Записки научных
семинаров ПОМИ
Том 251, 1998 г.

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ASYMPTOTICS OF THE FREDHOLM DETERMINANT ASSOCIATED WITH THE CORRELATION FUNCTIONS OF THE QUANTUM NONLINEAR SCHRÖDINGER EQUATION

ABSTRACT. The correlation functions of the quantum nonlinear Schrödinger equation can be presented in terms of a Fredholm determinant. The explicit expression for this determinant is found for the large time and long distance.

1. INTRODUCTION

In the present paper we derive an asymptotic formula for a Fredholm determinant of a linear integral operator of a special type. This problem arises in connection with calculation of correlation functions of integrable models. The example, we consider below, is related to a correlation function of local fields of the quantum Nonlinear Schrödinger equation out off free fermionic point. Let us remind in brief the basic definitions of this model.

The quantum Nonlinear Schrödinger equation can be described in terms of the canonical Bose fields $\Psi(x,t)$, $\Psi^\dagger(x,t)$, $(x,t\in\mathbf{R})$ obeying the standard commutation relations

$$[\Psi(x,t),\Psi^\dagger(y,t)]=\delta(x-y). \tag{1.1}$$

The Hamiltonian of the model is

$$H=\int dx\,\left(\partial_x\Psi^\dagger(x)\partial_x\Psi(x)+c\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)-h\Psi^\dagger(x)\Psi(x)\right). \tag{1.2}$$

Here $0<c<\infty$ is the coupling constant and h is the chemical potential. The Hamiltonian H acts in the Fock space with the vacuum vector $|0\rangle$, which is characterized by the relation:

$$\Psi(x,t)|0\rangle=0. \tag{1.3}$$

The evolution of the field Ψ with respect to time t is standard

$$\Psi(x,t)=e^{iHt}\Psi(x,0)e^{-iHt}. \tag{1.4}$$

The quantum Nonlinear Schrödinger equation describes a one-dimensional Bose gas with delta-function interactions. The basic thermodynamic equation of the model is the Yang–Yang equation [YY] for the energy of an one-particle excitation $\varepsilon(\lambda)$ in thermal equilibrium

$$\varepsilon(\lambda) = \lambda^2 - h - \frac{T}{2\pi}\int\limits_{-\infty}^{\infty} \frac{2c}{c^2+(\lambda-\mu)^2} \ln\left(1+e^{-\frac{\varepsilon(\mu)}{T}}\right) d\mu. \tag{1.5}$$

It is worth mentioning also the integral equation for the total spectral density of vacancies in the gas $\rho_t(\lambda)$:

$$2\pi\rho_t(\lambda) = 1 + \int\limits_{-\infty}^{\infty} \frac{2c}{c^2+(\lambda-\mu)^2}\vartheta(\mu)\rho_t(\mu)\, d\mu, \tag{1.6}$$

where

$$\vartheta(\lambda) = \left(1+\exp\left[\frac{\varepsilon(\lambda)}{T}\right]\right)^{-1} \tag{1.7}$$

is the Fermi weight. The value $\vartheta(\lambda)\rho_t(\lambda)$ defines the spectral density of particles in the gas.

The time-dependent temperature correlation function of the local fields is defined by

$$\langle\Psi(0,0)\Psi^\dagger(x,t)\rangle_T = \frac{\mathrm{tr}\left(e^{-\frac{H}{T}}\psi(0,0)\psi^\dagger(x,t)\right)}{\mathrm{tr}e^{-\frac{H}{T}}}. \tag{1.8}$$

Here the trace is taken with respect to all states.

The Fredholm determinant representation for the correlation function (1.8) was found in [2]. The classical differential equations and the Riemann–Hilbert problem, describing the Fredholm determinant, were formulated in [3, 4]. It was shown that the value of the determinant can be expressed in terms of solutions of the Riemann–Hilbert problem or the differential equations, which in turn can be solved asymptotically for large time and long distance separation. It is important that one can find not only the leading term of the asymptotics of the Fredholm determinant, but also the complete asymptotic expansion. However, there exists more simple method to calculate at least the leading term of the asymptotics of the Fredholm determinant. We would like to emphasize, that because of several reasons, which will be discussed in the next section, the leading term of the determinant does not describe the leading term of the asymptotics of the correlation function. Therefore, one should know the complete asymptotic expansion. As we have mentioned already, one can do this via the Riemann–Hilbert problem and the differential equations methods. Nevertheless, the direct evaluation of the asymptotics from the Fredholm determinant seems to be useful also. First, the direct evaluation can be considered as an independent confirmation of the results obtained in the framework of the Riemann–Hilbert problem. Second, in this case the calculations become rather simple and evident so it is clear why the Fredholm determinant can be simplified for the large distance and time.

2. The Fredholm determinant

The determinant representation for the correlation function (1.8) is given by the formula

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_T = -\frac{e^{-iht}}{2\pi}(0|\frac{\det\left(\tilde I+\widetilde V\right)}{\det\left(\tilde I-\frac{1}{2\pi}\widetilde K_T\right)}\int_{-\infty}^{\infty}\hat b_{12}(u,v)\,du\,dv|0). \tag{2.1}$$

One can find the detail description of all objects entering the r.h.s. of (2.1) in [2–4]. Here we restrict ourselves with only necessary explanation.

We are interested in the large time and long distance asymptotics of the correlation function. The Fredholm determinant in the denominator of (2.1) does not depend on time t and distance x, therefore it can be considered as some constant. The important objects, which possess nontrivial dependency on x and t, are the factor $\hat b_{12}(u,v)$ and the Fredholm determinant $\det\left(\tilde I+\widetilde V\right)$. Both of them depend also on auxiliary quantum operators-dual fields. Let us describe, first of all, these auxiliary operators.

The dual fields $\psi(\lambda)$, $\phi_D(\lambda)$ and $\phi_A(\lambda)$ (originally the last two fields were denoted as $\phi_{D_1}(\lambda)$ and $\phi_{A_2}(\lambda)$) were introduced in [2] in order to remove two-body scattering and to reduce the model to free fermionic one. These fields act in an auxiliary Fock space having vacuum vector $|0)$ and dual vector $(0|$ (one should not confuse the auxiliary vacuum vector $|0)$ with the vector $|0\rangle$ in (1.3)). Each of these fields can be presented as a sum of creation and annihilation parts

$$\begin{aligned}\phi_A(\lambda)&=q_A(\lambda)+p_D(\lambda),\\ \phi_D(\lambda)&=q_D(\lambda)+p_A(\lambda),\\ \psi(\lambda)&=q_\psi(\lambda)+p_\psi(\lambda).\end{aligned} \tag{2.2}$$

Here $p(\lambda)$ are annihilation parts of dual fields: $p(\lambda)|0) = 0$; $q(\lambda)$ are creation parts of dual fields: $(0|q(\lambda) = 0$.

The only nonzero commutation relations are

$$\begin{aligned}
&[p_A(\lambda), q_\psi(\mu)] = [p_\psi(\lambda), q_A(\mu)] = \ln h(\mu,\lambda),\\
&[p_D(\lambda), q_\psi(\mu)] = [p_\psi(\lambda), q_D(\mu)] = \ln h(\lambda,\mu),\\
&[p_\psi(\lambda), q_\psi(\mu)] = \ln[h(\lambda,\mu)h(\mu,\lambda)],\\
&\qquad\text{where}\quad h(\lambda,\mu) = \frac{\lambda-\mu+ic}{ic}.
\end{aligned} \tag{2.3}$$

Recall that c is the coupling constant in (1.2). It follows immediately from (2.3) that the dual fields belong to an Abelian sub-algebra

$$[\psi(\lambda), \psi(\mu)] = [\psi(\lambda), \phi_a(\mu)] = [\phi_b(\lambda), \phi_a(\mu)] = 0, \tag{2.4}$$

where $a, b = A, D$. The properties (2.3), (2.4) in fact, permit us to treat the dual fields as complex functions, which are holomorphic in some neighborhood of the real axis.

From this point of view one can consider the factor $\hat{b}_{12}(u,v)$ and the determinant $\det\left(\tilde{I}+\widetilde{V}\right)$ in (2.1) as they depend on some functional parameters $\psi(\lambda)$, $\phi_A(\lambda)$ and $\phi_D(\lambda)$. With rather smooth restrictions (like analyticity of these functional parameters in a neighborhood of the real axis) one can find an asymptotic expansion of the factor, as well as the Fredholm determinant, for the large x and t separation. The terms of this asymptotic series functionally depend on dual fields. However, in order to find the asymptotics for the correlation function in the l.h.s. of (2.1) one has to calculate the vacuum expectation value of the expression obtained. The detailed analysis of the Riemann–Hilbert problem and the differential equations shows that some of the terms of the asymptotic series, containing negative powers of t (or x), provide a non-trivial contribution into the leading term of the asymptotics after the calculation of their vacuum expectation value. That is why, in order to describe the asymptotic behavior of the correlation function, one must find not only the leading term of the series, depending on dual fields, but some corrections to this term also.

In the present paper we are not going to calculate the asymptotics of the correlation function. Our main goal is to illustrate the principal possibility of a simplification of the Fredholm determinant $\det\left(\tilde{I}+\widetilde{V}\right)$ in the regime of large time and long distance. Therefore, in particular, we restrict ourselves with the calculation of the leading term and the first important correction of the determinant only. The factor $\hat{b}_{12}$ remains out off framework of our consideration. In order to describe this factor (as well as to find more precise estimates for the determinant) one has to apply more powerful (but not so evident!) methods of the Riemann–Hilbert problem and the differential equations.

The integral operator $\tilde{I}+\widetilde{V}$, which Fredholm determinant enters the r.h.s. of (2.1), acts on the real axis as

$$\left(\tilde{I}+\widetilde{V}\right)\circ f(\mu)=f(\lambda)+\int\limits_{-\infty}^{\infty}\widetilde{V}(\lambda,\mu)f(\mu)d\mu, \tag{2.5}$$

where $f(\lambda)$ is some trial function. Thus, $\tilde{I}$ is the identity operator.

The kernel $\widetilde{V}(\lambda,\mu)$ can be presented in the form

$$\widetilde{V}(\lambda,\mu)=\frac{1}{\lambda-\mu}\int\limits_{-\infty}^{\infty}du(E_+(\lambda|u)E_-(\mu|u)-E_-(\lambda|u)E_+(\mu|u)), \tag{2.6}$$

Here functions $E_\pm$ introduced in [KKS2] are equal to:

$$\begin{aligned}E_+(\lambda|u)=&\frac{1}{2\pi}\frac{Z(u,\lambda)}{Z(u,u)}\left(\frac{e^{-\phi_A(u)}}{u-\lambda+i0}+\frac{e^{-\phi_D(u)}}{u-\lambda-i0}\right)\sqrt{\vartheta(\lambda)}\\&\times e^{\psi(u)+\tau(u)+\frac{1}{2}(\phi_D(\lambda)+\phi_A(\lambda)-\psi(\lambda)-\tau(\lambda))},\end{aligned} \tag{2.7}$$

$$E_-(\lambda|u)=\frac{1}{2\pi}Z(u,\lambda)e^{\frac{1}{2}(\phi_D(\lambda)+\phi_A(\lambda)-\psi(\lambda)-\tau(\lambda))}\sqrt{\vartheta(\lambda)}, \tag{2.8}$$

where the function $Z(\lambda,\mu)$ is defined by

$$Z(\lambda,\mu)=\frac{e^{-\phi_D(\lambda)}}{h(\mu,\lambda)}+\frac{e^{-\phi_A(\lambda)}}{h(\lambda,\mu)}. \tag{2.9}$$

Here $\psi(\lambda)$, $\phi_D(\lambda)$ and $\phi_A(\lambda)$ are just the dual fields (2.2).

Recall also that the function $\vartheta(\lambda)$ is the Fermi weight (1.7). This function defines the dependence of the correlation function on temperature and chemical potential. The function $\tau(\lambda)$ is the only function depending on time and distance:

$$\tau(\lambda)=it\lambda^2-ix\lambda. \tag{2.10}$$

Later on we shall study the asymptotics of the determinant (1.8) in the limit $t \to \infty$, $x \to \infty$, however the ratio $x/2t = \lambda_0$ remains fixed. Therefore, it is convenient to rewrite the function $\tau(\lambda)$ in terms of variables t and λ_0:

$$\tau(\lambda) = it(\lambda - \lambda_0)^2 - it\lambda_0^2. \tag{2.11}$$

Thus, we have described the Fredholm determinant in the r.h.s. of (2.1). The auxiliary quantum operators-dual fields-enter the kernel $\widetilde{V}$. However, due to the property (2.4) the Fredholm determinant is well defined. It is important that the kernel $\widetilde{V}(\lambda, \mu)$ possesses no singularity at $\lambda = \mu$.

3. Properties of the integral kernels

For the investigation of the asymptotics of the Fredholm determinant we use well known representation

$$\det(I + \widetilde{V}) = \exp\left\{\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int \widetilde{V}(\lambda_1, \lambda_2)\widetilde{V}(\lambda_2, \lambda_3) \ldots \widetilde{V}(\lambda_n, \lambda_1)\, d^n\lambda\right\} \tag{3.1}$$

Recall, that we consider the case, when $t \to \infty$, $x \to \infty$ and the ratio $x/2t = \lambda_0$ remains fixed. In this case the original kernel $\widetilde{V}(\lambda, \mu)$ can be simplified. Indeed, for the large t one can take the integral in (2.6) with respect to u (see (A.12)). Up to terms of the order $t^{-1/2+\varepsilon}$, where ε is arbitrary positive, we have

$$\widetilde{V}(\lambda, \mu) = \frac{e^{\frac{1}{2}(\tau(\lambda) - \tau(\mu))} S_+(\lambda, \mu) + e^{\frac{1}{2}(\tau(\mu) - \tau(\lambda))} S_-(\lambda, \mu)}{\lambda - \mu}, \tag{3.2}$$

with

$$\begin{aligned} S_+(\lambda, \mu) = \frac{i}{2\pi} e^{\frac{1}{2}\left(\psi(\lambda) - \psi(\mu) + \phi_D(\lambda) + \phi_A(\lambda) + \phi_D(\mu) + \phi_A(\mu)\right)} Z(\lambda, \mu) \\ \times \left(e^{-\phi_D(\lambda)}\theta(\lambda - \lambda_0) - e^{-\phi_A(\lambda)}\theta(\lambda_0 - \lambda)\right) \sqrt{\vartheta(\lambda)}\sqrt{\vartheta(\mu)}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} S_-(\lambda, \mu) = -\frac{i}{2\pi} e^{\frac{1}{2}\left(\psi(\mu) - \psi(\lambda) + \phi_D(\lambda) + \phi_A(\lambda) + \phi_D(\mu) + \phi_A(\mu)\right)} Z(\mu, \lambda) \\ \times \left(e^{-\phi_D(\mu)}\theta(\mu - \lambda_0) - e^{-\phi_A(\mu)}\theta(\lambda_0 - \mu)\right) \sqrt{\vartheta(\lambda)}\sqrt{\vartheta(\mu)}. \end{aligned} \tag{3.4}$$

Let us consider some properties of integral operators possessing the kernels of type (3.2)

$$V(\lambda,\mu)=\frac{e^{\frac12(\tau(\lambda)-\tau(\mu))}S_+(\lambda,\mu)+e^{\frac12(\tau(\mu)-\tau(\lambda))}S_-(\lambda,\mu)}{\lambda-\mu}. \tag{3.5}$$

Here $S_\pm(\lambda,\mu)$ are some functions (in particular (3.3) and (3.4)), which assumed to be integrable at the real axis. These functions may depend on the ratio $\lambda_0=x/2t$, but not on x and t separately. We also demand that the kernel $V(\lambda,\mu)$ possesses no singularity at the point $\lambda=\mu$, i. e. the functions $S_\pm$ are smooth in the vicinity of $\lambda=\mu$ and

$$S_+(\lambda,\lambda)=-S_-(\lambda,\lambda), \tag{3.6}$$

The main goal of this section is to study the operator product of the kernels (3.5). In what follows we shall drop out all the terms, vanishing in the limit $t\to\infty$.

The following simple lemma will play an important role.

Lemma 3.1. *Let two kernels $V^{(j)}(\lambda,\mu)$, $\quad j=1,2$ of type (3.5) are given*

$$V^{(j)}(\lambda,\mu)=\frac{e^{\frac12(\tau(\lambda)-\tau(\mu))}S_+^{(j)}(\lambda,\mu)+e^{\frac12(\tau(\mu)-\tau(\lambda))}S_-^{(j)}(\lambda,\mu)}{\lambda-\mu}, \qquad j=1,2, \tag{3.7}$$

with

$$S_+^{(j)}(\lambda,\lambda)=-S_-^{(j)}(\lambda,\lambda),\qquad j=1,2. \tag{3.8}$$

Then the kernel $V^{(12)}(\lambda,\mu)$

$$V^{(12)}(\lambda,\mu)=\int\limits_{-\infty}^{\infty}V^{(1)}(\lambda,\nu)V^{(2)}(\nu,\mu)\,d\nu, \tag{3.10}$$

is equal to

$$V^{(12)}(\lambda,\mu)=\frac{e^{\frac12(\tau(\lambda)-\tau(\mu))}S_+^{(12)}(\lambda,\mu)+e^{\frac12(\tau(\mu)-\tau(\lambda))}S_-^{(12)}(\lambda,\mu)}{\lambda-\mu}, \tag{3.10}$$

and

$$S_+^{(12)}(\lambda,\lambda)=-S_-^{(12)}(\lambda,\lambda). \tag{3.11}$$

Proof. Let us notice that due to (3.8) one can replace the denominator $\lambda-\mu$ in (3.7) by, for example, $\lambda-\mu-i0$. Then, applying formulæ (A.12) of Appendix A, we immediately arrive at the representation (3.10) for $V^{(12)}(\lambda,\mu)$ with

$$
\begin{aligned}
S_+^{(12)}(\lambda,\mu) = \int\limits_{-\infty}^{\infty} d\nu S_+^{(1)}(\lambda,\nu)S_+^{(2)}(\nu,\mu)\\
\times\left(\frac{\theta(\nu-\lambda_0)}{\nu-\mu-i0}+\frac{\theta(\lambda_0-\nu)}{\nu-\mu+i0}-\frac{\theta(\nu-\lambda_0)}{\nu-\lambda+i0}-\frac{\theta(\lambda_0-\nu)}{\nu-\lambda-i0}\right),
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
S_-^{(12)}(\lambda,\mu) = \int\limits_{-\infty}^{\infty} d\nu S_-^{(1)}(\lambda,\nu)S_-^{(2)}(\nu,\mu)\\
\times\left(\frac{\theta(\nu-\lambda_0)}{\nu-\mu+i0}+\frac{\theta(\lambda_0-\nu)}{\nu-\mu-i0}-\frac{\theta(\nu-\lambda_0)}{\nu-\lambda-i0}-\frac{\theta(\lambda_0-\nu)}{\nu-\lambda+i0}\right).
\end{aligned} \tag{3.13}
$$

where $\theta(\lambda)$ is the step function. Obviously,

$$S_+^{(12)}(\lambda,\lambda) = 2\pi i \mathrm{sign}(\lambda-\lambda_0)S_+^{(1)}(\lambda,\lambda)S_+^{(2)}(\lambda,\lambda) = -S_-^{(12)}(\lambda,\lambda), \tag{3.14}$$

where the sign function as usual is equal to

$$\mathrm{sign}(\lambda) = \theta(\lambda)-\theta(-\lambda). \tag{3.15}$$

The lemma is proved.

The direct corollary of the lemma is that any power of the kernel (3.5)

$$V^n(\lambda,\mu) = \int V(\lambda,\lambda_1)V(\lambda_1,\lambda_2)\dots V(\lambda_{n-1},\mu)\,d^{n-1}\lambda \tag{3.16}$$

has just the same structure as the kernel $V(\lambda,\mu)$:

$$\Big(V(\lambda,\mu)\Big)^n = \frac{e^{\frac{1}{2}(\tau(\lambda)-\tau(\mu))}S_{+,n}(\lambda,\mu)+e^{\frac{1}{2}(\tau(\mu)-\tau(\lambda))}S_{-,n}(\lambda,\mu)}{\lambda-\mu}, \tag{3.17}$$

where we have set $S_{\pm,1}(\lambda,\mu)\equiv S_\pm(\lambda,\mu)$. Recall that we drop out all the terms of the order $t^{-1/2}$.

A very simple formula for $S_{\pm,n}(\lambda,\lambda)$ follows from (3.14). Indeed, we have

$$S_{\pm,n}(\lambda,\lambda) = 2\pi i \mathrm{sign}(\lambda-\lambda_0)S_{\pm,n-1}(\lambda,\lambda)S_{\pm,1}(\lambda,\lambda), \tag{3.18}$$

and hence,

$$S_{\pm,n}(\lambda,\lambda) = \Big(2\pi i \mathrm{sign}(\lambda-\lambda_0)\Big)^{n-1}\Big(S_\pm(\lambda,\lambda)\Big)^n. \tag{3.19}$$

The traces of powers of V can be described via formula (3.17)

$$
\begin{aligned}
\operatorname{tr}\Big(V(\lambda,\mu)\Big)^n &= \int\limits_{-\infty}^{\infty} \tau'(\lambda) S_{+,n}(\lambda,\lambda)\, d\lambda \\
&- \int\limits_{-\infty}^{\infty} \frac{\partial}{\partial\mu}\Big(S_{+,n}(\lambda,\mu)+S_{-,n}(\lambda,\mu)\Big)\Big|_{\mu=\lambda}\, d\lambda.
\end{aligned} \tag{3.20}
$$

The second term in (3.20) does not depend on t (it depends only on λ_0). The first term is proportional to t and, hence, just this term describes asymptotic of determinant. Due to (3.19) we have

$$
\operatorname{tr}\Big(V(\lambda,\mu)\Big)^n \longrightarrow \int\limits_{-\infty}^{\infty} \tau'(\lambda)\Big(2\pi i \operatorname{sign}(\lambda-\lambda_0)\Big)^{n-1}\Big(S_+(\lambda,\lambda)\Big)^n\, d\lambda. \tag{3.21}
$$

Substituting this estimate into (3.1) we obtain

$$
\begin{aligned}
\ln\det(\tilde I + V) &\longrightarrow \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \tau'(\lambda)\operatorname{sign}(\lambda-\lambda_0) \\
&\times \ln\Big(1+2\pi i \operatorname{sign}(\lambda-\lambda_0) S_+(\lambda,\lambda)\Big)\, d\lambda.
\end{aligned} \tag{3.22}
$$

It is sufficient now to substitute the explicit expression (3.3) into the last formula. Clearly that

$$
\tau'(\lambda) = 2it(\lambda-\lambda_0), \tag{3.23}
$$

$$
S_+(\lambda,\lambda) = \frac{i\vartheta(\lambda)}{2\pi}\operatorname{sign}(\lambda-\lambda_0)\left(1+e^{\big(\phi_A(\lambda)-\phi_D(\lambda)\big)\operatorname{sign}(\lambda-\lambda_0)}\right). \tag{3.24}
$$

Therefore we obtain for the leading term of the asymptotics of the Fredholm determinant (2.1)

$$
\begin{aligned}
\ln\det(\tilde I + \widetilde V) &\longrightarrow \frac{t}{\pi}\int\limits_{-\infty}^{\infty} |\lambda_0-\lambda| \\
&\times \ln\Big\{1-\vartheta(\lambda)\Big(1+e^{\big(\phi_{A_2}(\lambda)-\phi_{D_1}(\lambda)\big)\operatorname{sign}(\lambda-\lambda_0)}\Big)\Big\}\, d\lambda + O(1).
\end{aligned} \tag{3.25}
$$

Thus, we have obtained the leading term for the asymptotics of the Fredholm determinant. It is interesting to consider the free fermionic limit of this result. In this case the coupling constant c goes to infinity, and all commutation relations between annihilation and creation parts of dual fields vanish. Thus, dual fields do not contribute into the vacuum expectation value and, hence, one can put them equal to zero. The equation (1.5) can be solved explicitly, so we find for the Fermi weight

$$\vartheta(\lambda) = \left(e^{\frac{\lambda^2-h}{T}} + 1\right)^{-1}, \qquad c \to \infty. \tag{3.26}$$

Thus, we arrive at

$$\ln\det(\tilde{I} + \widetilde{V}) \stackrel{c\to\infty}{\longrightarrow} \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |x - 2\lambda t| \ln\left(\frac{e^{\frac{\lambda^2-h}{T}} - 1}{e^{\frac{\lambda^2-h}{T}} + 1}\right) d\lambda, \tag{3.27}$$

which exactly coincides with the result obtained in the [5].

4. Improved formula for the asymptotics

The leading term of the asymptotics of the Fredholm determinant is given by the formula (3.25). However, this formula is not complete due to the presence of dual fields and it does not provide the correct result for the asymptotics of the correlation function. Indeed, if we calculate the vacuum expectation of $\det(\tilde{I} + \widetilde{V})$ defined by (3.25), we obtain the result, which is very similar to the free fermionic one:

$$\begin{aligned}(0|\exp\left\{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |x - 2\lambda t| \ln\left\{1 - \vartheta(\lambda)\left(1 + e^{\left(\phi_A(\lambda)-\phi_D(\lambda)\right)\mathrm{sign}(\lambda-\lambda_0)}\right)\right\} d\lambda\right\}|0) \\ = \exp\left\{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |x - 2\lambda t| \ln\{1 - 2\vartheta(\lambda)\} \, d\lambda\right\}.\end{aligned} \tag{4.1}$$

Here we have used the fact that dual fields $\phi_D(\lambda)$ and $\phi_A(\lambda)$ do not contain non-commutative operators. However, it is easy to show that some corrections to the leading term strongly change the situation. Let us find one of these corrections. Namely, it is not difficult to find, how the Fredholm determinant depends on dual field $\psi(\lambda)$. To do it, let us pay attention to the fact that $\psi(\lambda)$ enters the kernel $\widetilde{V}(\lambda,\mu)$ (3.2) only in the combination with $\tau(\lambda)$: $\psi(\lambda) + \tau(\lambda)$. More explicitly

$$S_\pm(\lambda,\mu) = e^{\pm\frac{1}{2}(\psi(\lambda)-\psi(\mu))}\tilde{S}_\pm(\lambda,\mu), \tag{4.2}$$

where $\tilde{S}_\pm(\lambda,\mu)$ do not depend on $\psi(\lambda)$. One can rewrite now formulæ (3.20), (3.21) as

$$
\begin{aligned}
\operatorname{tr}\Big(\widetilde{V}(\lambda,\mu)\Big)^n &= \int_{-\infty}^{\infty} (\tau'(\lambda)+\psi'(\lambda))\tilde{S}_{+,n}(\lambda,\lambda)\,d\lambda \\
&- \int_{-\infty}^{\infty} \frac{\partial}{\partial\mu}\Big(\tilde{S}_{+,n}(\lambda,\mu)+\tilde{S}_{-,n}(\lambda,\mu)\Big)\Big|_{\mu=\lambda}\,d\lambda,
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
&\operatorname{tr}\Big(\widetilde{V}(\lambda,\mu)\Big)^n \longrightarrow C_0^{(n)} \\
&+\int_{-\infty}^{\infty}\Big(\tau'(\lambda)+\psi'(\lambda)\Big)\Big(2\pi i\operatorname{sign}(\lambda-\lambda_0)\Big)^{n-1}\Big(\tilde{S}_{+}(\lambda,\lambda)\Big)^n\,d\lambda,
\end{aligned}
\tag{4.4}
$$

where $C_0^{(n)}$ depend only on dual fields $\phi_D(\lambda)$ and $\phi_A(\lambda)$. Thus, we arrive at the improved formula for the asymptotics of the Fredholm determinant

$$
\begin{aligned}
\det(\tilde{I}+\widetilde{V})\longrightarrow C(\phi_D,\phi_A)\exp\Bigg\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(|x-2\lambda t| - i\operatorname{sign}(\lambda-\lambda_0)\psi'(\lambda)\Big) \\
\times\ln\Big\{1-\vartheta(\lambda)\Big(1+e^{\left(\phi_A(\lambda)-\phi_D(\lambda)\right)\operatorname{sign}(\lambda-\lambda_0)}\Big)\Big\}\,d\lambda\Bigg\}.
\end{aligned}
\tag{4.5}
$$

Now the r.h.s. of (4.5) contains non-commutative operators (in spite of all dual fields still commute with each other), and its vacuum expectation value becomes very nontrivial. The method of the calculation of the vacuum expectation of the expression (4.5) was developed in [6]. It was shown that the result can be expressed in terms of a solution of a nonlinear integral equation, closely related to the Thermodynamic Bethe Ansatz equations.

Acknowledgments

This work was supported by the Russian Foundation of Basic Research under Grant No. 96-01-00344.

A Asymptotics of singular integrals

Consider the asymptotics of the integral

$$I(\lambda,\lambda_0,t,[\varphi]) = \int_{-\infty}^{\infty} \frac{e^{\tau(u)}}{u-\lambda-i0}\varphi(u,\lambda)\,du. \tag{A.1}$$

where

$$\tau(u) = it(u-\lambda_0)^2 - it\lambda_0^2. \tag{A.2}$$

Recall, that $t\to+\infty$, while λ_0 is fixed. The parameter λ is arbitrary real, the function $\varphi(u,\lambda)$ is holomorphic with respect to u in some neighborhood of the real axis.

Using the steepest descent method one can estimate the integral (A.1)

$$I(\lambda,\lambda_0,t,[\varphi]) = \varphi(\lambda,\lambda)I_1(\lambda,x,t) + O(t^{-1/2}), \tag{A.3}$$

where

$$I_1(\lambda,\lambda_0,t) = \int_{-\infty}^{\infty} \frac{e^{\tau(u)}}{u-\lambda-i0}\,du. \tag{A.4}$$

The last integral can be expressed in terms of the "error function"

$$I_1(\lambda,\lambda_0,t) = i\pi e^{\tau(\lambda)}\left[\mathrm{erf}\left((\lambda-\lambda_0)\sqrt{t}e^{-i\frac{\pi}{4}}\right)+1\right], \tag{A.5}$$

where, by definition

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz. \tag{A.6}$$

Clearly, that the r.h.s. of (A.5) possesses no uniform asymptotic with respect to t for all λ. Its asymptotic behavior strongly depends of position of λ with respect to the λ_0. Indeed, if, for example, $\lambda-\lambda_0\gg t^{-1/2}$, then the argument of the error function goes to plus infinity, and we have

$$I_1(\lambda,\lambda_0,t)\to 2\pi i e^{\tau(\lambda)} \qquad \text{for} \qquad \lambda-\lambda_0\gg t^{-1/2}. \tag{A.7}$$

However, if $\lambda_0-\lambda\gg t^{-1/2}$, then

$$I_1(\lambda,\lambda_0,t)\to 0 \qquad \text{for} \qquad \lambda_0-\lambda\gg t^{-1/2}. \tag{A.8}$$

Finally, if the difference $\lambda-\lambda_0$ is of the order of $t^{-1/2}$, then we can not apply asymptotic formulæ for the error function. Nevertheless, it is easy to see, that for arbitrary smooth, integrable at the real axis function $f(\lambda)$ the following estimate holds

$$\int_{-\infty}^{\infty} I_1(\lambda,\lambda_0,t)f(\lambda)\,d\lambda = 2\pi i\int_{\lambda_0}^{\infty} e^{\tau(\lambda)}f(\lambda)\,d\lambda + O(t^{-1/2}). \tag{A.9}$$

Thus, one can say, that in a weak sense the original integral $I(\lambda,\lambda_0,t,[\varphi])$ possesses the following asymptotics for the large t

$$I(\lambda,x,t,[\varphi]) = 2\pi i e^{\tau(\lambda)}\varphi(\lambda,\lambda)\theta(\lambda-\lambda_0) + O(t^{-1/2+\varepsilon}), \tag{A.10}$$

where ε is arbitrary positive, and $\theta(\lambda)$ is a step function:

$$\theta(\lambda) = \begin{cases} 1, & \lambda > 0, \\ 0, & \lambda < 0. \end{cases} \tag{A.11}$$

Using the same way we can find asymptotic formulæ for similar singular integrals. Below the list of formulæ, which were used in the section 3, is given.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{\pm\tau(u)}}{u-\lambda\mp i0}\varphi(u,\lambda)\,du &= \pm 2\pi i e^{\pm\tau(\lambda)}\varphi(\lambda,\lambda)\theta(\lambda-\lambda_0) + O(t^{-1/2+\varepsilon}), \\ \int_{-\infty}^{\infty} \frac{e^{\pm\tau(u)}}{u-\lambda\pm i0}\varphi(u,\lambda)\,du &= \mp 2\pi i e^{\pm\tau(\lambda)}\varphi(\lambda,\lambda)\theta(\lambda_0-\lambda) + O(t^{-1/2+\varepsilon}). \end{aligned} \tag{A.12}$$

References

1. C. N. Yang and C. P. Yang, *Thermodynamics of one dimensional system of bosons with repulsive delta-interaction.* J. Math. Phys. **10**, (1969) 1115–1122.
2. T. Kojima, V. E. Korepin and N. A. Slavnov, *Determinant Representation for Dynamical Correlation Functions of the Quantum Nonlinear Schrödinger Equation.* Preprint ITP-SB-96-71 (1996), hep-th/9611216, (accepted to Commun. Math. Phys.).
3. T. Kojima, V. E. Korepin and N. A. Slavnov, *Completely Integrable Equation for the Quantum Correlation Function of Nonlinear Schrödinger Equation.* Preprint ITP-SB-96-75 (1996), hep-th/9612252, (accepted to Commun. Math. Phys.).
4. V. E. Korepin and N. A. Slavnov, *The Riemann–Hilbert problem associated with quantum nonlinear Schrödinger equation.* Preprint ITP-SB-97-33, (1997), hep-th/9706147, (accepted to J. Phys. A).
6. A. R. Its, A. G. Izergin, V .E. Korepin, and G. G. Varzugin, *Large time and distance asymptotics of the correlator of the impenetrable bosons at finite temperature.* Physica **D54** (1991), 351.

7. V. E. Korepin and N. A. Slavnov, *Normal ordering in the theory of correlation functions of exactly solvable models*. Preprint ITP-SB-97-51, (1997), hep-th/9709204, (accepted to J. Phys. A).

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Поступило 22 января 1998 года