Записки научных
семинаров ПОМИ
Том 251, 1998 г.

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Q-SYMMETRY IN CONDENSED MATTER PHYSICS: EFFECTIVE GENERALIZED HUBBARD MODEL WITH QUANTUM SYMMETRY

Abstract. We transform the Hubbard model with phonons, which has recently been shown to exhibit under particular circumstances a quantum symmetry $\mathrm{su}(2) \oplus [\mathrm{su}(2)]_q$, into an effective electron hamiltonian, by means of a generalized Lang-Firsov transformation. The effective hamiltonian is explicitly proved to mantain the quantum symmetry, which survives even when relaxing one of the constraints on the hamiltonian parameters which must hold in the model with phonons. We investigate under which circumstances some superconducting eigenstates of the effective hamiltonian, which can be built thanks to the quantum symmetry, could possibly become metastable.

1998 PACS

I. Introduction

The Hubbard model [1] provides the simplest description of itinerant interacting electron systems by assuming that the itinerant electrons interact only via an on-site Coulomb repulsion term. At half filling the model has a $\widetilde{\mathrm{su}}(2) \oplus \mathrm{su}(2)$ global symmetry algebra [2]. In [3], Yang constructed eigenstates of the Hubbard hamiltonian characterized by off-diagonal long-range order (ODLRO), labelled in terms of two quantum numbers connected with such symmetry, one of which is related to the existence of superconductivity. These eigenstates, however, are of high energy compared to the ground state. The addition of a phonon field to the Hubbard model, although breaking part of the symmetry, has been shown to restore it as a deformed (*quantum*) $[\mathrm{su}(2)]_q$ symmetry, where the deformation parameter q is related to the strength λ of electron-phonon coupling. Here, by means of the deformed symmetry, we construct explicitly some exact eigenstates of the interacting electron-phonon system, which exhibit ODLRO. Successively, in order to investigate under which circumstances they could possibly become metastable states, we first reduce the hamiltonian, by a standard Lang–Firsov transformation, to an effective electron hamiltonian. The latter is shown to have a more general quantum symmetry than the original hamiltonian, in that the symmetry survives even when relaxing the conditions on the original hamiltonian parameters. Finally, the superconducting eigenstates constructed resorting to the q-symmetry are compared with eigenstates with a lower number of pairs, in order to investigate their possible (meta)stability with respect to breaking of pairs.

II. Hubbard model with phonons and quantum symmetry

The grand-canonical Hubbard hamiltonian has the form $H_{\rm Hub} = H_{el}^{(\rm loc)} + H_{el}^{(\rm hop)}$. The first factor represents the local contributions to the electron energy,

$$H_{el}^{(\rm loc)} = \sum_{\mathbf{j}\in\Lambda} \left(-\mu (n_{\mathbf{j},\uparrow} + n_{\mathbf{j},\downarrow}) + U\, n_{\mathbf{j},\uparrow} n_{\mathbf{j},\downarrow} \right) , \tag{1}$$

the second the non-local kinetic part of the energy,

$$H_{el}^{(\rm hop)} = -t \sum_{<\mathbf{j},\mathbf{k}\rangle} \sum_{\sigma\in\{\Uparrow,\Downarrow\}} \left(c_{\mathbf{j},\sigma}^\dagger c_{\mathbf{k},\sigma} + c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{j},\sigma} \right) . \tag{2}$$

Here $n_{\mathbf{j},\sigma} = c_{\mathbf{j},\sigma}^\dagger c_{\mathbf{j},\sigma}$, $c_{\mathbf{j},\sigma}^\dagger, c_{\mathbf{j},\sigma}$ are fermionic creation and annihilation operators over a d-dimensional lattice Λ of N sites, and $< \mathbf{j}, \mathbf{k}\rangle$ stands for couples of nearest neighbour sites in Λ. μ is the chemical potential, U the on-site Coulomb repulsion energy [1]. t denotes the hopping integral of the band electrons, given (in units such that $\hbar = 1$) by

$$t = \int d\mathbf{r}\, \phi^*(\mathbf{r} - R_{\mathbf{j}}) \left[-\frac{1}{2m} \triangle + V(\mathbf{r}, \{R_{\mathbf{l}}\}) \right] \phi(\mathbf{r} - R_{\mathbf{k}}), \tag{3}$$

where $\phi(\mathbf{r} - R_{\mathbf{j}})$ is the atomic wave function for an electron of mass m with respect to the ion at $R_{\mathbf{j}}$ in Λ, and $V(\mathbf{r}, \{R_{\mathbf{l}}\})$ the single electron potential.

By inclusion of phonons, treated as an ensemble of independent Einstein oscillators [4] with frequency ω, describing the ion vibrations, the Hubbard hamiltonian is changed to

$$H = H_{el}^{(\rm loc)} + H_{ph} + H_{\text{el-ph}}^{(\rm hop)}, \tag{4}$$

where

$$H_{ph} = \sum_{\mathbf{j}} \left(\frac{p_{\mathbf{j}}^2}{2M} + \frac{1}{2} M \omega^2 x_{\mathbf{j}}^2 \right) , \tag{5}$$

M denoting the ion mass, and $x_{\mathbf{j}}$, $p_{\mathbf{j}}$ the local ion displacement and momentum operators respectively ($[x_{\mathbf{j}}, p_{\mathbf{k}}] = i\delta_{\mathbf{j},\mathbf{k}}$) commuting with the Fermi operators.

$$H^{(\text{hop})}_{\text{el-ph}} = -\sum_{<\mathbf{j},\mathbf{k}\rangle} \sum_{\sigma} \left(t_{\mathbf{j},\mathbf{k}}\, c^{\dagger}_{\mathbf{k},\sigma} c_{\mathbf{j},\sigma} + \text{h.c.} \right), \tag{6}$$

with $t_{\mathbf{j},\mathbf{k}}$ given by the obvious generalization of (3) taking into account the fact that the orbitals should now be centered at the displaced ion sites:

$$\begin{aligned} t_{\mathbf{j},\mathbf{k}} &= \int d\mathbf{r} \phi^*(\mathbf{r} - R_{\mathbf{j}} - x_{\mathbf{j}}) \\ \times &\left[-\frac{1}{2m} \triangle + V(\mathbf{r}, \{R_{\mathbf{l}} - x_{\mathbf{l}}\}) \right] \phi(\mathbf{r} - R_{\mathbf{k}} - x_{\mathbf{k}}). \end{aligned} \tag{7}$$

The hopping amplitude $t_{\mathbf{j},\mathbf{k}}$ depends now explicitly on the phonon degrees of freedom, which will appear in both the contributions to the hopping hamiltonian (6), $H^{(\text{hop})}_{\text{el-ph}} = H^{(\text{loc})}_{\text{el-ph}} + H^{(\text{non-loc})}_{\text{el-ph}}$. The former, due to the direct contribution of $V(\mathbf{r}, \{R_{\mathbf{l}}\})$ to (7), consists of purely local terms (of which, for simplicity, we keep only the first [5, 6]),

$$H^{(\text{loc})}_{\text{el-ph}} \approx -\lambda \sum_{\mathbf{j}} n_{\mathbf{j}}\, x_{\mathbf{j}}, \tag{8}$$

with $n_{\mathbf{j}} = n_{\mathbf{j},\uparrow} + n_{\mathbf{j},\downarrow}$, which describes coupling with strength λ of the phonons to the charge density of the band electrons. $\lambda \neq 0$ dynamically breaks the su(2), <u>not</u> the $\widetilde{\text{su}}(2)$ symmetry. $H^{(\text{non-loc})}_{\text{el-ph}}$ contains instead the contribution of the ion displacements to the hopping amplitude $t_{\mathbf{j},\mathbf{k}}$ induced in (7) by the laplacian. A simple change of the variable $\mathbf{r}$ in (7) allows us to rewrite this latter contribution as

$$\begin{aligned} &- \int d\mathbf{r}' \phi^* \left(\mathbf{r}' + \frac{1}{2} (R_{\mathbf{j}} + R_{\mathbf{k}}) - x_{\mathbf{k}} \right) \\ &\times \frac{\hbar^2 \triangle_{\mathbf{r}'}}{2m} \phi \left(\mathbf{r}' + \frac{1}{2} (R_{\mathbf{k}} + R_{\mathbf{j}}) - x_{\mathbf{j}} \right) \\ &= -\mathrm{e}^{-\frac{i}{2\hbar}(R_{\mathbf{j}} - R_{\mathbf{k}}) \cdot (p_{\mathbf{k}} - p_{\mathbf{j}})} \int d\mathbf{r} \phi^* (\mathbf{r} - x_{\mathbf{k}}) \frac{\hbar^2 \triangle_{\mathbf{r}}}{2m} \phi (\mathbf{r} - x_{\mathbf{j}}), \end{aligned} \tag{9}$$

where of course $\phi(x)$ is a wave function, $\phi(x) \equiv < x|\phi >$, namely the representative of a component of the state vector $|\phi >$ in the so called

$\{x\}$-representation, in which $< x|p|\phi >= -i\hbar\frac{\partial}{\partial x}\phi(x)$. The last integral in (9) is then reconducted – upon integrating by parts and recalling the form of $\phi(x)$ out of the ion core – to the exponential form $t\exp\left[\zeta\left(\mathbf{x_j}-\mathbf{x_k}\right)\right]$, in which $\mathbf{j}$ and $\mathbf{k}$ are assumed to be ordered. When this is done,

$$H_{\text{el-ph}}^{(\text{non}-\text{loc})} = -t\sum_{<\mathbf{j},\mathbf{k}>}\sum_{\sigma}\Big(\exp\left\{\zeta\left(x_{\mathbf{j}}-x_{\mathbf{k}}\right)\right\} \times\exp\left\{i\kappa\left(p_{\mathbf{j}}-p_{\mathbf{k}}\right)\right\}c^{\dagger}_{\mathbf{k},\sigma}c_{\mathbf{j},\sigma}+\text{h.c.}\Big). \tag{10}$$

We proved in [7] that a global symmetry of the model described by H exists at the <u>*quantum*</u> level in one dimension in the form of $\widetilde{\text{su}}(2)\oplus[\text{su}(2)]_q$. We propose here a different proof of such q-symmetry, which will be in the next section the basis for the successive study of an effective purely electronic hamiltonian equivalent to (4).

Following a standard procedure [4] we first rotate the hamiltonian H by means of a Lang–Firsov transformation, which formally decouples the electron and phonon parts (see below). The phonon degrees of freedom of the resulting hamiltonian are then eliminated by projecting the rotated hamiltonian onto the phonon vacuum (see next section). In fact such a procedure has been shown to be equivalent [8] to projecting the original hamiltonian over a set of coherent states constructed in the phonon algebra $\mathcal{A}_p$ generated by $\left\{n_{\mathbf{j}}b^{\dagger}_{\mathbf{j}}\,,\,n_{\mathbf{j}}b_{\mathbf{j}}\,,\,b^{\dagger}_{\mathbf{j}}b_{\mathbf{j}}\,,\,n^2_{\mathbf{j}}\right\}$, where $b_{\mathbf{j}}=2^{-\frac{1}{2}}\left(x_{\mathbf{j}}+ip_{\mathbf{j}}\right)$, $\left[b_{\mathbf{i}},b^{\dagger}_{\mathbf{j}}\right]=\delta_{\mathbf{i},\mathbf{j}}$, $n^2_{\mathbf{j}}$ being a central element, and to minimize the energy of the resulting effective hamiltonian with respect to the parameter labelling the coherent states. $\mathcal{A}_p$, to which H_{ph} and $H^{(\text{loc})}_{\text{el-ph}}$ belong, is the central extension by $n^2_{\mathbf{j}}$ of the Weyl-Heisenberg boson algebra.

In practice, in our case the Lang-Firsov transformation is implemented by acting with the unitary operator $\mathcal{U}=\prod_{\mathbf{j}}\exp\left(i\frac{\lambda}{M\omega^2}p_{\mathbf{j}}n_{\mathbf{j}}\right)$ onto H. The rotated hamiltonian $\widetilde{H}$ reads

$$\begin{aligned}\widetilde{H} &= \mathcal{U}H\mathcal{U}^{-1}=\widetilde{H}^{(\text{loc})}_{el}+\widetilde{H}_{ph}\\ &-t\sum_{<\mathbf{j},\mathbf{k}>}\sum_{\sigma}\left(\mathrm{e}^{\zeta\left(x_{\mathbf{j}}-x_{\mathbf{k}}\right)}\mathrm{e}^{i\left(\kappa-\frac{\lambda}{M\omega^2}\right)\left(p_{\mathbf{j}}-p_{\mathbf{k}}\right)}c^{\dagger}_{\mathbf{k},\sigma}c_{\mathbf{j},\sigma}+\text{h.c.}\right)\\ &\times\mathrm{e}^{2\zeta\frac{\lambda}{M\omega^2}\left(n_{\mathbf{k},-\sigma}-n_{\mathbf{j},-\sigma}-1\right)},\end{aligned} \tag{11}$$

where, as expected, the local part of the electron-phonon coupling has formally disappeared, and $\widetilde{H}^{(\mathrm{loc})}_{el}$, $\widetilde{H}_{ph}$ are the same as $H^{(\mathrm{loc})}_{el}$, H_{ph} with the coefficients μ and U changed into $\widetilde{\mu} = \mu + \frac{\lambda^2}{2M\omega^2}$, and $\widetilde{U} = U - \frac{\lambda^2}{M\omega^2}$, respectively. It is interesting to notice that the hopping term in (11) has acquired, thanks to the transformation, an extra purely electronic exponential in the local electron number operator. Remarkably, the exponent changes sign depending on the relative occupation of the two sites between which the electron hops. Such a behaviour could possibly imply that, due to the local electron-phonon coupling, the system prefers a staggered ground-state[9], a well known feature of similar systems.

It is easily checked that hamiltonian (11) is invariant under the algebra $\widetilde{\mathrm{su}}(2)$ generated by

$$S_+ = \sum_{\mathbf{j}} c^\dagger_{\mathbf{j},\uparrow} c_{\mathbf{j},\downarrow}, \quad S_- = S_+^\dagger, \quad S_z = \frac{1}{2} \sum_{\mathbf{j}} (n_{\mathbf{j},\uparrow} - n_{\mathbf{j},\downarrow}). \qquad (12)$$

The symmetry under $[\mathrm{su}(2)]_q$ is first checked by constructing local $[\mathrm{su}(2)_{\mathbf{j}}]_q$, $\mathbf{j} \in \Lambda$, commuting with the $\mathbf{j}$-th part of the local hamiltonian, $H^{(\mathrm{loc})} = H_{ph} + H^{(\mathrm{loc})}_{el}$, followed by its extension to a global $[\mathrm{su}(2)]_q$ symmetry of $\widetilde{H}$ resorting to the coalgebra.

We shall now turn to the one-dimensional case. Dealing first with the local symmetry, we set

$$K_j^{(+)} = \mathrm{e}^{i\pi j} c^\dagger_{j,\uparrow} c^\dagger_{j,\downarrow} \;,\; K_j^{(-)} = K_j^{(+)\,\dagger} \;,\; K_j^{(z)} = \frac{1}{2} \left(n_{j,\uparrow} + n_{j,\downarrow} - 1 \right) \;. \qquad (13)$$

The three operators $K_j^{(\gamma)}$, $\gamma = \pm, z$, generate a local quantum $[\mathrm{su}(2)]_q$ algebra [10]:

$$[K_j^{(z)}, K_j^{(\pm)}] = \pm K_j^{(\pm)},$$

$$[K_j^{(+)}, K_j^{(-)}] = \frac{\sinh\left(2\alpha K_j^{(z)}\right)}{\sinh(\alpha)} \equiv \left[2\alpha K_j^{(z)}\right]_q, \; \alpha \in \mathbb{C}, \qquad (14)$$

with $q = e^{-\alpha}$ arbitrary and $[X]_q \doteq \dfrac{q^X - q^{-X}}{q - q^{-1}}$, as well as a su(2) Lie algebra ($\alpha = 0$).

$\widetilde{H}_j^{(\mathrm{loc})}$ commutes with all $K_\ell^{(\gamma)}$'s, provided $\widetilde{\mu} = \frac{1}{2}\widetilde{U}$. Such condition is similar to that of half-filling for the pure Hubbard model. Otherwise, the local hamiltonian commutes only with $K_\ell^{(z)}$, and the local symmetry reduces to $u(1)$. Noticeably, the local $[\mathrm{su}(2)]_q$ is in any case an ideal of $\widetilde{H}$, and this will allow us the construction of eigenstates of $\widetilde{H}$ at *any* filling.

Extension to the whole lattice of the above local algebras gives rise to a physically non-trivial symmetry of $\widetilde{H}$ only in the quantum algebra case. Indeed, the quantum group structure implies that the correct extension must be performed in terms of the coproduct. For $[\mathrm{su}(2)]_q$ the Cartan element $K^{(z)}$ is primitive, and its coproduct $\Delta(K^{(z)}) = \mathbb{I}\otimes K^{(z)} + K^{(z)}\otimes\mathbb{I}$, whereas

$$\Delta(K^{(+)}) = \mathrm{e}^{\alpha K^{(z)}} \otimes K^{(+)} + K^{(+)} \otimes \mathrm{e}^{-\alpha^* K^{(z)}},$$
$$\Delta(K^{(-)}) = \left[\Delta(K^{(+)})\right]^\dagger . \tag{15}$$

The coalgebra on Λ of the local $[\mathrm{su}(2)]_q$ is then given by the iterated coproduct $\widehat{K}^{(\gamma)} = \Delta^{(N)}(K^{(\gamma)})$, $\gamma = \pm, z$. Explicitly

$$\widehat{K}^{(z)} = \sum_j \mathbb{I}\otimes\cdots\mathbb{I}\otimes K^{(z)}\otimes\mathbb{I}\cdots\otimes\mathbb{I} = \sum_j K_j^{(z)}, \tag{16}$$

$$\widehat{K}^{(+)} = \sum_j \mathrm{e}^{i\pi j}\mathrm{e}^{\alpha K^{(z)}} \otimes\cdots\otimes \mathrm{e}^{\alpha K^{(z)}} \otimes K^{(+)} \otimes \mathrm{e}^{-\alpha^* K^{(z)}} \otimes\cdots\otimes \mathrm{e}^{-\alpha^* K^{(z)}}$$

$$= \sum_j \mathrm{e}^{i\pi j} \prod_{k<j} \mathrm{e}^{\alpha K_k^{(z)}} K_j^{(+)} \prod_{k>j} \mathrm{e}^{-\alpha^* K_k^{(z)}}; \quad \widehat{K}^{(-)} = \left[\widehat{K}^{(+)}\right]^\dagger . \tag{17}$$

The tensor products in (16) [(17)] consist of N factors, the single one of which not equal to the identity $\mathbb{I}$ [or to the exponential of $K^{(z)}$] is at position j.

$[\mathrm{su}(2)]_q$ as defined in (16), (17) commutes with $\widetilde{H}_{el}^{(\mathrm{loc})}$, and obviously with $\widetilde{H}_{ph}$. $\widetilde{H}$ has then $\widetilde{\mathrm{su}}(2)\oplus[\mathrm{su}(2)]_q$ as global symmetry if the remaining genealized hopping term in (11) commutes with $\widehat{K}^{(z)}$ – which is always the case as the latter, due to (16) and (13), is nothing but the total electron number operator – and with $\widehat{K}^{(\pm)}$. Explicit calculation shows that this happens if the physical parameters satisfy a single constraint,

$$\kappa = \frac{\lambda}{M\omega^2}, \tag{18}$$

and the parameter α has non-vanishing real part Re $\alpha = 2\zeta\kappa$. Notice that $\zeta = 0$ implies $\alpha = 0$, thus the quantum symmetry can be ignored only if the effect of ion displacements on the hopping amplitude is neglected. It should be remarked, on the other hand, that setting $\zeta = 0$ would let us recover a pure Hubbard hamiltonian plus uncorrelated phonons. We shall show in the next section that in fact the constraint connecting λ and κ for q-symmetry can be relaxed. ζ instead has merely the role of fixing the quantum group deformation parameter q.

If the condition over the chemical potential is not fulfilled, $[\widetilde{H}, \widehat{K}^{(\pm)}] = (\widetilde{U} - 2\widetilde{\mu})\, \widehat{K}^{(\pm)}$. Starting from any known eigenstate $|E\rangle$ of $\widetilde{H}$ one can then construct a sequence of eigenstates

$$|E_n\rangle = \frac{1}{\sqrt{\mathcal{N}}}\, [\widehat{K}^{(\pm)}]^n |E\rangle\, , \; n \leq N, \quad \mathcal{N} = \frac{[N-n]_q!}{[N]_q!\,[n]_q!},$$

where $[X]_q! = [X]_q\,[X-1]_q!\, ,\, [1]_q! \equiv [1]_q = 1$, with energy eigenvalues $E_n = E + n\left(\widetilde{U} - 2\widetilde{\mu}\right)$. It must be remarked that $|E_n\rangle$ has a lower energy than the original eigenstate $|E\rangle$ whenever the condition $\widetilde{\mu} > \dfrac{1}{2}\widetilde{U}$ is realized. Such a condition can always be implemented by adjusting the chemical potential, *i.e.*, for appropriate values of the filling.

A first example of obvious eigenstate $|E\rangle$ of $\widetilde{H}$ is the vacuum, $|\mathcal{E}_0\rangle \doteq |0\rangle_{ph}|0\rangle_{el}$, for which $\mathcal{E}_0 = \frac{1}{2}\omega$. More generally, it is straightforward that the states

$$|\mathcal{E}_\ell\rangle \doteq |\ell\rangle_{ph}|0\rangle_{el} \quad \ell = 0, 1, \ldots, \tag{19}$$

are eigenstates of $\widetilde{H}$ as well, with eigenvalues $\mathcal{E}_\ell = (\ell + \frac{1}{2})\omega$. Acting on $|\mathcal{E}_\ell\rangle$ with $[\widehat{K}^{(\pm)}]^n$, a whole class of new eigenstates is hence obtained. One may wonder whether any of these states (in particular that generated out of $|\mathcal{E}_0\rangle$) could possibly be the ground state for $\widetilde{H}$, at least for some values of the physical parameters.

In analogy with the pure Hubbard model case, the answer is negative, because it is always possible to construct states which are not eigenstates of $\widetilde{H}$ and have the same energy as $\left[\widehat{K}^{(+)}\right]^n |\mathcal{E}_0\rangle$ (see [7]). However, the question of possible (meta)stability of these states with respect to pair breaking remains open. Such question cannot be investigated at this level following the procedure developed in [3] by Yang, because of the presence of the phonon field, which even at a local level complicates enormously the structure of the dynamical algebra. In the next section we shall discuss a possible way out, in the frame of a purely electronic hamiltonian equivalent to $\widetilde{H}$.

Notice that the states $\left[\widehat{K}^{(+)}\right]^n |\mathcal{E}_0\rangle$, in order to be eigenstates of the original hamiltonian H, have to be rotated back by an inverse Lang–

Firsov transform. When this is done, it is easily realized that they are non-trivial many-phonon states.

III. Effective electron hamiltonian and quantum symmetry

It is now possible, following the scheme developed in [4], to trace away the phonon degrees of freedom by projecting the hamiltonian $\widetilde{H}$ over the phonon vacuum. In this way one obtains an effective purely electronic hamiltonian $\mathcal{H}$, which reads, apart from the constant factors which have been eliminated for the sake of compactness,

$$\mathcal{H} = \widetilde{H}_{el}^{(\mathrm{loc})} - \widetilde{t} \sum_{j,\sigma} \mathrm{e}^{2\zeta \frac{\lambda}{M\omega^2}(n_{\mathbf{k},-\sigma} - n_{\mathbf{j},-\sigma} - 1)} \left(c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma} \right) , \quad (20)$$

with $\widetilde{t} = \mathrm{e}^{-(\zeta + \frac{\lambda}{M\omega^2} - \kappa)^2} t$. The effective hopping amplitude $\widetilde{t}$ has an overall factor which is responsible for the increasing of the effective electron mass (proportional to the inverse of the hopping amplitude) due to the presence of phonons. Such effect is known in the literature (where it was obtained for $\zeta = \kappa = 0$) as polaron formation. It is interesting to notice that here such effect is partially modified by the exponential operator $\mathrm{e}^{2\zeta \frac{\lambda}{M\omega^2}(n_{\mathbf{k},-\sigma} - n_{\mathbf{j},-\sigma} - 1)}$, whose appearence is due to to the inclusion of ion displacement in the hopping amplitude, but remarkably does not depend at all on κ.

The latter feature has the important consequence of removing the constraint among the parameters (18) which guaranteed invariance of $\widetilde{H}$ under $[\mathrm{su}(2)]_q$. Such quantum symmetry, however, survives, as it is easily checked observing that $\mathcal{H}$ has in fact vanishing commutators with all the generators of $[\mathrm{su}(2)]_q$ given in (17), no matter which is the actual value of κ. The only constraint which has to hold in order to mantain the symmetry is the one fixing the chemical potential. However, in full analogy with what we checked at the end of previous section for $\widetilde{H}$, when such filling condition is relaxed, then $\mathcal{H}$ still belongs to an ideal of $[\mathrm{su}(2)]_q$. Thus the states constructed by the ladder operators of the algebra remain eigenstates of $\mathcal{H}$ also in this latter case.

Another remarkable feature of $\mathcal{H}$ is that, for $\lambda = 0$, the q-symmetry transforms into the standard su(2) superconducting symmetry of the Hubbard model, no matter how large ζ and κ are. The same is true for $\zeta = 0$, whatever values λ and κ assume. This is interesting because, as we shall see in the next section, such feature is reflected in the structure of some eigenstates of $\mathcal{H}$, which turn out to significantly differ from those constructed for the standard Hubbard model only when both ζ and λ are simultaneously non-vanishing.

IV. Eigenstates with pair-breaking

New eigenstates of $\mathcal{H}$ can now be constructed by acting with $[K^{(+)}]^n$ on known eigenstates, which now belong to the Hilbert space of fermions alone. In particular, starting from the empty state $|0\rangle$, the states $|\eta\rangle_n \doteq [K^{(+)}]^n|0\rangle$ contain $2n$ electrons, and have energy eigenvalue $(U-2\mu)n$. Even though such states, as already explained, cannot be the ground state, one may wonder whether they are stable with respect to breaking of one of the local pairs.

Eigenstates of $\mathcal{H}$ near $|\eta\rangle_n$ can be obtained by following a procedure close to that developed by Yang in [3]. In fact, it is easily checked that states constructed by acting with the operator

$$\eta_a^\dagger = \sum_j (-)^j c^\dagger_{j+a,\uparrow} c^\dagger_{j,\downarrow}, \quad a \neq 0, \pm 1, \tag{21}$$

onto the electron vacuum, are eigenstates of $\mathcal{H}$, with eigenvalue -2μ. Such states consist of pairs of electrons with correlation length greater than one lattice spacing, as opposed to those created by $K^{(+)}$. Of course, the states $|\eta\rangle_n$ and $[K^{(+)}]^{(n-1)}\eta_a^\dagger|0\rangle$ have now the same number of electrons, but different energy eigenvalues, the latter differing from the former by a term $\widetilde{U}$. Hence whenever $\widetilde{U} < 0$ one may conclude that $|\eta\rangle_n$ is metastable with respect to the above local pair breaking. Nevertheless in order to check metastability one should analyse all possible states which realize local pair breaking. Indeed also $\eta_{\pm 1}|0\rangle$ create a non-local pair of electrons, even though in general they are *not* eigenstates of $\mathcal{H}$.

A more accurate analysis shows that it is possible to construct three eigenstates of $\mathcal{H}$ out of η_0, η_1 and η_{-1}. They read

$$\begin{aligned} |\lambda_0\rangle &= \frac{1}{\sqrt{2}}(\eta_{+1} + \eta_{-1})|0\rangle, \\ |\lambda_\pm\rangle &= \frac{1}{\mathcal{N}_\pm}\left[\frac{\lambda_\pm + 2\mu}{2\sinh\alpha}\eta_0 + (\eta_{+1} - \eta_{-1})\right]|0\rangle, \end{aligned} \tag{22}$$

where

$$\mathcal{N}_\pm = \sqrt{2 + \left(\frac{\lambda_\pm + 2\mu}{2\sinh\alpha}\right)^2},$$

$$\text{and} \quad \lambda_\pm = -2\mu + \frac{1}{2}\left(\widetilde{U} \pm \sqrt{\widetilde{U}^2 + 32(\sinh\alpha)^2}\right)$$

are the energy eigenvalues corresponding to the eigenvectors $|\lambda_\pm\rangle$. It is easily checked that the energy eigenvalue corresponding to $|\lambda_0\rangle$ is $\lambda_0 =$

-2μ, and that $|\lambda_0\rangle$, $|\lambda_\pm\rangle$ form an orthonormal basis for the local space spanned by η_0, η_1 and η_{-1}.

It is remarkable that among the eigenstates given by (22) one ($|\lambda_-\rangle$) has an energy eigenvalue lower than that corresponding to $K_+|0\rangle$, for any $\alpha \neq 0$, and $\widetilde{U} < 0$. This means that when the latter conditions hold the states $|\eta\rangle_n$ are not metastable with respect to the local pair breaking implemented as $[K^{(+)}]^{(n-1)}|\lambda_-\rangle$. This state however still contains n local pairs, of which $n-1$ are weighted by the exponential factors in K_+, and one is a superposition of zero-length and one lattice spacing length couples. It would be interesting - even though beyond the purposes of present paper – to investigate stability of this state with respect to pair breaking.

For $\alpha = 0$ K_+ reduces to η_0, the states $|\lambda_\pm\rangle$ collapse into $\eta_0|0\rangle$, and $\eta_{\pm 1}|0\rangle$ become eigenstates of $\mathcal{H}$ as well, with eigenvalue -2μ. Hence in this case $[\eta_0]^{(n-1)}\eta_a|0\rangle$ are eigenstates of $\mathcal{H}$ of higher energy if compared with $[\eta_0]^n|0\rangle$ for *all* $a \neq 0, \widetilde{U} < 0$, and one recovers precisely the behaviour of the pure Hubbard model. This is obvious, as $\mathcal{H}$ in this case becomes a Hubbard hamiltonian (even though the hamiltonian parameters are scaled with respect to the original ones). Again, such observation could have a relevant physical meaning in that it implies that the simultaneous non vanishing of both ζ and λ, which allows $\alpha \neq 0$, could change the structure of the phase space.

V. Conclusions

In this paper we have analyzed how the $\mathrm{su}(2) \oplus [\mathrm{su}(2)]_q$ symmetry of the Hubbard model with phonons survives when the hamiltonian is reduced – by means of projection onto Lang–Firsov like phonon coherent states – to an effective electron model. It has been shown that in this case the quantum symmetry survives even though there is no longer any constraint on the physical parameters of the resulting hamiltonian. The only condition which must be satisfied is that both the parameters describing the coupling of electrons and phonons are different from zero (otherwise the symmetry reduces to the ordinary $\mathrm{su}(2) \oplus \mathrm{su}(2)$ symmetry of the Hubbard model). The effective electronic hamiltonian $\mathcal{H}$ obtained in this way presents unusual features, first of all an exponential term in the difference of electron number operators on neighbouring sites which multiplies the usual hopping term. Similar kind of extended Hubbard hamiltonians with quantum symmetries has recently been investigated in relation to their possible integrability in one dimension [11, 12]. In that case the deformed symmetry algebra leading to integrability was the deformed superalgebra $[gl(2,1)]_q$, whereas we expect that for the present model a more appropriate supersymmetry algebra would be $[\mathrm{su}(2,2)]_q$, of which $su(2) \oplus [\mathrm{su}(2)]_q$ generates the bosonic part of the superalgebra. A deeper analysis of this point would certainly be of interest.

Finally, we have shown that the eigenstates of $\mathcal{H}$ which can be generated thanks to the quantum symmetry, and consist of local pairs, are metastable for $\widetilde{U} < 0$ with respect to pair breaking generated by η_a, $a > 1$. Again, as opposite to the pure Hubbard case, for any $\alpha \neq 0$ it is possible however to construct other eigenstates of $\mathcal{H}$ which preserve the number of local pairs of electrons and have energy lower than those generated by means of the quantum symmetry. It will be worth investigating how such states could be eventually generalized in order to obtain a true ground state of $\mathcal{H}$ in some region of the phase space.

Work is in progress with respect to the above points.

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Поступило 6 апреля 1998 г.