})^{\dagger} = \sqrt{\frac{[N+1]}{N+1}}b\,, \qquad (3)$$

$$N = b^{\dagger}b,$$

the 'box' is

$$[n] = \frac{1-q^{-2n}}{1-q^{-2}}. \qquad (4)$$

The exponential phase or Susskind–Glogowers operators [2] ϕ, $\phi^{\dagger}$ were introduced as the decomposition of the bose operators

$$b = \sqrt{N+1}\phi,\ \ b^{\dagger} = \phi^{\dagger}\sqrt{N+1}. \qquad (5)$$

and are intensively studied in quantum optics and solid state physics [3]. These operators satisfy the commutation relations:

$$\begin{gathered}[\phi, \phi^\dagger] = \pi, \\ [N, \phi] = -\phi, \ \ [N, \phi^\dagger] = \phi^\dagger, \end{gathered} \tag{6}$$

in which π is the vacuum projector $\pi = \mid 0\rangle\langle 0 \mid$ and N is the number operator. The operators ϕ, $\phi^\dagger$ can be expressed in terms of the Fock number states $\mid n\rangle$, as:

$$\phi = \sum_{n=0}^{\infty} \mid n\rangle\langle n+1 \mid, \ \phi^\dagger = \sum_{n=0}^{\infty} \mid n+1\rangle\langle n \mid . \tag{7}$$

In fact, the operator ϕ is "one-sided unitary" or an isometric, although

$$\phi\phi^\dagger = 1,$$

one has

$$\phi^\dagger\phi = 1 - \mid 0\rangle\langle 0 \mid .$$

From the representations (3), (5) it follows that the q-boson operators (1) are connected with the exponential phase operators:

$$B = \sqrt{[N+1]}\phi, \ \ B^\dagger = \phi^\dagger\sqrt{[N+1]}.$$

In the limit $\gamma \to \infty$:

$$B \to \phi, \ B^\dagger \to \phi^\dagger. \tag{8}$$

The algebra (1) transfers into the algebra (6), since $q^{-2N} \to \pi$ when $\gamma \to \infty$. When $\gamma \to 0$ the q-bosons become ordinary bosons. It means that the q-bosons provide the smooth deformation from the ordinary bosons (2) to the phase operators (6).

The q-coherent states $\mid z\rangle_q$ are defined as the eigenvectors of the annihilation operator B [4]:

$$B \mid z\rangle_q = z \mid z\rangle_q. \tag{9}$$

The normalized q-coherent states are equal to

$$\begin{gathered}\mid z\rangle_q = E_q^{-\frac{1}{2}}(\mid z \mid^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}} \mid n\rangle, \\ \mid z \mid < (1-q^{-2})^{-\frac{1}{2}}, \ z \in \mathcal{C}, \end{gathered} \tag{10}$$

where the q-exponent $E_q(z)$ is defined by the equality

$$E_q(z) = \sum_{n=0}^{\infty} \frac{z}{[n]!} \tag{11}$$

and $[n]! \equiv \prod_{n=0}^{\infty}[n]$. When $q \to 1$ the states (11) transfer into the standard coherent states of the bose oscillator

$$\begin{aligned} \mid z\rangle_1 &= e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \mid n\rangle, \\ b \mid z\rangle_1 &= z \mid z\rangle_1 \,, \end{aligned} \tag{12}$$

while in the limit $q \to 0$ they are equal to:

$$\begin{aligned} \mid z\rangle_0 &= \sqrt{1- \mid z \mid^2} \sum_{n=0}^{\infty} z^n \mid n\rangle, \\ \phi \mid z\rangle_0 &= z \mid z\rangle_0 \,. \end{aligned} \tag{13}$$

These eigenstates of the exponential phase operator (13) were discussed in [5, 6]. The action of the projector operator on these states is equal to:

$$\pi \mid z\rangle_0 = \sqrt{1- \mid z \mid^2} \mid 0\rangle. \tag{14}$$

Let us consider now the expectation value of operators in coherent states (12). It is easy to obtain the following expressions

$$\begin{aligned} {}_1\langle z \mid B \mid z\rangle_1 &= \Psi_q(\rho) e^{i\varphi}, \\ {}_1\langle z \mid B^\dagger \mid z\rangle_1 &= \Psi_q(\rho) e^{-i\varphi}. \end{aligned} \tag{15}$$

Here we have used the representation $z = \sqrt{\rho} e^{i\varphi}$ for the complex parameter and

$$\Psi_q(\rho) = e^{-\rho} \sqrt{\rho} \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \sqrt{\frac{[n+1]}{n+1}} \tag{16}$$

is the q-deformation of the Carruthers and Nieto function [5]

$$\Psi(\rho) = e^{-\rho} \sqrt{\rho} \sum_{n=0}^{\infty} \frac{\rho^n}{n! \sqrt{n+1}}.$$

It is clear that when $\gamma \to \infty$

$$\Psi_q(\rho) \to \Psi(\rho), \tag{17}$$

and

$$\Psi_q(\rho) \to \sqrt{\rho}, \tag{18}$$

for $\gamma \to 0$. In the classical limit ($\rho \to \infty$):

$$\Psi(\rho) \to 1 + \mathcal{O}\left(\frac{1}{\rho}\right). \tag{19}$$

2. Non-Hermitian integrable models

In the present work we shall discuss the model defined by the Hamiltonian on a chain [7]

$$\begin{aligned} H_\phi &= -\frac{1}{2}\sum_{n=1}^{M}(\phi^\dagger_{n+1}\phi_n + g(1-\pi_n)) \\ &= -\frac{1}{2}\sum_{n=1}^{M}(\phi^\dagger_{n+1}\phi_n + g\phi^\dagger_n\phi_n) \end{aligned} \tag{20}$$

under the periodic boundary conditions $j + M = M$. The exponential phase operators $\phi^\dagger_n$, ϕ_n satisfy the commutation relations

$$\begin{gathered} [\phi_n, \phi^\dagger_m] = \pi_n\delta_{nm}, \\ [N_n, \phi_m] = -\phi_n\delta_{nm}, \;\; [N_n, \phi^\dagger_m] = \phi^\dagger_m\delta_{nm}, \end{gathered} \tag{21}$$

where π_n is the vacuum projector $\pi_n = (\mid 0\rangle_n\langle 0 \mid_n)$ and N_n is the number of particles operator on the n-th lattice site. In (20) g is the interaction constant.

The kinetic term of the Hamiltonian (20) describes the nearest-neighbour hopping from the left to right with equal amplitudes and the site term is proportional to the vacuum projector. When the constant $g = 0$, (20) reduces to the free phase model introduced and discussed in [8–10], and the corresponding free Hamiltonian commutes with its Hermitian conjugate. The introduction of the term proportional to the vacuum projector breaks this symmetry and: $[H_\phi,\, H^\dagger_\phi] \neq 0$.

The non-Hermitian phase-differences model (20) is the $\gamma \to \infty$ limit of the q-boson non-Hermitian integrable model defined by the Hamiltonian:

$$H_{NQB} = -\frac{1}{2}\sum_{n=1}^{M}\left(B^\dagger_{n+1}B_n + \frac{g}{\chi}B^\dagger_n B_n\right), \tag{22}$$

where $B^\dagger_n$, B_n and N_n are independent operators of q-boson algebra (1), and $\chi = 1 - q^{-2}$. The solution of this model will be presented elsewhere.

When $\gamma \to 0$ the Hamiltonian $H_{NQB} + \frac{g}{2\chi}\hat{N}$ transfers into the integrable non-Hermitian boson model

$$H_{NB} = -\frac{1}{2}\left(\sum_{n=1}^{M} b_{n+1}^{\dagger} b_n - \frac{g}{2} N_n^2\right), \tag{23}$$

where $b_n^{\dagger}$, b_n are canonical boson operators and $N_n = b_n^{\dagger} b_n$.

One can calculate the expectation value of the introduced Hamiltonians in the coherent basis

$$\begin{aligned} \mid z\rangle_1 &= \otimes_{n=1}^{M} \mid z_n\rangle_1^n, \\ z_n &= \rho_n e^{i\varphi_n}, \end{aligned}$$

where $\mid z_n\rangle_1^n$ are (12). Applying (15) to the Hamiltonian (22) we have

$$\begin{aligned} &{}_1\langle z \mid H_{NQB} \mid z\rangle_1 = \\ &= -\frac{1}{2}\sum_{n=1}^{M}\left(\Psi_q(\rho_{n+1})\Psi_q(\rho_n)e^{i(\varphi_{n+1}-\varphi_n)} + \frac{g}{\chi^2}(1 - e^{-\chi\rho_n})\right), \end{aligned}$$

where $\Psi_q(\rho)$ is (16). In the phase limit ($\gamma \to \infty$)

$${}_1\langle z \mid H_\phi \mid z\rangle_1 = -\frac{1}{2}\sum_{n=1}^{M}\left(\Psi(\rho_{n+1})\Psi(\rho_n)e^{i(\varphi_{n+1}-\varphi_n)} + g(1 - e^{-\rho_n})\right),$$

which in the classical limit ($\rho \to \infty$), due to (19), transfers into the Hamiltonian of the XY classical magnet chain.

3. Solution of the non-Hermitian phase model

The model (20) is solved for its eigenstates and eigenvalues by the algebraic Bethe ansatz in [7]. The $L-$operator of the model is defined as

$$L_n(u) = \begin{pmatrix} u^{-1} - gu\pi_n & \phi_n^{\dagger} \\ \phi_n & u \end{pmatrix}, \tag{24}$$

where $u \in \mathcal{C}$, and $\phi_n^{\dagger}$, ϕ_n, π_n are (21). This operator-valued matrix satisfies the bilinear relation:

$$R(u,v)L_n(u) \otimes L_n(v) = L_n(v) \otimes L_n(u)R(u,v), \tag{25}$$

in which $R(u,v)$ is the 4×4 R-matrix with the non-zero elements equal to

$$\begin{aligned} R_{11}(u,v) &= R_{44}(u,v) = f(v,u), \\ R_{22}(u,v) &= R_{33}(u,v) = g(v,u), \\ R_{23} &= 1, \end{aligned} \tag{26}$$

with

$$f(v,u)=\frac{u^2}{u^2-v^2},\quad g(v,u)=\frac{uv}{u^2-v^2}. \tag{27}$$

The monodromy matrix is introduced in the usual way as

$$T(u)=L_M(u)\dots L_1(u)=\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u)\end{pmatrix}. \tag{28}$$

The commutation relations of the elements of the monodromy matrix are given by the equation (25) with the same $R-$matrix (26). The transfer matrix, trace of the monodromy, $\tau(u)=TrT(u)=A(u)+D(u)$, satisfies the commutation relation

$$[\tau(u),\tau(v)]=0 \tag{29}$$

for all $u,\ v\in\mathcal{C}$. The transfer matrix plays the role of the generating function of the integrals of motion, the introduced Hamiltonian (20) being one of these.

The monodromy matrix (28) satisfies the relation

$$e^{\zeta\hat{N}}T(u)e^{\frac{1}{2}\zeta\sigma^3}=e^{\frac{1}{2}\zeta\sigma^3}T(u)e^{\zeta\hat{N}}, \tag{30}$$

where

$$\hat{N}=\sum_{n=1}^{M}N_n \tag{31}$$

is the total number operator. It follows from (30) that the transfer matrix commutes with the total number operator:

$$[\tau(u),\hat{N}]=0. \tag{32}$$

From the definitions (24) and (28) one finds by the direct calculation that

$$u^M A(u)=1+u^2\left\{\sum_{n=1}^{M-1}\phi^\dagger_{n+1}\phi_n-g\pi_n\right\}+\ldots+u^{2M}(-g)^M\prod_{n=1}^{M}\pi_n$$

and likewise

$$u^M D(u)=u^2\phi_1^\dagger\phi_M+\ldots+u^{2M}.$$

From (29) and (32) it follows that the operator

$$H_\phi=-\frac{1}{2}\left(gM+\frac{\partial u^M\tau(u)}{\partial u^2}\,|_{u=0}\right)=-\frac{1}{2}\sum_{n=1}^{M}(\phi^\dagger_{n+1}\phi_n+g(1-\pi_n)), \tag{33}$$

which is the Hamiltonian of the introduced model (20), commutes with the transfer matrix and with the total number operator

$$[H_\phi, \tau(u)] = [H_\phi, \hat{N}] = 0. \tag{34}$$

The operator $B(u)$ (28) is interpreted as a creation operator as it follows from the relation (30) that $\hat{N}B(u) = B(u)(\hat{N}+1)$. The N-particle eigenstates of the transfer matrix $\tau(u)$ are taken to be of the form

$$\mid \Phi_N(u_1, u_2, \ldots, u_N\rangle = \prod_{j=1}^{N} B(u_j) \mid \Omega\rangle, \tag{35}$$

with $\mid \Omega\rangle = \prod_{n=1}^{M} \mid 0\rangle_n$, where the state $\mid 0\rangle_n$ is the local Fock vacuum state: $\phi_n \mid 0\rangle_n = 0$. The parameters u_j have to satisfy the Bethe equations:

$$(u_n^{-2} - g)^M = (-1)^{N-1} \prod_{m=1}^{N} \frac{u_n^2}{u_m^2}. \tag{36}$$

The eigenvalues $\Theta_N(v)$ of $\tau(v)$ to which the eigenvalues (35) belong are such that

$$\tau(v) \mid \Phi_N(u_1, u_2, ldots, u_N\rangle = \Theta_N(v) \mid \Phi_N(u_1, u_2, \ldots, u_N\rangle$$

and one finds that

$$\Theta_N(v) = (v^{-1} - gv)^M \prod_{m=1}^{N} \frac{u_m^2}{u_m^2 - v^2} + v^M \prod_{m=1}^{N} \frac{v^2}{v^2 - u_m^2}. \tag{37}$$

By using (33) it is easy to calculate the eigenspectrum of the operator H_ϕ in (20). The N-particle eigenenergies are equal to:

$$E_N = -\frac{1}{2}\left(gM + \frac{\partial u^M \Theta_N(u)}{\partial u^2} \mid_{u=0}\right) = -\frac{1}{2}\sum_{n=1}^{N} u_n^{-2}. \tag{38}$$

The state conjugated to (35) is $\langle \bar{\Phi}_N(u_{1,} u_2, \ldots, u_N \mid = \langle 0 \mid \prod_{n=1}^{N} C(u_n)$ where $C(\lambda)$ is the element of the monodromy matrix (28) with the dual pseudovacuum defined by the equation $\langle 0 \mid \phi^\dagger = 0$. This state is the eigenvector of the transfer matrix $\tau(v)$ with the same eigenvalue $\Theta_N(v)$ (37), and thus of the considered Hamiltonian (20) with the eigenenergy (38).

4. Dynamical roughening models

The Heisenberg chain defined by the Hamiltonian

$$H_{\rm spin} = -\sum_{j=1}^{M}\left(\sigma_j^z\sigma_{j+1}^z - \frac{1}{4}\right) - \sum_{j=1}^{M}\sigma_j^+\sigma_{j+1}^-, \tag{39}$$

is widely used as the dynamical model for interface growth [11–12]. This model is connected with the R-matrix (26). The bilinear relation (25) is satisfied by the L-operator

$$L(n \mid u) = \begin{pmatrix} u\sigma_n^+\sigma_n^- & \sigma_n^- \\ \sigma_n^+ & u - \frac{1}{u}\sigma_n^+\sigma_n^- \end{pmatrix}. \tag{40}$$

The Hamiltonian (39) of the spin model is expressed through the transfer matrix $\tau(u)$ constructed by the L-operators (40):

$$H_{\rm spin} = -\frac{1}{2}\frac{\partial}{\partial u}\ln\tau(u)\mid_{u=1} + \frac{M}{2}. \tag{41}$$

The N-particle eigenenergy of the spin Hamiltonian (39) is equal to:

$$E_N = -\sum_{n=1}^{N}\frac{1}{u_n^2 - 1}, \tag{42}$$

where the parameters u_n satisfy the Bethe equations

$$(1 - u_n^{-2})^M = (-1)^{N-1}\prod_{m=1}^{N}\frac{u_m^2}{u_n^2}. \tag{43}$$

One can think that a spin configuration $\mid \sigma\rangle$ describes a set of particles on a chain: $+$ corresponds to a particle and $-$ to a vacant site. The Hamiltonian (39) creates then the hopping of particles to the left, provided the final state is empty. In the surface growth it is known as the single-step asymmetric exclusion model. If we define height variables through $h_{j+1} - h_j = \frac{1}{2}[\sigma_{j+1} + \sigma_j]$, then this rule describes how material attaches to and evaporates from the surface. It is argued [11] that the system (39) corresponds to a discretized version of noisy Burgers equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial x}(+\text{noise}). \tag{44}$$

If one interprets $u = h'$ as the slope of a one-dimensional surface, then (44) becomes the Kardar–Parisi–Zhang equation [13], which governs the shape fluctuations of various growth models. Thus the discussed spin system belongs to the Kardar–Parisi–Zhang universality class with the dynamic exponent $\frac{3}{2}$. Accordingly the energy gaps of low lying energy levels scale with the system size M as $M^{-3/2}$ as opposed to the M^{-1} scaling at criticality.

Let us discuss the possible connection of the introduced phase-differences model with the dynamical roughening models. The Hamiltonian of the phase model (20) create a persistent current around the ring. The particles hop with the rate equal to one to their right. The exclusion process is governed by the term proportional to $(1-\pi_n)$ The surface profiles are defined by the set of occupation numbers of particles at the sites of a chain. The microscopic state of the system is given by the Fock vector $\mid n_{1,}\dots,n_M\rangle$. The local height of the profile is equal to the occupation number $N_j \mid n_{1,}\dots,n_M\rangle = n_j \mid n_{1,}\dots,n_M\rangle$. The macroscopic state $\mid \Upsilon\rangle = \sum v(n_{1,}\dots,n_M)\mid n_{1,}\dots,n_M\rangle$ is then defined by the probability $v(n_{1,}\dots,n_M)$ of finding the system in the microscopic state $\mid n_{1,}\dots,n_M\rangle$. The evolution is given by a master equation of the form $\frac{\partial}{\partial t}\mid \Upsilon\rangle = -H\mid \Upsilon\rangle$. It is evident that the central issue of the theory is the long distance behavior of the correlation function

$$G(t,j) =< N_j e^{-|t|H} N_1 > -N^2.$$

From this point of view it is interesting to calculate the finite size corrections for the non-Hermitian phase model which are connected with the critical exponents of the theory. In the thermodynamic limit ($M, N\to\infty$ with $M/N = D > 0$) the Bethe equations in the logarithmic form transfer into the linear integral equations. One assumes that the momenta $\{\lambda_n\}$ become densely packed along a smooth curve Γ in the complex plain, which is symmetric with respect to the imaginary axis and has end points Q and $-\bar{Q}$. For the energy of the ground state of the model in the limit when $M,N\to\infty$, and $\frac{N}{M}=D$ is finite, we get the following expression

$$E_M + N(1+g) = Me - \frac{\pi}{12M}v + O(M^{-x})\ ,\ x>1, \tag{45}$$

where e is the ground energy and v is the Fermi velocity. This expansion is typical for the Bethe ansatz solvable models with nontrivial ground state. In this case it is reasonable to study the spectrum for the low-lying excitations [14]. The changes in the energy for these excitations are equal to

$$\mathrm{Re}\,\delta E = \frac{\pi v}{M}\left[N^+ + (Zd)^2 + \left(\frac{\Delta}{2Z}\right)^2\right] + O(M^{-2}), \tag{46}$$

where N^+, d and Δ are some integers specifying the types of the elementary excitations. The function Z describes the reaction of the vacuum particles (the particles which form the ground state) on the excitations and is equal to [7]:

$$Z = \frac{2\pi}{2\pi - (\phi(Q) - \phi(-\bar{Q}))}, \tag{47}$$

where $\phi(\lambda) = i\log(e^{-i\lambda} + g)$. In the limiting cases it is possible to find the explicit expressions for this function. When $g \to 0$

$$Z = (1 + D)\left\{1 - \frac{g}{\pi}\sin\frac{\pi D}{1 + D}\right\},$$

and

$$Z = 1 + \frac{1}{\pi g}\sin \pi D$$

for $g \to \infty$. The answer (47) indicates that the critical exponents of the phase-difference model in the considered case continuously depend on the parameters of the system.

It is worth mentioning that the Z-function for the chain (39)

$$Z = \frac{2\pi}{2\pi + (\phi(Q) - \phi(-\bar{Q}))},$$

where $\phi(\lambda) = i\log(e^{-i\lambda} - 1)$ and Q is the end point of the correspondent integration curve. The excitation spectrum is given by (39) in the antiferromagnetic region.

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