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SOME PROPERTIES OF ELEMENTARY HAMILTONIAN MATRICES

Abstract. Certain Hamiltonian matrices, which form the building blocks for the canonical form, have a particularly simple form. In this paper we look at some properties of these matrices.

1. Introduction

A symplectic linear space is a real vector space V of even dimension $2n$ equipped with a non-degenerate, antisymmetric bilinear form ω. A basis (v_i) of such a space is said to be symplectic if

$$\begin{aligned} \omega(v_i, v_j) &= +1 \quad 1 \le i \le n, j = n+i \\ &= -1 \quad n+1 \le i \le 2n, j = i-n \\ &= \ \ 0 \quad \text{otherwise} \end{aligned}$$

Every symplectic space has a symplectic basis [ ][6].

Two types of operator on symplectic spaces are of particular importance. We say that the operator $\mathcal{L}$ is symplectic if

$$\omega(\mathcal{L}v_i, \mathcal{L}v_j) = \omega(v_i, v_j)$$

and Hamiltonian if

$$\omega(\mathcal{L}v_i, v_j) = -\omega(v_i, \mathcal{L}v_j).$$

We will write $\mathcal{T}$ for symplectic operators and $\mathcal{A}$ for Hamiltonian operators.

A real, square $2n \times 2n$ matrix M is said to be symplectic if $M^tJM = J$ and Hamiltonian if $M^tJ + JM = 0$ where

$$J = J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We will write T for symplectic matrices and A for Hamiltonian matrices. It is easy to see that the matrix of a symplectic (resp. Hamiltonian) operator in a symplectic basis is symplectic (resp. Hamiltonian).

A matrix A is Hamiltonian if and only if it can be written $A = JS$ where S is symmetric. A linear Hamiltonian vector field is of the form $f(x) = Ax$, where A is a Hamiltonian matrix; its Hamiltonian is the quadratic form $H(x) = \frac{1}{2}x^tSx$, where S is as above.

A symplectic matrix is invertible and its determinant is $+1$. (There is no difficulty in seeing that the determinant is ± 1; however showing that it is $+1$ requires a little work: see for example [4].) If a matrix T is symplectic then so is its transpose and we have the relation $T^{-1} = -JT^tJ$.

If we conjugate a Hamiltonian matrix A with a symplectic matrix T we obtain a matrix $\tilde{A} = T^{-1}AT$, which is also Hamiltonian. In this case we say that A and $\tilde{A}$ are symplectically similar or symplectically conjugate. In the same way we say that the symmetric matrices S and $\tilde{S}$ are symplectically congruent if there is a symplectic matrix T such that $\tilde{S} = T^tST$. Writing $\tilde{A} = J\tilde{S}$ we have the relation $\tilde{S} = T^tST$. Thus two Hamiltonian matrices are symplectically similar if and only if their corresponding symmetric matrices are symplectically congruent.

Consider two symplectic symplectic spaces (V_1, ω_1) and (V_2, ω_2). A linear map from $L : V_1 \longrightarrow V_2$ is a symplectomorphism if it preserves the bilinear form, i.e.,

$$\forall u, v \in V_1 \; \omega_2(Lu, Lv) = \omega_1(u, v).$$

In this case we say that the two spaces are symplectomorphic.

Suppose now that we have triples $(V_1, \omega_1, \mathcal{A}_1)$ and $(V_2, \omega_2, \mathcal{A}_2)$, where $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) is a Hamiltonian operator on the symplectic space (V_1, ω_1) (resp. (V_2, ω_2)). If there exists a symplectomorphism $L : V_1 \longrightarrow V_2$ such that $L \circ \mathcal{A}_1 = \mathcal{A}_2 \circ L$ we say that the two triples are of the same type and we write $(V_1, \omega_1, \mathcal{A}_1) \sim (V_2, \omega_2, \mathcal{A}_2)$. Clearly $\sim$ defines an equivalence relation: we call the equivalence classes types. Clearly all vector spaces of triples in a given type have the same dimension so we can speak of the dimension of a type.

A subspace W of a symplectic space (V, ω) is said to be symplectic if $\omega|W$ is non-degenerate. This amounts to saying that $(W, \omega|W)$ is a symplectic space. Clearly the dimension of W must be even. Two subspaces W and Z of V are said to be orthogonal if

$$\forall w \in W, \; \forall z \in Z \; \omega(w, z) = 0.$$

If in addition $V = W \oplus Z$ then Z is said to an orthogonal complement of U. It can be shown that a subspace is symplectic if and only if it has an orthogonal complement [7].

A triple $(V, \omega, \mathcal{A})$ is decomposable if V can be written as the direct sum of non-trivial, orthogonal, $\mathcal{A}$-invariant subspaces. Otherwise we say that the triple is indecomposable. The decomposition of a triple α gives rise to two triples α_1 and α_2 and we can write $\alpha = \alpha_1 + \alpha_2$. If the triple β is equivalent to α then it too is decomposable and we have $\beta = \beta_1 + \beta_2$ with $\alpha_1 \sim \beta_1$ and $\alpha_2 \sim \beta_2$. It follows that all the members of a type are either decomposable or indecomposable and so we can speak of decomposable and indecomposable types. Also we can define the decomposition of a type Δ unambiguously in the following way: if $\alpha \in \Delta$ and $\alpha = \alpha_1 + \alpha_2$ then we write $\Delta = \Delta_1 + \Delta_2$ where $\alpha_1 \in \Delta_1$ and $\alpha_2 \in \Delta_2$.

There is no difficulty in seeing that for any type Δ we can write

$$\Delta = \Delta_1 + \Delta_2 + \ldots + \Delta_s$$

where the Δ_i are indecomposable types. N. Burgoyne and R. Cushman have proved in [2] that this decomposition is unique (up to order).

Let the triples $(V_i, \omega_i, \mathcal{A}_i)$, $i = 1, 2$ belong to the same type and $L : V_1 \longrightarrow V_2$ be a symplectomorphism making them equivalent. If (e_i) is a symplectic basis of V_1 and A is the (Hamiltonian) matrix of $\mathcal{A}_1$ in (e_i) then (Le_i) is a symplectic basis of V_2 and A is also the matrix of $\mathcal{A}_2$ in (Le_i). Thus we can associate to a type a certain family of Hamiltonian matrices, namely those representing the operator of any triple belonging to the type in a symplectic basis. As the transformation matrix from one symplectic basis to another is symplectic, this family is just the collection of Hamiltonian matrices which are symplectically similar to any matrix of the operator of any triple in a symplectic basis. A consequence of this is that we can associate to any type Δ a polynomial P_Δ, the characteristic polynomial of a matrix associated with Δ.

If Δ is a decomposable (resp. indecomposable) type then we will will say that any matrix associated with it is decomposable (resp. indecomposable). It is not difficult to see that if a Hamiltonian matrix A is decomposable there exists a symplectic matrix T and a permutation matrix K such that

$$K^t T^{-1} A T K = \mathrm{diag}(A_1, \ldots, A_s)$$

where $s > 1$ and the blocks $A_1, \ldots, A_s$ are Hamiltonian matrices (see appendix).

We will be concerned here with particularly simple indecomposable matrices which we will call elementary Hamiltonian matrices.

2. Elementary Hamiltonian matrices

The characteristic polynomial of a Hamiltonian operator is even [7]. This means that if λ is an eigenvalue then so is $-\lambda$ and $\bar{\lambda}$ and these with the same multiplicities as λ. Writing gen (λ) for the generalized eigenspace of λ we can group the generalized eigenspaces to obtain subspaces of the following types:

$$\begin{aligned}
P &= \mathrm{gen}\,(\alpha) \oplus \mathrm{gen}\,(-\alpha) \\
Q' &= \mathrm{gen}\,(\gamma + \mathrm{i}\delta) \oplus \mathrm{gen}\,(\gamma - \mathrm{i}\delta) \oplus \mathrm{gen}\,(-\gamma + \mathrm{i}\delta) \oplus \mathrm{gen}\,(-\gamma - \mathrm{i}\delta) \\
R' &= \mathrm{gen}\,(\mathrm{i}\beta) \oplus \mathrm{gen}\,(-\mathrm{i}\beta) \\
S &= \mathrm{gen}\,(0)
\end{aligned}$$

where $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}_+^*$. (Of course the spaces Q' and R' are in the complexification of V.)

If $\mathcal{A}$ is a Hamiltonian operator on a symplectic space (V, ω) and λ and μ are eigenvalues such that $\lambda + \mu \neq 0$ then the corresponding generalized eigenspaces are orthogonal [5]. This will also be the case if we replace Q' (resp. R') by its real part which we will write Q (resp. R). It follows that the characteristic polynomial of an indecomposable type has one of four forms:

$$\begin{aligned}
&(X^2 - \alpha^2)^m \\
&[X - (\gamma^2 + \delta^2)]^m [X + (\gamma^2 + \delta^2)]^m \\
&(X^2 + \beta^2)^m \\
&X^m, m \ \text{ even}.
\end{aligned}$$

It turns out that for a given number $2n$ there are two indecomposable types with characteristic polynomial X^{2n}, if n is even, or $(X^2 + \beta^2)^n$ and one in the case where the polynomial characteristic is $(X^2 - \alpha^2)^n$ or X^{2n} with n odd. For a given number $4n$ there is one indecomposable type with the polynomial $[X - (\gamma^2 + \delta^2)]^n [X + (\gamma^2 + \delta^2)]^n$ [Cush2]. We can associate to each one of these types a particularly simple matrix; we will call these matrices elementary Hamiltonian matrices.

All Hamiltonian matrices have the following block structure:

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

(if A is $2n \times 2n$, then the A_i are $n \times n$) where $A_4 = -A_1^t$ and A_2,A_3 are symmetric. Hence we can describe a Hamiltonian matrix by giving the form of the blocks A_1, A_2 and A_3. The elementary Hamiltonian matrices have the following forms:

I/

$$A_1 = \begin{pmatrix} \alpha & & & & \\ 1 & \alpha & & & \\ & 1 & \alpha & & \\ & & \ddots & \ddots & \\ & & & 1 & \alpha \end{pmatrix} \qquad A_2 = A_3 = 0$$

II/

$$A_1 = \begin{pmatrix} \Delta & & & & \\ I_2 & \Delta & & & \\ & I_2 & \Delta & & \\ & & \ddots & \ddots & \\ & & & I_2 & \Delta \end{pmatrix} \qquad A_2 = A_3 = 0$$

where

$$D = \begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix} \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

III/

$$A_2 = \begin{pmatrix} & & & +1 & +\beta \\ & & \iddots & +\beta & \\ & \iddots & \iddots & & \\ +1 & +\beta & & & \\ +\beta & & & & \end{pmatrix} \qquad A_3 = \begin{pmatrix} & & & & -\beta \\ & & & -\beta & -1 \\ & & \iddots & \iddots & \\ & \iddots & \iddots & & \\ -\beta & -1 & & & \end{pmatrix}$$

$$S_2 = S_3 = 0$$

or III', the negative of III.

$IV(e)/$ (n even)

$$A_1 = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \quad A_2 = 0$$

or $IV(e)'$, which is the same as $IV(e)$ with the exception that -1 replaces 1 in the block A_3.

$IV(o)/$ (n odd)

$$A_1 = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \qquad A_2 = A_3 = 0$$

Remark. The elementary matrices of I and IV are Burgoyne-Cushman and Bruno normal forms[2, 1] (the forms are the same); that of II is a Burgoyne-Cushman form and those of III Bruno forms. We have chosen what we consider to be the simplest matrix of each type.

We have already remarked that a Hamiltonian matrix A can be written as the product of the matrix J and a symmetric matrix S. We now propose to study the spectra of these symmetric matrices when A is an elementary Hamiltonian matrix. For want of another term we will refer to these matrices as elementary symmetric matrices.

3. Spectra of elementary symmetric matrices

We will now analyze the spectra of the elementary symmetric matrices introduced in the preceding section. If $A = JS$ then S can be written in block form:

$$S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$$

where

$$S_1 = -A_3 \quad S_2 = -A_4 \quad S_3 = A_1 \quad S_4 = A_2.$$

Before looking at the spectra of these matrices we will prove an elementary result.

Proposition. *Let X be a $2n \times 2n$ matrix with the block form*

$$X = \begin{pmatrix} U & V \\ W & Z \end{pmatrix}$$

where U, V, W, and Z are $n \times n$ matrices. If W and Z commute then we have

$$\det X = \det(UZ - VW).$$

Proof. Suppose first that Z is invertible. Then we have

$$\begin{pmatrix} U & V \\ W & Z \end{pmatrix} \begin{pmatrix} Z & 0 \\ W & Z^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} UZ - VW & VZ^{-1} \\ WZ - ZW & I_n \end{pmatrix} = \begin{pmatrix} UZ - VW & VZ^{-1} \\ 0 & I_n \end{pmatrix}$$

because W and Z commute.

However

$$\det \begin{pmatrix} Z & 0 \\ W & Z^{-1} \end{pmatrix} = 1$$

and so

$$\det \begin{pmatrix} U & V \\ W & Z \end{pmatrix} = \det(UZ - VW).$$

Now consider the general case. The matrix $Z - \lambda I_n$ is invertible for an infinite number of values of λ. (Z has only a finite number of eigenvalues.) Thus for an infinite number of values of λ we have

$$\det \begin{pmatrix} U & V \\ W & Z - \lambda I_n \end{pmatrix} = \det(U(Z - \lambda I_n) - VW).$$

This means that the polynomials

$$\det \begin{pmatrix} U & V \\ W & Z - xI_n \end{pmatrix} \quad \text{and} \quad \det(U(Z - xI_n) - VW)$$

take the same value at an infinite number of points and so are equal. On setting $x = 0$ we find the result. $\square$

Now let us look at the spectra of the elementary symmetric matrices.

I/ In this case we have

$$S_2 = \begin{pmatrix} \alpha & 1 & & & \\ & \ddots & \ddots & & \\ & & \alpha & 1 & \\ & & & \alpha & 1 \\ & & & & \alpha \end{pmatrix} \qquad S_3 = \begin{pmatrix} \alpha & & & & \\ 1 & \alpha & & & \\ & 1 & \alpha & & \\ & & \ddots & \ddots & \\ & & & 1 & \alpha \end{pmatrix}$$

$$S_1 = S_4 = 0.$$

Writing Φ_S for the characteristic polynomial of S, from the proposition we have

$$\Phi_S(\lambda) = \det(S - \lambda I_{2n}) = \det \begin{pmatrix} -\lambda I_n & S_2 \\ S_3 & -\lambda I_n \end{pmatrix}$$

$$= \det(\lambda^2 I_n - S_2 S_3) = \det(S_2 S_3 - \lambda^2 I_n).$$

Notice however that

$$M = S_2 S_3 = \begin{pmatrix} a & \alpha & & & \\ \alpha & a & \alpha & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha & a & \alpha \\ & & & \alpha & \alpha^2 \end{pmatrix}$$

where $a = \alpha^2 + 1$.

M is tridiagonal with non-zero values on the upper and lower semi-diagonals. Therefore the eigenvalues of M are distinct; in addition, if M' is the corresponding matrix when the matrix S is $2(n+1) \times 2(n+1)$, then the eigenvalues of M lie between those of M' [3]. If we write $\mu_1 < \mu_2 \ldots < \mu_n$ for the eigenvalues of M then

$$\Phi_S(\lambda) = \Phi_M(\lambda^2) = \prod_{i=1}^{n} (\lambda^2 - \mu_i)$$

Hence Φ_S is an even polynomial having distinct roots: if λ is a root then so is $-\lambda$. In addition, the characteristic polynomials of the $2n \times 2n$ and $2(n+1) \times 2(n+1)$ matrices S are distincts.

Remark. Notice that the μ_i must be strictly positive.

II/ Here we have

$$S_2 = \begin{pmatrix} \Delta^t & I_2 & & & \\ & \ddots & \ddots & & \\ & & \Delta^t & I_2 & \\ & & & \Delta^t & I_2 \\ & & & & \Delta^t \end{pmatrix} \qquad S_3 = \begin{pmatrix} \Delta & & & & \\ I_2 & \Delta & & & \\ & I_2 & \Delta & & \\ & & \ddots & \ddots & \\ & & & I_2 & \Delta \end{pmatrix}$$

$$S_1 = S_4 = 0$$

Using the proposition again we obtain

$$\Phi_S(\lambda) = \det(S - \lambda I_{4n}) = \det \begin{pmatrix} -\lambda I_{2n} & S_2 \\ S_3 & -\lambda I_{2n} \end{pmatrix}$$

$$= \det(\lambda^2 I_{2n} - S_2 S_3) = \det(S_2 S_3 - \lambda^2 I_{2n}).$$

Notice also that if we write $\epsilon = \gamma^2 + \delta^2$ then

$$\Delta^t \Delta = \epsilon I_{2n}$$

and so

$$M = S_2 S_3 = \begin{pmatrix} \sigma I_2 & \Delta & & & \\ \Delta^t & \sigma I_2 & \Delta & & \\ & \ddots & \ddots & \ddots & \\ & & \Delta^t & \sigma I_2 & \Delta \\ & & & \Delta^t & \epsilon^2 I_2 \end{pmatrix}$$

where $\sigma = \epsilon^2 + 1$.

Let us look at the spectrum of M. If μ is an eigenvalue and $X^t = (X_1, \ldots, X_n)^t$ a corresponding eigenvector, where $X_i \in \mathbb{R}^2$, then we have the following system $(*)$

$$\begin{aligned} \epsilon X_1 + \Delta X_2 &= \mu X_1 \\ \Delta^t X_1 + (1+\epsilon) X_2 + \Delta X_3 &= \mu X_2 \\ \vdots &= \vdots \\ \Delta^t X_{n-2} + (1+\epsilon) X_{n-1} + \Delta X_n &= \mu X_{n-1} \\ \Delta^t X_{n-1} + (1+\epsilon) X_n &= \mu X_n . \end{aligned}$$

If we multiply the ith equation by Δ^{i-1}, for $i = 2, \ldots, n-1$, we obtain

$$\begin{aligned} \Delta X_2 &= (\mu - \epsilon) X_1 \\ \Delta^2 X_3 &= (\mu - \epsilon - 1)\Delta X_2 - \epsilon X_1 \\ \Delta^3 X_4 &= (\mu - \epsilon - 1)\Delta^2 X_3 - \epsilon \Delta X_2 \\ \vdots &= \vdots \\ \Delta^{n-1} X_n &= (\mu - \epsilon - 1)\Delta^{n-2} X_{n-1} - \epsilon \Delta^{n-3} X_{n-2} \end{aligned}$$

Letting $P_1(x) = x - \epsilon$ we have

$$\Delta X_2 = P_1(\mu) X_1 .$$

Also

$$\Delta^2 X_3 = (\mu - \epsilon - 1)\Delta X_2 - \epsilon X_1 = (\mu - \epsilon - 1) P_1(\mu) X_1 - \epsilon X_1 .$$

We now let $P_2(x) = (x - \epsilon - 1) P_1(x) - \epsilon$ and so obtain

$$\Delta^2 X_3 = P_2(\mu) X_1 .$$

Continuing in this way we get a sequence of polynomials $P_1, \dots, P_{n-1}$ with $\deg P_i = i$ such that

$$\forall i \quad \Delta^i X_{i+1} = P_i(\mu) X_i.$$

Now multipying the last equation in the system $(*)$ by Δ^{n-1} we have

$$\epsilon \Delta^{n-2} X_{n-1} + (1+\epsilon)\Delta^{n-1} X_n = \mu X_n$$

$$\begin{aligned} &\Longrightarrow (\mu - \epsilon - 1)\Delta^{n-1} X_n = \epsilon \Delta^{n-2} X_{n-1} \\ &\Longrightarrow (\mu - \epsilon - 1) P_{n-1}(\mu) = \epsilon P_{n-2}(\mu) X_1. \end{aligned}$$

Setting

$$Q(x) = (x - \epsilon - 1) P_{n-1}(x) - \epsilon P_{n-2}(x)$$

we have $\deg Q = n$ and

$$Q(\mu) X_1 = 0.$$

Using the system $(*)$ and the fact that X is an eigenvector we see that $X_1 \neq 0$ and so $Q(\mu) = 0$.

Now consider the matrix $M - \mu I_{2n}$. If we eliminate the first two lines and the last two columns we obtain the matrix

$$\begin{pmatrix} \Delta^t & \eta I_2 & \Delta & & & \\ & \Delta^t & \eta I_2 & \Delta & & \\ & & \ddots & \ddots & \ddots & \\ & & & \Delta^t & \eta I_2 & \Delta \\ & & & & \Delta^t & \eta I_2 \\ & & & & & \Delta^t \end{pmatrix}$$

where $\eta = \epsilon^2 + 1 - \mu$.

This matrix is clearly non-singular. It follows that the dimension of any eigenspace cannot be greater than 2. Notice however that all eigenvalues are roots of the polynomial Q so that there can be at most n. As M is $2n \times 2n$ and symmetric and therefore diagonalizable, each root of Q must be an eigenvalue of multiplicity 2 of M.

If $\mu_1 \dots, \mu_n$ are the distinct roots of Q then we have

$$\Phi_S(x) = \prod_{i=1}^{n} (x^2 - \mu_i)^2.$$

Every eigenvalue of S is of multiplicity 2 and if λ is an eigenvalue then so is $-\lambda$.

Remark. Notice that the μ_i must be strictly positive.

III/ In this case we have

$$S_1 = \begin{pmatrix} & & & 1 & \beta \\ & & 1 & \beta & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ 1 & \beta & & & \\ \beta & & & & \end{pmatrix} \qquad S_4 = \begin{pmatrix} & & & & \beta \\ & & & \beta & 1 \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ & \beta & 1 & & \\ \beta & 1 & & & \end{pmatrix}$$

$$S_2 = S_3 = 0$$

or S', the negative of S.

We have

$$\det(S - \lambda I_{2n}) = \det(S_1 - \lambda I_n)\det(S_4 - \lambda I_n) = \det(S_1 - \lambda I_n)^2$$

Now let us look at the spectrum of the matrix S_1. First notice that

$$M = S_1^2 = \begin{pmatrix} b & \beta & & & \\ \beta & b & \beta & & \\ & \ddots & \ddots & \ddots & \\ & & \beta & b & \beta \\ & & & \beta & \beta^2 \end{pmatrix}$$

where $b = 1 + \beta^2$.

M is a tridiagonal matrix with non-zero values on the semi-diagonals. Therefore its eigenvalues are distinct; also if N is the corresponding matrix when S is $2(n+1) \times 2(n+1)$ then the eigenvalues of M lie between those of N. This means that the eigenvalues of S_1 are distinct and different from those of S_1 when S is $2(n+1) \times 2(n+1)$.

We would like to know the number of positive and negative eigenvalues of S_1. Consider the matrix

$$S(\epsilon) = \begin{pmatrix} & & & \epsilon & \beta \\ & & \epsilon & \beta & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ \epsilon & \beta & & & \\ \beta & & & & \end{pmatrix}$$

Clearly $S(1) = S_1$. Let $\Lambda^t(\epsilon) = (\lambda_1(\epsilon), \ldots, \lambda_n(\epsilon))^t$ be the vector of eigenvalues. As $S(\epsilon)$ is symmetric $\Lambda(\epsilon) \in \mathbb{R}^n$. Thus the map $\phi : \epsilon \longmapsto \Lambda(\epsilon)$ is a continuous map from $\mathbb{R}$ into $\mathbb{R}^n$. For any value of ϵ the eigenvalues $\lambda_1(\epsilon), \ldots, \lambda_n(\epsilon)$ are all non-zero so it follows that $\Lambda(\epsilon)$ stays in the same quadrant, i.e. the number of positive and negative eigenvalues do not change. Therefore to find the number of positive and negative eigenvalues of S_1 it is sufficient to look at the matrix $S(0)$.

If $n = 2k$ we have

$$F_k = S(0) - \lambda I_n = \begin{pmatrix} -\lambda & & & \beta \\ & \ddots & \iddots & \\ & \iddots & \ddots & \\ \beta & & & -\lambda \end{pmatrix}$$

It is not difficult to find the recurrence relation

$$\det F_k = (\lambda^2 - \beta^2) \det F_{k-1}$$

and so deduce that $\det F_k = (\lambda^2 - \beta^2)^k$. Thus the number of positive eigenvalues of S_1 is the same as the number of negative ones.

Suppose now that $n = 2k + 1$. Then

$$G_k = S(0) - \lambda I_n = \begin{pmatrix} -\lambda & & & & \beta \\ & \ddots & & \iddots & \\ & & \beta - \lambda & & \\ & \iddots & & \ddots & \\ \beta & & & & -\lambda \end{pmatrix}$$

If we develop the determinant of this matrix by the $(k+1)$th column we obtain

$$\det G_k = (\beta - \lambda) \det F_k = (\beta - \lambda)^{k+1} (\beta + \lambda)^k$$

Hence in this case the number of positive eigenvalues of S_1 is one greater than that of the negative eigenvalues.

Let us now return to S. We have shown that S has n eigenvalues each of multiplicity 2; if n is even the number of positive and negative eigenvalues is the same; if n is odd the number of positive eigenvalues exceeds that of the negative ones by 1.

The matrix S' is the negative of S so its eigenvalues are the negatives of those of S. Here also there are n eigenvalues of multiplicity 2; as before, if n is even, the number of positive and negative eigenvalues is the same. On the other hand, if n is odd, the number of negative eigenvalues of S is one greater than that of the positive eigenvalues.

Notice also that the eigenvalues of S and S' must be distinct. If S and S' have an eigenvalue λ in common then both λ and $-\lambda$ are eigenvalues of S. However the squares of the eigenvalues of S (and of S') are the eigenvalues of M. As $\lambda^2 = (-\lambda)^2$ the number of distinct eigenvalues of M in this case could not exceed $n-1$, a contradiction.

$IV(e)/$ This time we have

$$S_1 = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \qquad S_3 = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

$$S_2 = S_3^t \qquad S_4 = 0$$

Let λ be an eigenvalue and $(X,Y)^t = (x_1, \dots, x_n, y_1, \dots, y_n)^t$ a corresponding eigenvector. Then we have the systems

$$\begin{aligned} y_2 &= \lambda x_1 \\ \vdots &\quad \vdots \\ y_n &= \lambda x_{n-1} \\ x_n &= \lambda x_n \end{aligned}$$

and

$$\begin{aligned} 0 &= \lambda y_1 \\ x_1 &= \lambda y_2 \\ \vdots &\quad \vdots \\ x_{n-1} &= \lambda y_n \end{aligned}$$

As JS is singular and J non-singular, S is singular; therefore 0 is an eigenvalue of S. Looking at the systems above it is easy to see that in this case we must have

$$x_1 = \ldots = x_n = y_2 = \ldots = y_n = 0.$$

It follows that the eigenspace of 0 is the vector subspace of $\mathbb{R}^{2n}$ generated by the $(i+1)$th element of the canonical basis.

Now suppose that $\lambda^2 \neq 1$, then from the two systems we obtain

$$\begin{aligned} y_2 &= \lambda^2 y_2 \\ \vdots &\quad \vdots \\ y_n &= \lambda^2 y_n \end{aligned}$$

and so

$$y_2 = \ldots = y_n = 0.$$

It follows easily that $(X,Y)^t = 0$, which contradicts the fact that it is an eigenvector and hence non-zero. Therefore the only non-zero possible eigenvalues are ± 1.

If $\lambda = 1$ we have the relations

$$\begin{aligned} y_1 &= 0 \\ y_2 &= x_1 \\ \vdots\ &\ \ \vdots \\ y_n &= x_{n-1} \end{aligned}$$

We deduce that $+1$ is an eigenvalue whose eigenspace is the vector subspace generated by the vectors $e_1 + e_{n+2}, \ldots, e_{n-1} + e_{2n}, e_n$, where $(e_i)_{1 \le i \le 2n}$ is the canonical basis of $\mathbb{R}^{2n}$.

In the case of $\lambda = -1$ we have

$$\begin{aligned} x_n &= 0 \\ y_1 &= 0 \\ y_2 &= -x_1 \\ \vdots\ &\ \ \vdots \\ y_n &= -x_{n-1} \end{aligned}$$

We deduce that -1 is an eigenvalue whose eigenspace is the vector subspace generated by the vectors $e_1 - e_{n+2}, \ldots, e_{n-1} - e_{2n}$. Notice that the eigenspaces of $+1$ and -1 do not have the same dimensions.

The case of $IV(e)'$ can be treated in the same way as that of $IV(e)$. Once again the eigenvalues are 0 and ± 1 and the dimension of the eigenspace of 0 is 1. However this time the dimension of the eigenspace of -1 is one greater than that of $+1$.

$IV(o)$/ In this case we have

$$S_3 = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \quad S_2 = S_3^t \quad S_1 = S_4 = 0.$$

If λ be an eigenvalue and $(X,Y)^t = (x_1,\ldots,x_n,y_1,\ldots,y_n)^t$ a corresponding eigenvector, then we have the systems

$$\begin{aligned} y_2 &= \lambda x_1 \\ \vdots &\quad \vdots \\ y_n &= \lambda x_{n-1} \\ 0 &= \lambda x_n \end{aligned}$$

and

$$\begin{aligned} 0 &= \lambda y_1 \\ x_1 &= \lambda y_2 \\ \vdots &\quad \vdots \\ x_{n-1} &= \lambda y_n \end{aligned}$$

If $\lambda = 0$ then

$$x_1 = \ldots = x_{n-1} = y_2 = \ldots = y_n = 0$$

and we deduce that 0 is an eigenvalue whose eigenspace is generated by the vectors e_n and e_{n+1} of the canonical basis.

It easy to see that $\lambda^2 \neq 1$ is impossible. If $\lambda = 1$ we have the relations

$$\begin{aligned} x_n &= 0 \\ y_1 &= 0 \\ y_2 &= x_1 \\ \vdots &\quad \vdots \\ y_n &= x_{n-1} \end{aligned}$$

We find that $+1$ is an eigenvalue whose eigenspace is generated by the vectors $e_1 + e_{n+2}, \ldots, e_{n-1} + e_{2n}$.

A similar analysis shows that -1 is also an eigenvalue and that the vectors $e_1 - e_{n+2}, \ldots, e_{n-1} - e_{2n}$ generate its eigenspace.

4. Discussion

A natural question to ask concerning Hamiltonian matrices is the relation between the spectrum of the Hamiltonian matrix and that of the corresponding symmetric matrix. From what we have seen here the two spectra may have very different properties. For example a $2n \times 2n$ Hamiltonian matrix A may have two eigenvalues and its corresponding symmetric matrix $2n$ (case I) or n (case III). It is particularly interesting to look at case IV, that of a nilpotent Hamiltonian matrix. Here the matrix A has the unique eigenvalue 0 and the matrix S has this eigenvalue with multiplicity at most 2.

We should also observe the fundamental difference between the cases I and III. When we have a real pair $\pm\alpha$ all the eigenvalues are distinct, half being negative and half positive. However in the case of an imaginary pair $\pm i\beta$ all eigenvalues have multiplicity 2 and, in the case where n is odd, we do not have the same number of positive and negative eigenvalues. At least in this case it could not be argued that the difference comes from the choice of the simple Hamiltonian matrix: as the symmetric matrices corresponding to a given elementary Hamiltonian matrix are all symplectically congruent, they all have the same number of positive and negative eigenvalues.

This brings us to another point, namely the rank and signature of the quadratic form $H(x) = \frac{1}{2}x^tSx$ associated with an elementary Hamiltonian matrix. If we write r (resp. s) for the number of positive (resp. negative) eigenvalues then the couple $(r+s, r-s)$ gives us the rank and signature of the quadratic form. Here we have:

$$I : (2n, 0) \quad II : (4n, 0) \quad III(n \;\; \text{even}) : (2n, 0)$$
$$III(n \;\; \text{odd}) : (2n, 2) \;\; \text{or} \quad (2n, -2)$$
$$IV(e) : (2n-1, 1) \;\; \text{or} \;\; (2n-1, -1) \quad IV(o) : (2n-2, 0).$$

As the rank and signature of the quadratic form associated with any one member of a type are those of the quadratic form associated with any other member we can associate with any type a rank and signature, in particular with the indecomposable types. The ranks and signatures we have found here are those of the indecomposable types.

Let us look in a little more detail at the symmetric matrices of I and III. In our analysis we have seen that in both cases we obtain a matrix M of the same form

$$M = \begin{pmatrix} p & \pi & & & \\ \pi & p & \pi & & \\ & \ddots & \ddots & \ddots & \\ & & \pi & p & \pi \\ & & & \pi & \pi^2 \end{pmatrix}$$

where $p = \pi^2 + 1$.

In case I the eigenvalues of the symmetric matrix which interests us are all the square roots of the eigenvalues of M; however in case III we obtain half of them and in case III' the other half.

To close let us consider another property of the quadratic form $H(x) = \frac{1}{2}x^t Sx$ associated with an elementary Hamiltonian matrix. In all cases, except in the trivial ones where A is the 2×2 zero matrix or is the 2×2 matrix (a_{ij}) with

$$a_{ii} = a_{22} = 0 \quad a_{12} = \beta = -a_{21}$$

the quadratic form is not convex, because a quadratic form is convex if and only if it is positive (see appendix). Hence the only vector fields corresponding to indecomposable Hamiltonian matrices which have convex Hamiltonians are those which are symplectically conjugate to these two simple matrices.

Appendix

In this appendix we will prove two elementary results referred to in the main text.

If K is the permutation matrix $[e_1, e_{n+1}, \dots, e_n, e_{2n}]$, where $(e_i)_{1 \le i \le 2n}$ is the canonical basis of $\mathbb{R}^{2n}$, then we have

Proposition A1. *A Hamiltonian matrix A is decomposable if and only if there exists a symplectic matrix T such that*

$$K^t T^{-1} A T K = \operatorname{diag}(A_1, \dots, A_s)$$

where $s > 1$ and the blocks $A_1, \dots, A_s$ are Hamiltonian matrices.

Proof. First suppose that A is decomposable. Let (V, ω) be a symplectic space and $\mathcal{A}$ a Hamiltonian operator on this space such that A is the matrix of this operator in a symplectic basis $(y_i, x_i)_{1 \le i \le n} = (y, x)$. Let

$$V = U_1 \oplus \dots \oplus U_s$$

a decomposition of V into symplectic subspaces on which $\mathcal{A}$ is stable. For $i \in \{1, \dots, s\}$ let (q^i, p^i) be a symplectic basis of U_i. (The q^i and p^i are each composed of n_i vectors.) The matrix of $\mathcal{A}$ in the basis (q^i, p^i) is a Hamiltonian matrix A_i.

The basis $(q^1, \dots, q^s, p^1, \dots, p^s) = (q, p)$ is a symplectic basis of V and so the transformation matrix T from the basis (y, x) to the basis

(q,p) is symplectic. Let $\tilde{A}$ be the matrix of $\mathcal{A}$ in the basis (q,p). Then $\tilde{A} = T^{-1}AT$ and

$$\tilde{A} = \begin{pmatrix} A_{11} & & & A_{21} & & \\ & \ddots & & & \ddots & \\ & & A_{1s} & & & A_{2s} \\ A_{31} & & & A_{41} & & \\ & \ddots & & & \ddots & \\ & & A_{3s} & & & A_{4s} \end{pmatrix}$$

where the A_{ij} are the $n_i \times n_i$ blocks of the matrix A_i:

$$A_i = \begin{pmatrix} A_{1i} & A_{2i} \\ A_{3i} & A_{4i} \end{pmatrix}.$$

The matrix of $\mathcal{A}$ in the basis $(q^1, p^1, \ldots, q^s, p^s)$ is $\mathrm{diag}(A_1, \ldots, A_s)$, hence the result.

Now suppose that A is a Hamiltonian matrix and that there exists a symplectic matrix T such that

$$K^t T^{-1} A T K = \mathrm{diag}(A_1, \ldots, A_s)$$

where $s > 1$ and the blocks $A_1, \ldots, A_s$ are Hamiltonian matrices.

The matrix $T^{-1}AT$ has the same form as $\tilde{A}$ above and is Hamiltonian because T is symplectic. In addition the columns T_i of T form a symplectic basis of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ where ω_0 is the canonical antisymmetric bilinear form:

$$\forall\, x, y \in \mathbb{R}^{2n} \quad \omega_0(x,y) = x^t J_{2n} y.$$

If the columns of the block A_{1i} have the indices $k_i, k_i+1, \ldots, l_i$, then the subspace V_i of $\mathbb{R}^{2n}$ generated by the columns T_j, T_{n+j} where $k_i \leq j \leq l_i$ is symplectic, the Hamiltonian operator $\mathcal{A}$ defined by

$$\forall\, x \in \mathbb{R}^{2n} \quad \mathcal{A}(x) = Ax$$

is stable on this space and the matrix of $\mathcal{A}|V_i$ is A_i. Thus the triple $(\mathbb{R}^{2n}, \omega_0, \mathcal{A})$ is decomposable, which proves the converse. □

Proposition A2. *A real quadratic form is convex if and only if it is positive.*

Proof. Let q be a real quadratic form on a vector space V and ϕ its polar form. If $x, y \in V$ and $t \in [0,1]$ then

$$q[(1-t)x + ty] = (1-t)^2 q(x) + 2(1-t)t\phi(x,y) + t^2 q(y)$$

hence

$$(1-t)q(x)+tq(y)-q[(1-t)x+ty]$$
$$=(1-t)t[q(x)-2\phi(x,y)+q(y)]=(1-t)tq(x-y)$$

and the result follows. □

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