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INDUCED REPRESENTATIONS OF THE ONE-DIMENSIONAL QUANTUM GALILEI GROUP

ABSTRACT. We apply the induced representation technique to study the one dimensional quantum Galilei group. After a brief sketch of the general theory, we develop the representation at an algebraic level. Then we prove the existence of a quasi in invariant measure on the homogeneous space and the corresponding square integrable functions. The unitarity of the induced representations is first studied in the physically meaningful case of real quantum parameter, where the involution is not standard. The imaginary case, where $(*\circ S)^2 = id$ exhibits a behaviour which is analogue to the classical one.

1. THE QUANTUM GALILEI GROUP

The quantum Galilei can be obtained by deforming the algebra $\mathcal{F}$ of the functions on $G(1)$ by means of a non trivial 1-cocycle η with values in $\wedge^2\mathrm{Lie}\,(G(1))$ that defines a Lie–Poisson structure. Its description is as follows. Let $\{\mu, x, t, v\}$ be a set of canonical coordinates of the second kind on $G(1)$; the fundamental fields on the group manifold corresponding to the generators of $\mathrm{Lie}\,(G(1))$ are

$$\chi_M = i\frac{\partial}{\partial\mu}, \quad \chi_P = i\frac{\partial}{\partial x},$$
$$\chi_T = -\frac{\partial}{\partial t}, \quad \chi_B = -x\frac{\partial}{\partial\mu} + it\frac{\partial}{\partial x} + i\frac{\partial}{\partial v}.$$

We have

Proposition 1.1. *Let $g \equiv (\mu, x, t, v) \in G(1)$. Then $\eta : G(1) \to \wedge^2\mathrm{Lie}\,(G(1))$ given by*

$$\eta(g) = (2a\mu - 2axv + atv^2)\chi_M \wedge \chi_P - av^2\chi_M\chi_B - 2av\chi_P \wedge \chi_B$$

satisfies $\eta(gg') = \eta(g) + Ad_g\,\eta(g')$ and therefore is a (non trivial) 1-cocycle that defined the Lie–Poisson brackets

$$\{\mu, x\} = -2a\mu, \quad \{\mu, v\} = av^2, \quad \{x, v\} = 2av, \quad \{t, \cdot\} = 0.$$

Denote by $\mathcal{F}_q := \mathcal{F}(G(1))$ the Hopf algebra generated by $\{\mu, x, t, v\}$, whose relations are

$$[\mu, x] = -2a\mu, \quad [\mu, v] = av^2, \quad [x, v] = 2av,$$

t being a central element. Then $\mathcal{F}_q$ is a deformation of $\mathcal{F}$ in the direction of the Poisson brackets (2.1) and we obtain a Hopf algebra if we define the same comultiplication and antipode as in the classical case:

$$\begin{aligned} \Delta\mu &= \mu \otimes \mathbf{1} + \mathbf{1} \otimes \mu + v \otimes x + (1/2)v^2 \otimes t, & \Delta t &= t \otimes \mathbf{1} + \mathbf{1} \otimes t, \\ \Delta x &= x \otimes \mathbf{1} + \mathbf{1} \otimes x + v \otimes t, & \Delta v &== v \otimes \mathbf{1} + \mathbf{1} \otimes v. \end{aligned}$$

$$S(\mu) = -\mu + vx - (1/2)v^2 t, \quad S(x) = -x + tv, \quad S(t) = -t, \quad S(v) = -v.$$

The quantum enveloping algebra $\mathcal{U}_a := \mathcal{U}_a(G(1))$ is expressed in terms of the generators $\{M, K, K^{-1}, T, B\}$ with relations

$$[K, T] = 0, \quad KBK^{-1} = B + aM, \quad [B, T] = \frac{K - K^{-1}}{2a},$$

where M is central and $KK^{-1} = K^{-1}K = 1$. The Casimir reads

$$C = MT + \frac{1}{2a^2}\,[K + K^{-1} - 2]. \tag{1.1}$$

The coalgebra structure is given by

$$\begin{aligned} \Delta M &= M \otimes K + K^{-1} \otimes M, & \Delta K &= K \otimes K, \\ \Delta T &= T \otimes \mathbf{1} + b1 \otimes T, & \Delta B &= B \otimes K + K^{-1} \otimes B \end{aligned}$$

and

$$S(M) = -M, \quad S(K) = K^{-1}, \quad S(t) = -T, \quad S(B) = -B - aM.$$

There is a nondegenerate duality pairing between $\mathcal{F}_q$ and $\mathcal{U}_a$ given by

$$\langle \mu^\alpha x^\beta t^\gamma v^\alpha, I^{\alpha'} K^l T^{\gamma'} N^{\delta'} \rangle = i^{\alpha+\beta+\gamma+\delta} \alpha!\gamma!\delta!(al)^\beta \delta_{\alpha,\alpha'} \delta_{\gamma,\gamma'} \delta_{\delta,\delta'}$$

where $I = K^{-1}M$, $N = KB$ and $l \in \mathbf{Z}$ while the other indices are in $\mathbf{N}$.

To define the classical limit we write $K = e^{iaP}$ and the Casimir (1.1) for $a \to 0$ reproduced the classical quadratic form $C_{\rm cl} = MT - P^2/2M$.

Let us discuss the involution.

(i) Real parameter case

Let $a \in \mathbf{R}$. In this case there is a natural interpretation of $\mathcal{U}_a(G(1))$ as the symmetry of a free particle on a one dimensional lattice, see [1]. We shall define the following involution $M^* = M$, $K^* = K^{-1}$, $T^* = T$, $B^* = B$. The compatibility with the Hopf structure results then in

$$(XY)^* = Y^* X^*, \quad \Delta(X^*) = (* \otimes *) \circ \tau \circ \Delta X,$$

where $\tau(X \otimes Y) = Y \otimes X$. In the classification of $*$-structures given in [3], this is referred to as to the case (1,1). For this type of involution we have $* \circ S = S \circ *$ and we see that $(* \circ S)^2$ is different from identity. Therefore the use of the duality relations and of the operator $* \circ S$ for determining an involution on $\mathcal{F}_q$ requires some modifications. A direct calculation shows that if $Q : \mathcal{U}_a \to \mathcal{U}_a$ is the morphism defined on the generators by

$$Q(M) = M, \quad Q(K) = K, \quad Q(T) = T, \quad Q(B) = B + aM,$$

then Q is invertible and $Q^2 = (* \circ S)^2$. Therefore $\sigma = Q^{-1} \circ * \circ S$ is an antilinear multiplicative map of $\mathcal{U}_a$ with $\sigma^2 = \mathrm{id}$. Moreover $\Delta \circ \sigma = (\sigma \otimes \sigma) \circ \Delta$.

The involution on $\mathcal{F}_q$, again denoted by $*$, can now be defined as follows.

Proposition. *Let f^* be defined by the relation*

$$\langle f^*, X \rangle = \overline{\langle f, \sigma(X) \rangle}.$$

Then the map $f \mapsto f^*$ is an involution on $\mathcal{F}_q$. The generators $\{\mu, x, t, v\}$ are real under $*$ and we have:

- $(f_1 f_2)^* = f_1^* f_2^*$;
- $\Delta(f^*) = (* \otimes *)\Delta f$.

According to [3] *the involution is therefore of the $(0,0)$-type.*

(ii) Imaginary parameter case

Let $w = ia$ be the imaginary deformation parameter. Then

$$M^* = M, \quad K^* = K, \quad T^* = T, \quad B^* = B,$$

is an involution of type $(1,0)$ according to [3], i.e. such that $(* \circ S)^2 = \mathrm{id}$. Then, on $\mathcal{F}_q$ the involution can be defined in the usual way,

$$\langle X, f^* \rangle = \overline{\langle * \circ S(X), f \rangle},$$

which implies that x, v are real, while

$$\mu^* = \mu - iwv.$$

Contrary to the real case, the involution $*$ is now antimultiplicative also on $\mathcal{F}_q$, i.e. $(fg)^* = g^* f^*$.

2. Induced representations

In this section we are driefly going to review the main facts about induced representation theory in a way that can be easily reproduced in the quantum case.

For the undeformed Galilei group, due to the existence of maximal polarizing subalgebras, any irreducible representation is always induced from a character ω_J of a subgroup J. The representative space H_ω is the space of complex valued functions that are ω_J-covariant along the J-cosets,

$$f(jg) = \omega_J(j) f(g), \quad j \in J,\ g \in G, \tag{2.2}$$

and square integrable with respect to an invariant measure ν on the homogeneous space $J \setminus G$. Therefore H_ω is isometric to $L^2(J \setminus G, \nu)$ and $\rho_\omega = \mathrm{Ind}(J \uparrow G(1), \omega_J)$ reduces to the restriction to H_ω of the regular representation

$$(\rho_\omega(g_2) f)(g_1) = f(g_1 g_2), \quad g_1, g_2 \in G. \tag{2.3}$$

We reformulate the classical scheme in a Hopf algebra framework. Let us examine forst the covariance condition (2.2). Observe that $\mathcal{F}(j)$ is a Hopf algebra and $\pi_J : \mathcal{F}(G) \to \mathcal{F}(J)$, defined by $\pi_J(f) = f\big|_J$, is a Hopf algebra homomorphism. The character ω_J determines a corepresentation $1 \to \omega_J : \mathbf{C} \to \mathbf{C} \otimes \mathcal{F}(J) \simeq \mathcal{F}(J)$. Consider the right action of $j \in J$ on $f \in \mathcal{F}$ given by $(f \cdot j)(g) = f(jg)$ and let $\Lambda : \mathcal{F}(G) \to \mathcal{F}(J) \otimes \mathcal{F}(G)$ be the corresponding left coaction. The actual form of Λ is given by the standard dualization procedure, namely $\Lambda = (\pi_J \otimes \mathrm{id}) \circ \Delta$. The space H_ω of the induced representation will be defined as the subspace of functions in $\mathcal{F}(G)$ that satisfy the equivariance condition (2.2) rewitten as

$$\Lambda(f) = \omega_J \otimes f, \quad f \in \mathcal{F}(G). \tag{2.4}$$

On H_ω the comultiplication Δ determines a corepresentation Ψ_ω.

Provided a quantum subgroup exists, this procedure is naturally extended to the non commutative case, although it can't be considered general theory of induced representations of quantum groups. In fact, also at a completely algebraic level, the richness of the quantum case essentially is contained in the definition of homogeneous space (see [5]), or equivalently of quantum analogue of subgroups. What is commonly accepted is that the notion of quantum subgroup (i.e. quotient by Hopf ideal) is not enough to produce all existing *good* quantum homogeneous spaces. Quantum spheres in [4] are good examples of this feature.

Nevertheless in our quantum Galilei group a quaantum subgroup exists and the previous discussion is enough.

We define the Hopf algebra $\mathcal{F}_q(J)$ generated by the three primitive elements $\{\hat{\mu}, \hat{x}, \hat{t}\}$ with relations $[\hat{\mu}, \hat{x}] = -2a\hat{\mu}$ and $[\hat{t}, \cdot] = 0$. We also assume the involution where these generators are real. The map π_j : $\mathcal{F}_q \longrightarrow \mathcal{F}_q(J)$, defined on the generators by

$$\pi_J(\mu) = \hat{\mu},\ \pi_J(x) = \hat{x},\ \pi_J(t) = \hat{t},\ \pi_J(v) = 0, \tag{2.5}$$

is a surjective Hopf morphism. Being the coalgebra symmetric, π is a $*$ morphism both in the case of real and imaginary parameter. It is easy to verify that $\omega_{m,u} = \exp[-i(m\hat{\mu} + u\hat{t})]$ defines a one-dimensional corepresentation of $\mathcal{F}_q(J)$.

As it could be expected from the beginning, it is impossible to develop the theory in the framework of the polynomial functions $\mathcal{F}_q$. Indeed analytical problems must be considered. Being $G(1)$ not compact, no general definition of square integrable functions exists in the non commutative case, both on the group manifold and on the homogeneous space. The following procedure, far from being general, has useful. First we study the problem in the space of formal series, where we find all the algebraic relations. We then change the representative space treating some infinite series as new generators and considering the polynomials in this new set of generators. We then define the scalar product. In this way we obtain the unitary irreducible representations.

We refer to [2] for the details, here we summarize the results in the following propositions.

Proposition 2.1. *Let $\mathcal{H}_{m,u} = \{\phi_{m,u} f(v)\}$, where $\phi_{m,u} = \exp[-i(m\mu + ut)]$ and f is a formal series in v.*

– *Each element $\phi_{m,u} f(v) \in \mathcal{H}_{m,u}$ is equivariant according to (2.4);*

– *on $\mathcal{H}_{m,u}$ we can define a coaction by*

$$\begin{aligned}\Psi_{m,u}(\phi_{m,u} f(v)) = \exp[-im(\mu \otimes \mathbf{1} + \mathbf{1} \otimes \mu + v \otimes x + (1/2)v^2 \otimes t)] \\ \cdot \exp[-iu(t \otimes \mathbf{1} + \mathbf{1} \otimes t)] f(v \otimes \mathbf{1} + \mathbf{1} \otimes v).\end{aligned}$$

The corresponding action of an element $X \in \mathcal{U}_a$ is given by $\rho_{m,u}(X) = (\mathrm{id} \otimes X)\Psi_{m,u}$. Letting $\rho^{\uparrow}_{m,u}(X) = \phi^{-1}_{m,u}\rho_{m,u}(X)\phi_{m,u}$, with $X \in \mathcal{U}_a$, we have

$$\rho^{\uparrow}_{m,u}(M)f(v) = mf(v), \qquad \rho^{\uparrow}_{m,u}(T)f(v) = \left[\frac{mv^2}{2(1-iamv)} + u\right] f(v),$$
$$\rho^{\uparrow}_{m,u}(K)f(v) = \frac{1}{1-iamv}\, f(v), \;\; \rho^{\uparrow}_{m,u}(B)f(v) = i(1-iamv)\frac{\partial}{\partial v}\, f(v).$$

By direct check we see that the following relations hold

$$(1+iamv)\phi_{m,u}v\phi^{-1}_{m,u} = v$$

and, if $v_k = (\phi_{m,u})^{-k}v(\phi_{m,u})^k$, then

$$iam(k-l)v_k v_l = v_k - v_l.$$

Consider the linear space freely generated over the set of finite works in $\phi_{m,u}$, $\phi^{-1}_{m,u}$, and v with the above relations. Let H_0 be the subspace generated over the works whose total degree in $\phi_{m,u}$ is equal to that in $\phi^{-1}_{m,u}$. If $H_0^{(k)} = v_k\mathbf{C}[v_k]$, i.e. the space of polynomials in v_k with vanishing constant term, then, as a direct consequence of the definitions and of the above relations, we have

$$H_0 = \mathbf{C} \oplus \left(\bigoplus_{k=-\infty}^{\infty} H_0^{(k)}\right).$$

Let $H^{(k)}_{m,u} = \phi_{m,u}H_0^{(k)}$ and $H_{m,u} = \phi_{m,u}H_0$. Since the coproduct of $\phi_{m,u}$ is not polynomial in $\phi_{m,u}$, $\phi^{-1}_{m,u}$, and v no coaction of the quantum Galilei group can be defined on H_0 and $H_{m,u}$. However the action obtained as in ptoposition (2.1) admits a restriction to $H_{m,u}$ and therefore determines a representation of $\mathcal{U}_a$. The representation on $H_{m,u}$ is reducible but not completely reducible. The only irresucible component is the restriction to

$$H^{\mathrm{irr}}_{m,u} = \mathbf{C} \oplus H^{(0)}_{m,u} \oplus H^{(1)}_{m,u},$$

which is described in the following proposition.

Proposition 2.2. *Let $\chi_m = (1+iamv_1) \in \mathbf{C} \oplus H_0^{(1)}$. This element is invertible in H_0 with $\chi_m^{-1} = (1-iamv_0) \in \mathbf{C} \oplus H_0^{(0)}$. Then $\{\phi_{m,u}\chi_m^l\}_{l\in\mathbf{Z}}$*

is a basis for $H^{\rm irr}_{m,u}$ *and the irreducible representations given read*

$$\begin{aligned}\rho_{mu}(K^{\pm1})\phi_{m,u}\chi^l_m &= \phi_{m,u}\chi^{l\pm1}_m, \quad \rho_{mu}(B)\phi_{m,u}\chi^l_m = ml\phi_{m,u}\chi^l_m,\\ \rho_{mu}(T)\phi_{m,u}\chi^l_m &= \phi_{m,u}\chi^l_m\left(\frac{1}{2a^2m}(1-\chi_m-\chi_m^{-1}+u\right),\end{aligned} \tag{2.6}$$

while $\rho_{mu}(M)$ *is the multiplication by* m.

3. The real parameter case

Let $H^{\rm irr}_0 = \mathbf{C}\oplus H^{(0)}_0 \oplus H^{(1)}_0$. Define a linear functional $\nu_a : H^{\rm irr}_0 \to \mathbf{C}$ by

$$\nu_a(1)=1, \quad \nu_a(v^n_0) = \frac{1}{(iam)^n}, \quad \nu_a(v^n_1) = \frac{1}{(-iam)^n},$$

so that $\nu_a(\chi^l_m) = \delta_{l,0}$.

Proposition 3.1. *The following holds:*

– *if* $a, b\in H^{irr}_{m,u}$ *then* $a^*b \in H^{\rm irr}_0$ *and* $\langle a,b\rangle = \nu_a(a^*b)$ *defines a scalar product on* $H^{\rm irr}_{m,u}$*;*

– *the Hilbert space obtained by completing* $H^{\rm irr}_{m,u}$ *with respect to this s-scalar product is isometric to* $L^2([0,2\pi/a), a\,dp/(2\pi))$ *and, Fourier transform, to* $L^2(\mathbf{Z})$*;*

– *the representation given in* (2.6) *is unitary.*

Notice that in this case we have $\phi^*_{m,u} = \phi^{-1}_{m,u}$ and

$$\langle \phi_{m,u}\chi^l_m, \phi_{m,u}\chi^{l'}_m\rangle = \nu_a(\phi^*_{m,u}\chi^{l^*}_m\phi_{m,u}\chi^{l'}_m) = \nu(\chi_m^{-l+l'}) = \delta_{l,l'}.$$

Corollary 3.1. *The meassure* ν_a *is quasi-invariant with respect to the restriction to* $H^{\rm irr}_0$ *of the regular representation* $\rho(X) = ({\rm id}\otimes X)\Delta$ *of* $\mathcal{U}_a$*. Indeed for each* $f\in H^{\rm irr}_0$ *and* $X\in\mathcal{U}_a$ *we have*

$$\nu_a(\rho(X)f) = \nu_a(f\xi(X))$$

where

$$X\mapsto \xi(X) = \sum_{n=0}^{\infty}(iam)^n\langle v^n, X\rangle\chi^n_m : \mathcal{U}_a \to H^{\rm irr}_0.$$

Some remarks are in order to point out some peculier feature.

(i) First of all the measure ν_a is finite on the coordinate v, at difference with the classical case where it diverges, as can be checked in the limit $a\to 0$. Noncommutatively means here regularization.

(ii) Contrary to the classical situation, ν_a is only quasi-invariant and becomes invariant only in the continuum limit. Nevertheless the induced representations are unitary. This feature is directly connected with the non standard properties of the involution. In the next section we will show that if the involution is standard then the induced representation is only equivalent to an unitary one.

(iii) On $H_0^{\rm irr}$ we have a scalar product by $\langle a,b\rangle = \nu_a(a^\dagger b)$ where $a^\dagger = \phi^*_{m,u} a^* \phi_{m,u}$. The Hilbert space obtained is isometric with the space given in (4.6). If we formally write $\chi_m = \sum_{k=0}^{\infty}(1-\chi_m^{-1})$, we note that this series doesn't converge in norm.

4. The imaginary parameter case

We can induce from the same unitary corepresentation ω_{mu} of $\mathcal{F}_q(J)$. All the algebraic relations found in the previous section are still valid, provided we substitute a with $-iw$. Repeating the same construction we arrive to determine the induced representation space as

$$H^{\rm irr}_{m,u} = \langle \phi_{m,u}\chi_m^l\rangle_{l\in\mathbf{Z}},$$

where

$$\chi_m^l = \begin{cases} (1+wmx_{-1})^l & l\geqslant 0 \\ (1-wmx_0)^{-l} & l<0, \end{cases}$$

and $x_0 = v$, $x_{-1} = \phi_{m,u} v \phi^{-1}_{m,u}$. Notice that in this case $\phi^*_{m,u} = (1-mwv)^{-1}\phi^{-1}_{m,u}$; as a consequence x_{-1} and χ^n_m are real. Finally, the induced representation ρ_{mu} is easily obtained as formally the same written in (2.6).

The discussion of unitarity is completely different from the real parameter case. Let's define $\nu_w : H_0^{\rm irr} \to \mathbf{C}$,

$$\nu_w(1) = 1, \quad \nu_w(x_0^n) = \frac{1}{(wm)^n}, \quad \nu_w(x_1^n) = \frac{1}{(-wm)^n},$$

and the corresponding (pseudo) scalar product on $H^{\rm irr}_{m,u}$ by $\langle a,b\rangle = \nu_w(a^* b)$ for $a,b\in H^{\rm irr}_{m,u}$. Then, because $\phi^*_{m,u}\phi_{m,u} = \chi_m$, we have that

$$\langle \phi_{m,u}\chi_m^l, \phi_{m,u}\chi_m^n\rangle = \nu_w(\chi_m^{l+n+1}) = \delta_{l+n+1,0}. \qquad (4.7)$$

The corollary (3.1) is still valid provided we substitute a with $-iw$, so that the measure ν_w can be said also in this case quasi invariant. But it is easy to verify that, analogously to the classical case, the induced representation ρ_{mu} is not unitary. To construct unitary representation we need also the following result about the weight functions ξ.

Corollary 4.1. *Let $\varphi : \mathcal{U}_a \to H_0^{\mathrm{irr}}$ be defined by*

$$\varphi(X) = \sum_{n=0} c_n \chi_m^n \langle X, v^n \rangle,$$

where

$$c_n = (-wm)^n \frac{1}{n!} \frac{d^m}{dx^n} \frac{1}{\sqrt{1+x}}\bigg|_{x=0}.$$

Then

$$\xi(X) = \sum_{(X)} \varphi(X_{(1)}^*)^* \varphi(S(X_{(2)}));$$

$$\varphi(XY) = \sum_{(X)} \rho(X_{(1)}) \varphi(Y) \varphi(X_{(2)}).$$

Let's define $\tilde{\rho}_{mu}(X)a = \sum_{(X)} \rho(X_{(1)}) a \varphi(X_{(2)})$, with $a \in H_{m,u}^{\mathrm{irr}}$. Then $\tilde{\rho}_{mu}$ is different from ρ_{mu} only on B, where we have

$$\tilde{\rho}_{mu}(B)\phi_{m,u}\chi_m^n = iwm\left(n + \frac{1}{2}\right)\phi_{m,u}\chi_m^n.$$

Proposition 4.1. *The map $\tilde{\rho}_{mu}$ is a representation of CU_a on $H_{m,u}^{\mathrm{irr}}$ and for each $X \in CU_a$ we have that*

$$\langle a, \tilde{\rho}_{mu}(X)b \rangle = \langle \tilde{\rho}_{mu}(X^*)a, b \rangle.$$

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