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ALGORITHMS IN ALGEBRAIC TOPOLOGY AND HOMOLOGICAL ALGEBRA: THE PROBLEM OF THE COMPLEXITY

ABSTRACT. This survey tackles the problem of the high computational complexity lying in most of the algorithms in Algebraic Topology and Homological Algebra. We deal with three particular algorithms: the computation of the homology of commutative differential graded algebras, the homology of principal twisted cartesian products of Eilenberg–Mac Lane spaces and a combinatorial method computing Steenrod Squares.

1. INTRODUCTION

Since the notion of homeomorphism was clearly settled, the classification of topological spaces up to homeomorphism arose as the main problem in Topology. This soon became a hopeless job, and so it was born what we know today as *Algebraic Topology*, in which topological problems are studied by casting them into simpler form by means of some algebraic structures. An example of this is given by the *invariant algebraic structures* associated to topological spaces, such as homology, cohomology or homotopy groups.

This survey fits into the field of Algebraic Topology and, more explicitly, of Homological Algebra, which studies "the pure algebraic properties of algebraic objects in Algebraic Topology." The description of efficient algorithms of homological computation might be considered as a principal question in Homological Algebra, in order to use those processes mainly in the resolution of problems on Algebraic Topology; though this subject also influences directly on the development of non so closed areas as Cohomological Physics (in this sense, we find useful references in [22, 45], and [46]) or Secondary Calculus ([24, 53, 52]).

There are in the literature some papers in which the foundations for building up a Constructive Algebraic Topology are given (see, for example, [41] or [42]). In this paper, we follow several ideas of [42]. More precisely, we adopt the Eilenberg–Mac Lane philosophy sketched in the remarkable papers [14, 15], and [27], what is to work in a combinatorial setting (provided by the Simplicial Topology [31]) and to take as initial data explicit homotopy equivalences between differential graded modules. All this is complemented by the homological perturbation machinery ([19, 20, 23]). In this way, a suitable framework for computing homology in Algebraic Topology is raised.

If the question of computability in Algebraic Topology is a delicate one (see [7, 30, 54, 29, 33, 49, 27, 35, 50, 41, 42], etc.), a first approach to the problem of the complexity reveals a disappointed panorama. In most of the algorithms in this area, there are high computational costs and the problems of efficiency amount to improving parts of the algorithms, so that the number of steps is minimized.

We analyze three algorithms that we have already designed in previous papers from the point of view of the complexity. The first one is an algorithm for computing the homology of commutative differential graded algebras (briefly, CDGAs) [2]. This process can be seen as a refined version of the method outlined in [25]. In the second one, combinatorial fibrations are treated. More concretely, a "constructive" version of the Serre spectral sequence (see [1]) for twisted cartesian products (briefly, TCPs) of Eilenberg–Mac Lane spaces is established. Finally, starting from a manageable combinatorial formulation of the morphisms measuring the strong homotopy commutativity of the cup product in cohomology (see [37, 21]), we discuss the complexity of computing Steenrod squares of cochains.

2. Preliminaries

Although relevant notions of Homological Algebra are explained through the exposition of this survey, most of common concepts are not explicitly given and will be used with no further explanations (they might be consulted for instance in [10] and [28], otherwise).

Next, we recall the notion of DG-module and DG-algebra.

The ground ring Λ will be $\mathbb{Z}$ or $\mathbb{Z}$ localized at a prime p. A *differential graded module* (M, d) is a module M graded on the positive integers endowed with a differential operator d of degree -1 such that applied twice becomes null. Its *homology* $H_*(M)$ consists of the quotient groups $H_n(M) = \text{Ker } d_n / Im\ d_{n+1}$. Whenever this object is endowed, also, with an inner product which is compatible with both the graduation and the differential, then a new structure arises, a *DG-algebra*.

For instance, we may consider the exterior algebra with one generator

u on degree $2n+1$ and trivial product; and the polynomial algebra, which consists in one generator v on degree $2n$ and product the natural one of monomials.

The notion of *augmented graded differential algebra* (DGA-algebra) A concerns a DG-algebra endowed with two graded morphisms $\xi_A : A \longrightarrow \Lambda$ (augmentation) and $\eta_A : \Lambda \longrightarrow A$ (co-augmentation) such that $\xi_A \circ \eta_A = 1_\Lambda$ and $\xi_A \circ d_1 = 0$.

In the sequel, the notation $(A, d_A, *_A)$ means that A is DGA-algebra endowed with a differential operator d_A and an associative product $*_A$. If there is no confusion, subindexes or even operators will be omitted. The degree of an element $a \in A_n$ is simply denoted by $|a| = n$ or a_n.

Algebraic Topology analyzes transformations from continuous data to discrete data. These transformations usually involve combinatorial methods, which one of the earliest and most significant of them concerns to the triangulation of a topological space.

A *simplicial set* ([31]) consists in a graded set upon the non negative integers endowed with two families of morphisms (face operators and degeneracy operators), satisfying some commutativity relations:

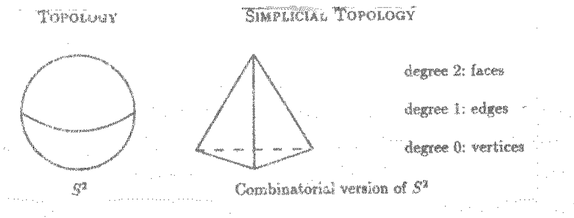
$$\partial_i \partial_j = \partial_{j-1} \partial_i, \quad \text{if } i < j; \quad s_i s_j = s_{j+1} s_i, \quad \text{if } i \leq j;$$

$$\partial_i s_j = s_{j-1} \partial_i, \quad \text{if } i < j; \quad \partial_i s_j = s_j \partial_{i-1}, \quad \text{if } i > j+1;$$

$$\partial_j s_j = 1_K = \partial_{j+1} s_j.$$

Depending on the initial category, many other simplicial objects arise (for instance, simplicial modules, simplicial groups, etc.), whenever both face and degeneracy operators belong to the same category.

For instance, we now show a combinatorial description of the sphere.



The next step towards Algebra is that every simplicial set X has canonically associated an algebraic object $C_*(X)$, which is endowed with a structure of DG-module: the n-component is the free Λ-module $C_n(X) = \Lambda[X_n]$ generated by the n-simplices of X, and the differential operator $d_n : C_n(X) \rightarrow C_{n-1}(X)$ is given by $d_n = \sum_{0 \leq i \leq n} (-1)^i \partial_i$.

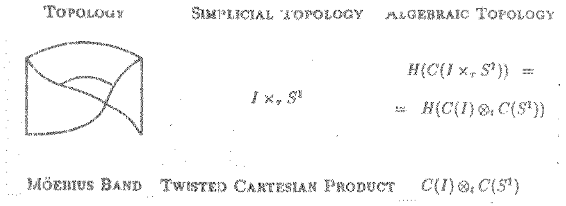
The subset consisting of all the degenerated simplices of $C_\star(X)$, $s(C_\star(X))$, is a graded submodule which is preserved by the action of the differential (that is, $d_n(s(C_\star(X))_n) \subset s(C_\star(X))_{n-1}$), so that the quotient complex, simply denoted by $C(X) = \{C_\star(X)/s(C_\star(X))\}$, becomes also a DG-module, called the *normalized DG-module* canonically associated to X.

The homology of X is defined by $H_\star(X) = H_\star(C(X))$.

In the example above, we may set the n-simplices to be the set of weakly increasing $(n+1)$-sequences concerning to the four ordered vertices; the i^{th} face of an n-simplex consists in removing the i^{th} vertex; analogously, the i^{th} degeneracy operator arises from duplicating the i^{th} vertex.

Attending to the proper context of the Simplicial Topology, the fiber bundles simplify their structures and are considered in a factorized fashion as a *twisted cartesian product* $F \times_\tau B$ of two simplicial sets F (fiber) and B (base), where the τ operator means a "twisting" action on the 0-face operator of $F \times B$.

For instance, let us consider the fiber bundle whose fiber is S^1 and base is the unit segment I, that is: the cylinder. The *Möebius Band* is obtained from that cylinder by applying a suitable twisting function τ:



Notice that homology is preserved in a suitable way.

The main tool in order to reduce the homology of a twisted cartesian product to that of a twisted tensor product is to consider a particular type of homotopy equivalence, which is due to Eilenberg and Zilber ([17]): we refer to the notion of *Eilenberg–Zilber contraction*.

The main input data in our framework are the *contractions* ([14, 23]): a contraction c from a DG-module N to a DG-module M, sometimes simply denoted by $N \overset{c}{\Rightarrow} M$ or (f, g, ϕ), consists in an homotopy equivalence determined by three morphisms f, g and ϕ; being $f : N_\star \rightarrow M_\star$ (projection) and $g : M_\star \rightarrow N_\star$ (inclusion) two DG-module morphisms and $\phi : N_\star \rightarrow N_{\star+1}$ an homotopy operator. Moreover, these data are required to satisfy the following rules: $fg = 1_M$, $f\phi = 0$, $\phi g = 0$,

$\phi d_N + d_N\phi + gf = 1_N$ and $\phi\phi = 0$; so that both the homologies of the DG-modules involved becomes isomorphic.

Using as data explicit contractions, notice that the basic procedure to compute the homology of an object (see [42]) consists in the construction of an explicit contraction from an initial DG-module N to a free finite type DG-module M, so that the homology of N is actually computable from that of M, via a matricial algorithm due to Veblen [51] (based upon the normal Smith's form of the matrices representing the differentials at each degree).

Next we show a characteristic type of contractions. Let X and Y be simplicial sets and let AW, EML and SHI be the morphisms:
Alexander-Whitney,

$$AW : C(X \times Y) \longrightarrow C(X) \otimes C(Y),$$

Eilenberg–Mac Lane,

$$EML : C(X) \otimes C(Y) \longrightarrow C(X \times Y),$$

and Shih,

$$SHI : C(X \times Y) \longrightarrow C(X \times Y);$$

which are defined on positive degrees by the formulas:

$$AW(a_n \times b_n) = \sum_{i=0}^{n} \partial_{i+1} \cdots \partial_n a_n \otimes \partial_0 \cdots \partial_{i-1} b_n\,,$$

$$\begin{aligned} &EML(a_p \otimes b_q) = \\ &\quad = \sum_{(\alpha,\beta)\in\{(p,q)-shuffles\}} (-1)^{\mathrm{sg}(\alpha,\beta)} (s_{\beta_q} \cdots s_{\beta_1} a_p \times s_{\alpha_p} \cdots s_{\alpha_1} b_q), \end{aligned}$$

$$\begin{aligned} &SHI(a_n \times b_n) = \\ &\quad = -\sum (-1)^{m+\mathrm{sg}(\alpha,\beta)} (s_{\beta_q+m} \cdots s_{\beta_1+m} s_{m-1} \partial_{n-q+1} \cdots \partial_n a_n \times \\ &\qquad\qquad \times s_{\alpha_{p+1}+m} \cdots s_{\alpha_1+m} \partial_m \cdots \partial_{m+p-1} b_n); \end{aligned}$$

where

$$m = n - p - q, \quad \mathrm{sg}(\alpha,\beta) = \sum_{i=1}^{p} (\alpha_i - (i-1)),$$

and the last sum is taken over the indices $0 \le q \le n-1$, $0 \le p \le n-q-1$ and $(\alpha,\beta) \in \{(p+1,q)\text{-shuffles}\}$.

Notice that morphisms AW and EML coincide with the identity map on degree 1 and SHI becomes null when it is applied on degree 1.

Then, the data set

$$EZ_{X,Y} : \{C(X \times Y),\, C(X) \otimes C(Y),\, AW_{X,Y},\, EML_{X,Y},\, SHI_{X,Y}\} \quad (1)$$

becomes a contraction, which is often called an *Eilenberg–Zilber contraction*.

The explicit formula of the SHI operator given here was stated by Rubio in [40] and proved by Morace in the appendix of [38]. It is quite surprising the lack of interest shown until now in the study of the homotopy operator involved in an Eilenberg–Zilber contraction, not only from the point of view of getting its explicit formula (a recursive formula was determined by Eilenberg and Mac Lane in [15]) but also from the point of view of obtaining algebraic preservation results of this operator with regard to the underlying coalgebra structure on $C(X \times Y)$.

If we deeply analyze the complexity involved in the contraction above, it arises that the projection morphism AW acts in polynomial time ($O(n^2)$), though the inclusion and homotopy operators act in exponential time ($O(2^n)$), since the number of (p,q)-shuffles is given by the expression $\binom{p+q-1}{p-1}$. Notice that the size of the instance is given by the degree of the elements and the elementary operations are the simplicial operators.

We could dare to say that the problem of the computational complexity in our algorithmic approach can summarize in the previous example. To avoiding the categorically exponential complexity of EML and SHI in our algorithms will be our main motivation. So, it arises an interesting problem, which is to determine the of lack of certain classes of elements such that the morphisms above act in "reasonable" time and the homological information is completely localized over these elements.

In fact, the tool provided by contractions in general suggests new perspectives and approximations to classic problems of Algebraic Topology and Homological Algebra: Eilenberg and Mac Lane fit a peculiar way of working in Algebraic Topology, as it is summarized in their papers [14, 15], and [16]. Here, we follow these characteristic procedures rather than Cartan's ones. Indeed, the deep analysis of the morphisms, products and algebra structures involved in most of the *contractions* in Algebraic Topology leads into a new and unexplored field, which is where we are moving in: we refer to the Homological Perturbation Theory applied to the algebra or coalgebra category.

Since Shih ([43]) established first a perturbation process on Brown's

Theorem ([6]) in 1961, it is not until 1989 that a Perturbation Theory is finally described. Gugenheim, Lambe and Stasheff ([19, 20) set that under certain circumstances the process of "perturbing" a given contraction stops and leads to a new contraction, which connects both of the homologies of the respective modules. This technique is called "Basic Perturbation Lemma", which we describe next.

We first recall the concept of a perturbation datum. Let N be a graded module and let $f : N \to N$ be a morphism of graded modules. The morphism f is defined to be *pointwise nilpotent* whenever for all $x \in N$, $x \neq 0$, a positive integer n exists such that $f^n(x) = 0$. Notice that the integer n generally depends on the element x. A *perturbation of a DGA-module* N consists in a morphism of graded modules $\delta : N \to N$ of degree -1, such that $(d_N + \delta)^2 = 0$ and $\xi_N \delta = 0$. A *perturbation datum of the contraction* $c : \{N, M, f, g, \phi\}$ is a perturbation δ of the DGA-module N verifying that the composition $\phi\delta$ is pointwise nilpotent.

We now state the main tool handling contractions. The **Basic Perturbation Lemma** ([19, 20]) is an algorithm such that from a contraction $c : \{N, M, f, g, \phi\}$ and a perturbation datum δ of c generates a new contraction

$$c_\delta : \{(N, d_N + \delta), (M, d_M + d_\delta), f_\delta, g_\delta, \phi_\delta\}$$

defined by the formulas

$$d_\delta = f\delta\Sigma_c^\delta g; \quad f_\delta = f(1 - \delta\Sigma_c^\delta\phi); \quad g_\delta = \Sigma_c^\delta g; \quad \phi_\delta = \Sigma_c^\delta \phi;$$

where

$$\Sigma_c^\delta = \sum_{i \geq 0} (-1)^i \; (\phi\delta)^i = 1 - \phi\delta + \phi\delta\phi\delta - \cdots + (-1)^i (\phi\delta)^i + \cdots .$$

The stop condition of this theorem is guaranteed by the pointwise nilpotency of the composition $\phi\delta$. Notice that the main difficulty to establish algorithms in our setting is to determine that the stop condition of BPL applied to a specified case is satisfied.

In the particular case that both of N and M are DGA-algebras, a new notion appears: an *algebra perturbation datum* δ of a contraction c from N to M consists in a perturbation datum δ of c which is also a derivation, that is, δ verifies the formulas

$$\delta *_N = *_N (1 \otimes \delta + \delta \otimes 1), \qquad \xi_N \delta = 0.$$

We may deal with perturbation results particularized to the DGA-algebra category, often applied to particular algebra contractions (called *semifull algebra contractions*), those where the injection preserves products and the projection and homotopy operators f and ϕ verify these formulas:

$$f *_A (\phi \otimes \phi) = 0, \quad f *_A (\phi \otimes g) = 0, \quad f *_A (g \otimes \phi) = 0, \tag{2}$$

$$\phi *_A (\phi \otimes \phi) = 0, \quad \phi *_A (\phi \otimes g) = 0, \quad \phi *_A (g \otimes \phi) = 0. \tag{3}$$

In this sense, we give a more specific version of the "Basic Perturbation Lemma" below.

Theorem 2.1 (SF-APL) [38]. *Let c be a semifull algebra contraction and δ an algebra perturbation datum of c. Then, the perturbed contraction c_δ is a semifull algebra contraction*

In this way, the general procedure we state here in order to compute the homology of a DGA-algebra concerns the existence of "small models" ([23]).

But all the techniques described before are not good enough: the main obstacle to the methods above is the high complexity of the procedures often involved. For instance, if $F \times_\tau B$ is a twisted cartesian product, Shih proved in [43] that if we perturb the Eilenberg–Zilber contraction $C(F \times B) \Rightarrow C(F) \otimes C(B)$, being

$$\delta(f, b) = (\tau b * \partial_0 f,\ \partial_0 b) - (\partial_0 f,\ \partial_0 b)$$

the perturbation datum, we recover Brown's theorem [6], in which a contraction from the normalized DG-module associated to the twisted cartesian product $F \times_\tau B$ onto the twisted tensor product $C(F) \otimes_t C(B)$ is established. Then, computing the homology of a combinatorial fiber bundle $F \times_\tau B$ reduces to the determination of a Brown's cochain t and the knowledge of some homological information (in terms of contractions) for the factors F and B.

Of course, this Brown cochain t is determined by the differential d_δ resulting of the perturbation process.

Now, recall the formula of the perturbed differential operator d_δ in terms of the proper morphisms AW, EML and SHI of the initial Eilenberg–Zilber contraction:

$$d_\delta = AW \circ \delta \circ (1 - SHI \circ \delta + SHI \circ \delta \circ SHI \circ \delta - \cdots) \circ EML;$$

hence, the formula above reveals an exponential character on the action of the differential operator, since EML and SHI so act. This is a common fact in general perturbation processes of Algebraic Topology.

Notice that in the early sixties, Szczarba gave an explicit formula for the cochain t involved in the differential operator d_δ ([48]).

In order to simplify the high computational charges of the techniques above, it is convenient to attend to the algebra structures present in the contractions. This approach allows us to get interesting results in our research, as we will show along the course of the paper.

3. First algorithm: homology of CDGAs

There already exist various methods computing the homology groups of commutative differential graded algebras (CDGAs), though they often involve high complexity processes.

There is one approximation which deals with the recursive construction of a resolution of the ground ring Λ over an initial algebra ([26]). Notice that a *resolution* of Λ over a DGA-algebra A consists in a differential A-module X which is projective as an A-module and such that the homology of X is zero except in degree 0, where it is Λ. If X is actually a free A-module, then X is called a *free resolution*.

Working in the context of CDGAs, Homological Perturbation Theory ([19, 20]) supplies a general algorithm computing the homology of these objects ([25]), which often bears high computational charges and actually restricts its application to the low dimensional homological calculus.

It is possible to improve this general method by applying SF-APL: the refining of the algorithm above, by means of preservation issues over the category of CDGAs when applying perturbation techniques ([38]), has already been partially shown in a paper of the authors ([2]).

It is commonly known that every commutative DGA-algebra A "factorizes" into a tensor product of exterior and polynomial algebras endowed with a differential-derivation; in the sense that there exists a weak homotopy equivalence, that is, an homomorphism connecting both structures which induces an isomorphism in homology (consult [8] if necessary).

Next, we recall what a "twisted tensor product of algebras" means.

A *twisted tensor product* (TTP) $\tilde{\otimes}^{\rho}_{i\in I} A_i$ is a CDGA satisfying the following conditions:

(i) It coincides with the usual tensor product $\otimes_{i\in I} A_i$ as a graded algebra.

(ii) The differential operator consists in the sum of the differential corresponding to the banal tensor product and a derivation ρ.

Actually, a TTP of DGA-algebras is a perturbed DGA-algebra in which the perturbation only affects its differential and not its product.

We now recall that the reduced bar construction associated to a CDGA-algebra A consists in the DGA-algebra

$$(\oplus_{n\geq 0} A^n, |\,|_s + |\,|_t, d_s + d_t),$$

defined by the graduation $|\,| = |\,|_s + |\,|_t$

$$|[a_1|\cdots|a_n]|_t = \sum_{i=1}^{n} |a_i|, \qquad |[a_1|\cdots|a_n]|_s = n;$$

and the differential operator $d = d_s + d_t$

$$d_t([a_1|\cdots|a_n]) = -\sum_{i=1}^{n}(-1)^{e_i - 1}[a_1|\cdots|d_A(a_i)|\cdots|a_n],$$

$$\begin{aligned} d_s([a_1|\cdots|a_n]) &= \xi_A(a_1)[a_2|\cdots|a_n]+ \\ &+\sum_{i=1}^{n-1}(-1)^{e_i}[a_1|\cdots|a_i *_A a_{i+1}|\cdots|a_n]+ \\ &+(-1)^{e_n}[a_1|\cdots|a_{n-1}]\xi_A(a_n); \end{aligned}$$

where $e_i = i + |a_1| + \cdots + |a_i|$.

The product on $\overline{B}(A)$ is given by the *shuffle product* $*_B$:

$$[a_1|\cdots|a_p] *_{\overline{B}} [b_1|\cdots|b_q] = \sum_{(\alpha,\beta)} (-1)^{\epsilon(\pi,a,b)}[c_{\pi(1)}|\cdots|c_{\pi(p+q)}],$$

where (α, β) moves on the (p, q)-shuffles, π is the permutation which determines them, $(c_1, \dots, c_{p+q}) = (a_1, \dots, a_p, b_1, \dots, b_q)$ and

$$\epsilon(\pi, a, b) = \sum_{\pi(i)>\pi(p+j)} |[a_i]||[b_j]|.$$

The importance of the reduced bar construction is shown up in the following definition of "small models" for DGA-algebras.

An *homological model* for a commutative DGA-algebra A consists in a free finite type DGA-algebra HBA and a semifull algebra contraction c, $\overline{A} \stackrel{c}{\Rightarrow} HBA$, from the reduced bar construction of A onto HBA ([2]).

The object we start from is a generic commutative DGA-algebra A, factorized into the model above as a TTP of exterior and polynomial algebras, $A = \tilde{\otimes}^{\rho}_{i\in I} A_i$.

Taking into account the ideas expressed in the section before, the principal goal that arises is to obtain a "chain" of contractions starting from the reduced bar construction of A, $\overline{B}(A)$, and ending up at a free DGA-algebra of finite type.

Three *almost-full algebra contractions* (i.e., semifull algebra contractions endowed with multiplicative projections) are mainly used in order to find the structure of graded module of an homological model for the DGA-algebra A:

- The contraction defined in [15] from $\overline{B}(A \otimes A')$ to $\overline{B}(A) \otimes \overline{B}(A')$, being A and A' two commutative DGA-algebras; which is briefly denoted by $C_{\overline{B}\otimes}$.

The morphisms (f, g, ϕ) which determine the contraction $C_{\overline{B}\otimes}$ stated above are given by the explicit formulas:

$$f[x_1 \otimes y_1|\cdots|x_n \otimes y_n] =$$

$$= \sum_{i=0}^{n} \xi_A(x_1 \cdots x_i)\xi_{A'}(y_{i+1} \cdots y_n)[x_1|\cdots|x_i] \otimes [y_{i+1}|\cdots|y_n],$$

$$g([x_1|\cdots|x_p] \otimes [y_1|\cdots|y_q]) =$$

$$= [x_1 \otimes 1|\cdots|x_p \otimes 1] \star [1 \otimes y_1|\cdots|1 \otimes y_q]$$

and the homotopy operator ϕ strongly depends on the SHI operator of a particular Eilenberg–Zilber contraction ($C_{\overline{B}\otimes}$ coincides up to signs with an Eilenberg–Zilber contraction [15]).

Given a tensor product $\otimes_{i \in I} A_i$ of commutative DGA-algebras, a contraction from $\overline{B}(\otimes_{i \in I} A_i)$ to $\otimes_{i \in I} \overline{B}(A_i)$ is easily determined, by only applying $C_{\overline{B}\otimes}$ several times in a suitable way. This new contraction is also denoted by $C_{\overline{B}\otimes}$.

- The isomorphism of DGA-algebras $C_{\overline{B}E}$ from $\overline{B}(E(x, 2n+1))$ to $\Gamma(\sigma(x), 2n+2)$, also described in [15]. Notice that the generator v of the previous divided power algebra is denoted by $\sigma(u)$, since $g_E(v) = \sigma(u)$, where σ is the Cartan's suspension operator, $\sigma : A \to \overline{B}(A)$, $\sigma(a) = [a]$. The homotopy operator ϕ_E becomes obviously null, since $C_{\overline{B}E}$ is an iso-contraction.

Last isomorphism might be considered as a *full algebra contraction* (that is, an almost-full algebra contraction endowed with an algebra homotopy operator), in a natural way.

- The contraction $C_{\overline{B}P}$ from $\overline{B}(P(y, 2n))$ to $E(\sigma(y), 2n+1)$ ([15]). The generator of the exterior algebra is denoted by $\sigma(y)$ so that both elements y and $\sigma(y)$ correspond one to each other by the projection f_P and inclusion g_P of the contraction.

All of the previous contractions have already been extensively described in the last decades, so we do not explain them any further.

Due to these three contractions, it is possible to establish by composition the following semifull algebra contraction $C_0 = (F, G, \Phi)$:

$$\overline{B}(\otimes_{i\in I} A_i) \Rightarrow \otimes_{i\in I}\overline{B}(A_i) \Rightarrow \otimes_{i\in I} HBA_i,$$

where HBA_i represents an exterior or a divided polynomial algebra, depending on whether A_i is a polynomial or an exterior algebra, respectively. Notice that the product on $\otimes_{i\in I} HBA_i$ is the natural one.

The next step is perturbing C_0, in order to obtain an homological model of the initial TTP $\tilde{\otimes}^{\rho}_{i\in I} A_i$. The morphism ρ produces a perturbation-derivation δ onto the tensor differential of $\overline{B}(\otimes_{i\in I} A_i)$. It seems necessary to emphasize that the pointwise nilpotence of $\phi\delta$ is guaranteed by the following facts:

- The homotopy operator ϕ increases the simplicial degree of $\overline{B}(\otimes_{i\in I} A_i)$ by one.
- The perturbation δ does not change the simplicial degree. Indeed, δ lowers the tensor degree by one.

Therefore, by applying **SF-APL** a new semifull algebra contraction $(F_\delta, G_\delta, \Phi_\delta)$ is established:

$$\overline{B}(\tilde{\otimes}^{\rho}_{i\in I} A_i) \overset{(C_0)_\delta}{\Rightarrow} (\otimes_{i\in I} HBA_i, d_\delta),$$

where the differential d_δ is determined by the perturbation process. That means that $HBA = \otimes_{i\in I}(HBA_i, d_\delta)$ is an **homological model** of $A = \tilde{\otimes}_{i\in I} A_i$. Notice that the homotopy operator Φ_δ increases the simplicial degree at least by one.

Theorem 3.1 [2]. *Given a finite type TTP A of exterior and polynomial algebras, there exists a semifull algebra contraction from the reduced bar construction $\overline{B}(A)$ to a free DGA-algebra H of finite type. That is, we have an homological model for A. Moreover, H consists in a TTP of exterior and divided power algebras such that, at graded algebra level, it is verified that:*

- *each $E(u, 2n-1)$ factor in A contributes with a $\Gamma(\sigma(u), 2n)$ factor in H.*
- *each $P(u, 2n)$ factor in A contributes with an $E(\sigma(u), 2n+1)$ factor in H.*

In this way, a general algorithm computing the homology of CDGAs is described. Notice that we may connect with the classical notion of homology, since given a TTP A of exterior and polynomial algebras, following Lambe's notation ([26]) there exists a free resolution $A\tilde{\otimes}^{\rho'} H$ that splits off of the reduced bar construction, being this splitting a semifull algebra contraction ([2]).

The computational cost for constructing the contraction $(C_0)_\delta$ is significant. Notice that both the inclusion and homotopy operators of the contractions $C_{\overline{B}\otimes}$ give an answer in exponential time, when evaluated on each element, since several Eilenberg–Zilber contractions are involved at this stage. So, a first impression is that the evaluation of d_δ upon any element becomes in general a process of exponential nature.

In spite of the discourse before, it is possible to take advantage of d_δ being a derivation in order to improve the complexity problem: indeed, the fact that d_δ is a derivation implies that it is only necessary to know the value of this morphism applied upon the generators of the model (notice that there are so many generators as the cardinal of the index set I indicates). This is an enormous improvement in the computation of the differential on the small model.

In order to clarify the general method, let us examine in detail what happens when we start from the particular case of the following CDGA-algebra. The ground ring is supposed to be $\mathbb{Z}$ in the sequel.

Let $A^s_{e_1}$ denote the CDGA-algebra which consists of the DG-module

$$E(x, 2s+1)\tilde{\otimes}^{\rho_1} P(y_1, 2s+2),$$

and the differential operator

$$\rho_1(y_1) = e_1\ x,$$

where $e_1 \in \mathbb{N}$. Consider as departure point the following contraction $C_0 = (F, G, \Phi)$:

$$\begin{array}{c} \overline{B}(E(x,2s+1) \otimes P(y_1, 2s+1)) \\ \Big\Downarrow{\scriptstyle C_{\overline{B}\otimes}} \\ \overline{B}(E(x,2s+1)) \quad \otimes \quad \overline{B}(P(y_1, 2s+2)) \\ \Big\Downarrow{\scriptstyle C_{\overline{B}E} \otimes C_{\overline{B}P}} \\ \Gamma(\sigma(x), 2s+2) \otimes E(\sigma(y_1), 2s+3). \end{array}$$

That is,

$$C_0 = \{(f_E \otimes f_P) \circ f, g \circ (g_E \otimes g_P), \phi + g \circ (1_\Gamma \otimes \phi_P) \circ f\},$$

following the notations referred before.

The question now is to apply the already described general method to this initial contraction. From now on, the degrees of the algebra generators are omitted in order to simplify the notation and lecture of the paper.

The differential ρ_1 produces a perturbation-derivation δ in

$$\overline{B}(E(x) \otimes P(y_1)),$$

in an obvious way.

Applying the perturbation machinery, the following semifull algebra contraction arises:

$$\overline{B}(A^s_{e_1}) \overset{(C_0)_\delta}{\Rightarrow} (\Gamma(\sigma(x)) \otimes E(\sigma(y_1)), d_\delta),$$

where d_δ is the differential determined in the perturbation process.

Hence,

$$H\overline{B}A^s_{e_1} = (\Gamma(\sigma(x)) \otimes E(\sigma(y_1)), d_\delta)$$

constitutes an **homological model** for this CDGA, where the differential operator is given by the formula

$$d_\delta = F \circ \delta \circ (\sum (-1)^i (\Phi \circ \delta)^i) \circ G =$$

$$= (f_E \otimes f_P) \circ f \circ \delta \circ (\sum (-1)^i ((\phi + g \circ (1_\Gamma \otimes \phi_P) \circ f) \circ \delta)^i) \circ g \circ (g_E \otimes g_P).$$

Taking advantage of d_δ being a derivation, it is only necessary to evaluate the differential over each of the generators $\sigma(x)$ and $\sigma(y_1)$.

In this particular case, the formula of d_δ reduces to $F \circ \delta \circ G$, since $\Phi \circ \delta \circ G = 0$. Going on calculating the image of the differential, it results that $d_\delta(\sigma(x)) = 0$ and $d_\delta(\sigma(y_1)) = -e_1 \cdot \sigma(x)$.

This procedure may be generalized in a proper way to every CDGA belonging to the family $A^{s,(r_1,r_2,\ldots,r_{n-1})}_{e_1,e_2,\ldots,e_n}$, which denotes the CDGA-algebra consisting of the DG-module

$$E(x, 2s+1) \tilde{\otimes}^{\rho_1} P(y_1, 2s+2) \tilde{\otimes}^{\rho_2} P(y_2, (2s+2)(r_1+1)) \tilde{\otimes}^{\rho_3}$$

$$\tilde{\otimes}^{\rho_3} \cdots \tilde{\otimes}^{\rho_n} P(y_n, (2s+2)(r_1+1)\cdots(r_{n-1}+1)),$$

and the differential operator

$$\rho_i(y_i) = e_i \; x \otimes y_1^{r_1} \otimes y_2^{r_2} \otimes \cdots \otimes y_{i-1}^{r_{i-1}}, \quad \forall 1 \leq i \leq n,$$

where $e_i \in \mathbb{N}$ with $e_1 > e_2 > \cdots > e_n$ and $r_i \in \mathbb{N} \cup \{0\}$.

We may consider a similar perturbation process on the corresponding C_0 contraction, taking the derivation morphism δ induced by the ρ_i as perturbation datum.

There is an outstanding fact to take into account: the perturbed projection F_δ (a quasi-algebra projection) is a multiplicative morphism when we take a particular class of elements. For example:

$$F_\delta([x \otimes {y_1}^{r_1}] *_{\overline{B}} \overset{m \text{ times}}{\cdots} *_{\overline{B}} [x \otimes {y_1}^{r_1}]) =$$

$$= F_\delta([x \otimes {y_1}^{r_1}]) *_{HB} \overset{m \text{ times}}{\cdots} *_{HB} F_\delta([x \otimes {y_1}^{r_1}]), \tag{4}$$

where $*_{HB}$ denotes the natural product on the homological model.

Working in this way, it is easy to explicit the action of d_δ upon each generator $\sigma(y_i)$:

$$d_\delta(\sigma(y_i)) =$$

$$= -r_{i-1}! \cdot \frac{\left[\prod_{k=1}^{i-1}(r_k+1)\right]!}{(r_{i-1}+1)! \prod_{k=1}^{i-2}(r_k+1)^{\prod_{j=k+1}^{i-1}(r_j+1)}} \cdot e_i \cdot {e_{i-1}}^{r_{i-1}} \cdot$$

$$\cdot \prod_{k=1}^{i-2} {e_k}^{r_k \prod_{j=k+1}^{i-1}(r_j+1)} \cdot \sigma(x)^{\left(\prod_{k=1}^{i-1}(r_k+1)\right)}, \tag{5}$$

where $i = 1, 2, \dots, n$.

This completes the study of the homology calculation of $A^{s,(r_1,\dots,r_{n-1})}_{e_1,\dots,e_n}$.

Notice that though coefficients are big enough, working with $\mathbb{Z}_{(p)}$ as ground ring simplifies enormously the difficulties on the real computations of the formulas above, which have already been implemented in Mathematica ([5]).

These and similar positive results let us be sincerely expectant in the achievement of an "efficient" general algorithm computing the homology of CDGAs.

4. Second algorithm: p-local homology of the $K(\pi, n) \times_\tau K(\pi', n')$ spaces

A classical result in Algebraic Topology is that every simplicial set factorizes, up to homotopy, into a twisted cartesian product of irreducible prime factors. These prime spaces $K(\pi, n)$, called Eilenberg–Mac Lane spaces, have non-trivial homotopy group π in only one dimension, n. On the other hand, a common technique in homotopy theory is to "localize" the homological information, that is, the homology is sensitive to only one prime.

Our second algorithm computes the p-local homology of a PTCP

$$X = X(\pi, m, \tau, \pi', n) = K(\pi, m) \times_\tau K(\pi' n),$$

where $K(\pi, m)$ and $K(\pi', n)$ are both Eilenberg–Mac Lane spaces and π and π' are finitely generated abelian groups, $m, n \in \mathbb{N}$. Notice that this study has been partially shown in a paper of the authors ([1]).

Whenever π is an ordinary (discrete) abelian group, one may consider $G\pi$ as a simplicial group, being π on each degree and all the face and degeneracy operators being the identity maps. Then the simplicial group $K(\pi, n)$ is defined as follows: $K(\pi, 0) = G\pi$ and $K(\pi, n+1) = \overline{W}K(\pi, n)$, where $\overline{W}$ denotes the classifying construction ([31]). Recall that the classifying construction $\overline{W}(G)$, which is canonically associated to a simplicial group G, consists in the following simplicial group:

$$\begin{aligned}
\overline{W}_0(G) &= \{[\,]\};\\
\overline{W}_n(G) &= G_{n-1} \times \cdots \times G_0, \quad n > 0;\\
s_0[\,] &= [e_0];\\
\partial_i[g_0] &= [\,], \quad i = 0, 1;\\
\partial_0[g_n, \dots, g_0] &= [g_{n-1}, \dots, g_0],\\
\partial_{i+1}[g_n, \dots, g_0] &= [\partial_i g_n, \dots, \partial_1 g_{n-i+1}, g_{n-i-1} + \partial_0 g_{n-i}, g_{n-i-2}, \dots, g_0],\\
s_0[g_{n-1}, \dots, g_0] &= [e_n, g_{n-1}, \dots, g_0],\\
s_{i+1}[g_n, \dots, g_0] &= [s_i g_n, \dots, s_0 g_{n-i}, e_{n-i}, g_{n-i-1}, \dots, g_0];
\end{aligned}$$

where $[\,]$ denotes the unique element of G_0, e_n denotes the identity element of $(G, +)$ and $[g_{n-1}, \dots, g_0]$ denotes a generic element of G_n, for $n > 0$.

Now, we recall several results which will allow us to determine the p-local homology of a PTCP of Eilenberg–Mac Lane spaces.

On one hand, there exists a semifull algebra contraction from $C(\overline{W}(G))$ to the reduced bar construction $\overline{B}(C(G))$, whichever commutative simplicial group G is ([39, 3]).

On the other hand, taking $\mathbb{Z}_{(p)}$ as ground ring, it is possible to establish a semifull algebra contraction from the reduced bar construction of a Cartan's elementary complex ([9]) to a tensor product of Cartan's elementary complexes ([38]).

This "endogamic" process will allow us to get a complete determination of the p-local homology of an Eilenberg–Mac Lane space. Notice that, given a contraction from the CDGA A to the CDGA A' with mutliplicative inclusion or projection, the contraction connecting the reduced bar constructions $\overline{B}(A)$ and $\overline{B}(A')$ is, indeed, a semifull algebra contraction ([38]).

In order to construct an homological model for $C(K(\pi, n))$, being π a finitely generated abelian group, we use two particular almost full algebra contractions, which are established in [15]. On one hand, that from $\overline{B}(\Lambda[\mathbb{Z}])$ to an exterior algebra $E(u, 1)$; on the other hand, that from $\overline{B}(\Lambda[\mathbb{Z}_{p^r}])$ to a twisted tensor product $E(u, 1) \otimes_\rho \Gamma(v, 2)$, where ρ is the derivation defined by $\rho(v) = p^r \cdot u$ and $\rho(u) = 0$.

Therefore, by combining the results before, it is possible to set up the following chain of DGA-algebra contractions:

$$C(K(\pi, 1)) \Rightarrow C(\overline{W}(K(\pi, 0))) \Rightarrow$$

$$\Rightarrow \overline{W}_N(C(K(\pi, 0))) \Rightarrow \overline{B}(\Lambda[\pi]) \Rightarrow G(\pi, 1);$$

and hence,

$$C(K(\pi, n+1)) \Rightarrow C(\overline{W}(\overline{W}^n(K(\pi, 0)))) \Rightarrow \overline{W}_N(\overline{W}^n_N(C(K(\pi, 0)))) \Rightarrow$$

$$\Rightarrow \overline{B}(\overline{W}^n_N(C(K(\pi, 0))) \Rightarrow \overline{B}(\overline{B}^n(\Lambda[\pi])) \Rightarrow \overline{B}(G(\pi, n)) \Rightarrow G(\pi, n+1)$$

Thanks to them, the following theorem extracted from [36] can be proved without any difficulties:

Theorem 4.1 [1]. *Let $\Lambda = \mathbb{Z}_{(p)}$ be the ground ring. Let π and n be a finitely generated abelian group and a non-negative integer, respectively. There is a semifull algebra contraction $c_{\pi,n}$ from $C(K(\pi, n))$ to a tensor product $G(\pi, n)$. The latter is composed by Cartan's elementary complexes corresponding to admissible sequences* ([9]).

Obviously, the p-local homology of $K(\pi, n)$ is the p-local homology of $G(\pi, n)$ and this last homology can be easily computed, since $G(\pi, n)$ is a free DGA-algebra of finite type.

Once the p-local homology of an Eilenberg–Mac Lane space is already established from our point of view, we inquire for the corresponding one of a PTCP ot two Eilenberg–Mac Lane spaces.

This homology is actually straightforward provided from an extension of Brown's Theorem.

Theorem 4.2 [1]. *There is a contraction c_τ from $C(X(\pi, m, \tau, \pi', n))$ to a free DG-module of finite type M, which may be expressed on the form*

$$(G(\pi, m) \otimes G(\pi', n), d_{G(\pi,m)} \otimes 1 + 1 \otimes d_{G(\pi',n)} + d_\tau).$$

But we may deal with particular PTCPs of Eilenberg–Mac Lane spaces, those which are naturally endowed with a group structure from that of their component factors. Besides, the twisting function τ constitutes itself a group homomorphism.

There is a classical result (see [31]) telling us that if X is a commutative simplicial group, then X is homotopically equivalent to a banal cartesian product of Eilenberg–Mac Lane spaces. Let us suppose that $X(\pi, m, \tau, \pi', n)$ is a simplicial group, considering as inner product the natural one of the cartesian product $K(\pi, m) \times K(\pi', n)$. In the light of the previous comment, it seems to be possible to transform a simplicial group $X(\pi, m, \tau, \pi', n)$ into a simple cartesian product by means of some simplicial techniques. We use here a different approach in order to compute the p-local homology of X in this case.

Thus, we may apply an improved version of Brown's Theorem concerning to the algebra category via SF-APL, so that the differential operator d_τ obtained by the perturbation process is completely determined whenever we know its images acting upon the generators of the final algebra $G(\pi, m) \otimes G(\pi', n)$.

Theorem 4.3 [1]. *Let us suppose that $X(\pi, m, \tau, \pi', n)$, endowed with the natural inner product of the cartesian product $X(\pi, m, 0, \pi', n)$, is a commutative simplicial group. Then, c_τ becomes a semifull algebra contraction.*

In this way, the improvement in the computation of the differential is enormous in comparison with the general case. We are trying to develop new methods based on the observance of some interesting properties to reduce the high exponential nature of the general algorithm.

In this sense, an interesting challenge would be to obtain an universal "twisting cochain" $\bar{t}$ in the sense of Prouté ([34]), such that there exists a contraction from $C(\overline{W}(G) \times_\tau \overline{W}(G'))$ to $\overline{B}(C(G)) \otimes_{\bar{t}} \overline{B}(C(G'))$. This result could be considered as an extension of Brown's Theorem in the particular case that we start from a TCP of classifying spaces.

5. Third algorithm: a new method computing Steenrod squares

As a third example, we present here a combinatorial method computing Steenrod squares of a simplicial set.

It is well-known that there are several methods for constructing Steenrod squares ([47]).

One of them consists in making these operations by using the cohomology of the Eilenberg–Mac Lane spaces (see, for instance, [31], p. 107).

Another method is determined by the construction of a family of morphisms $\{D_i\}$ measuring the lack of commutativity of the cup product on the cochain level. This sequence of morphisms is called a higher diagonal approximation, and its existence is always guaranteed by the acyclic models method ([13]). In this way, it is possible to derive a recursive procedure to obtain the formula of any D_i (see [12]; [32, Sec. 7]).

This last method may be improved by attending to some simplicial issues ([21]), such that new explicit formulas for the component morphisms of a higher diagonal approximation in terms only of the face operators are finally established, which lead into a new and powerful algebraic-topological tool.

We start from a higher diagonal approximation determined in [37], where the formula of D_i is established in terms of the component morphisms of an Eilenberg–Zilber contraction from $C_\star(X \times X)$ onto $C_\star(X) \otimes C_\star(X)$, where $C_\star(X)$ denotes the normalized chain complex of a given simplicial set X:

$$D_i = AW \circ (t \circ SHI)^i \circ \Delta,$$

where the morphisms Δ and t are defined by the formulas

$$\Delta : C_\star(X) \to C_\star(X \times X), \quad \Delta(x) = (x, x);$$

$$t : C_\star(X \times X) \to C_\star(X \times X), \quad t(x_1, x_2) = (x_2, x_1).$$

Now, on one hand, the component morphisms of the contraction above are defined in terms of the face (∂_i) and degeneracy operators (s_i) of the simplicial set X. On the other hand, the formula of any morphism D_i always carries the use of the homotopy operator of the Eilenberg–Zilber contraction referred before, and the explicit formula of this morphism is determined by shuffles (some special types of permutations) of degeneracy operators. In consequence, if we express directly the morphisms D_i in terms of the face and degeneracy operators of X, the number of summands appearing in the formulas of any morphism D_i evaluated over an element of degree n becomes at least 2^n, in general. Therefore, any algorithm designed to start from these formulas would become too slow and actually useless for practical implementation.

In spite of this, the idea of simplifying these formulas arises in a natural way. This simplification or normalization is based on the fact that any composition of face and degeneracy operators of every simplicial set may be factorized in a “canonical way” (see [31, p. 4]): any composition of this type can be expressed in the unique form:

$$s_{j_t} \cdots s_{j_1} \partial_{i_1} \cdots \partial_{i_s}, \tag{6}$$

where $j_t > \cdots > j_1 \geq 0$, $i_s > \cdots > i_1 \geq 0$ and any of the indexes exceeds the degree of the elements over which such morphism is defined to act.

Moreover, taking into account that the image of a morphism D_i lies on the tensor product $C_\star(X) \otimes C_\star(X)$, those summands of the simplified formula of D_i including factors which involve any degeneracy operators in their expressions, must be eliminated, since they become null at the normalized chain complex associated to X. So, the formulas for the morphisms D_i may be expressed as follows.

Theorem 5.1. *Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ be the ground ring. Then, the morphism*

$$h_n = AW \circ (t \circ SHI)^n : C_m(X \times X) \to (C_\star(X) \otimes C_\star(X))_{m+n}$$

may be expressed in the form:

- *If n is odd, then:*

$$AW(t\,SHI)^n = \sum_{i_n=n}^{m} \sum_{i_{n-1}=n-1}^{i_n-1} \cdots \sum_{i_0=0}^{i_1-1} \partial_{i_0+1} \cdots \partial_{i_1-1} \partial_{i_2+1} \cdots \cdot \partial_{i_{n-2}-1} \partial_{i_{n-1}+1} \cdots \partial_{i_n-1} \otimes \otimes\, \partial_0 \cdots \partial_{i_0-1} \partial_{i_1+1} \cdots \cdot \partial_{i_{n-1}-1} \partial_{i_n+1} \cdots \partial_m .$$

- *If n is even, then:*

$$AW(t\,SHI)^n = \sum_{i_n=n}^{m} \sum_{i_{n-1}=n-1}^{i_n-1} \cdots \sum_{i_0=0}^{i_1-1} \partial_{i_0+1} \cdots \partial_{i_1-1} \partial_{i_2+1} \cdots \cdot \partial_{i_{n-1}-1} \partial_{i_n+1} \cdots \partial_m \otimes \otimes\, \partial_0 \cdots \partial_{i_0-1} \partial_{i_1+1} \cdots \cdot \partial_{i_{n-2}-1} \partial_{i_{n-1}+1} \cdots \partial_{i_n-1} .$$

So, the classical definition of Steenrod squares:

$$Sq^i(c) = \begin{cases} (c \otimes c) D_{j-i}, & i \leq j; \\ 0, & i > j; \end{cases}$$

where $c \in \operatorname{Hom}(C_j(X), \mathbb{Z}_2)$, is complemented by a manageable combinatorial definition of the higher diagonal approximation.

Theorem 5.2. *Let $\mathbb{Z}_2$ be the ground ring. If $c \in Hom\,(C_j(X), \mathbb{Z}_2)$ and $x \in C_{j+i}(X)$, then the Steenrod squares $Sq_i : H^j(X;\mathbb{Z}_2) \to H^{j+i}(X;\mathbb{Z}_2)$ are defined by the formulas:*

- *If $i \leq j$ and $j \equiv i \pmod 2$ then:*

$$Sq^i(c)(x) = \sum_{i_n=S(n)}^{m} \sum_{i_{n-1}=S(n-1)}^{i_n-1} \cdots \sum_{i_1=S(1)}^{i_2-1}$$
$$c\,\partial_{i_0+1}\cdots\partial_{i_1-1}\partial_{i_2+1}\cdots\cdots\partial_{i_{n-1}-1}\partial_{i_n+1}\cdots\partial_m(x)\otimes$$
$$\otimes\, c\,\partial_0\cdots\partial_{i_0-1}\partial_{i_1+1}\cdots\cdots\partial_{i_{n-2}-1}\partial_{i_{n-1}+1}\cdots\partial_{i_n-1}(x).$$

- *If $i \leq j$ and $j+1 \equiv i \pmod 2$ then:*

$$Sq^i(c)(x) = \sum_{i_n=S(n)}^{m} \sum_{i_{n-1}=S(n-1)}^{i_n-1} \cdots \sum_{i_1=S(1)}^{i_2-1}$$
$$c\,\partial_{i_0+1}\cdots\partial_{i_1-1}\partial_{i_2+1}\cdots\cdots\partial_{i_{n-2}-1}\partial_{i_{n-1}+1}\cdots\partial_{i_n-1}(x)\otimes$$
$$\otimes\, c\,\partial_0\cdots\partial_{i_0-1}\partial_{i_1+1}\cdots\cdots\partial_{i_{n-1}-1}\partial_{i_n+1}\cdots\partial_m(x).$$

- *If $i > j$, then $Sq^i(c)(x) = 0$. In the formulas above, $n = j - i$, $m = j + i$ and*

$$S(k) = i_{k+1} - i_{k+2} + \cdots + (-1)^{k+n-1} i_n + (-1)^{k+n} \lfloor \frac{m+1}{2} \rfloor + \lfloor \frac{k}{2} \rfloor,$$

for all $0 \leq k \leq n$, being $i_0 = S(0)$.

It is clear that, at least in the case in which X presents a finite number of non-degenerated simplices at each dimension, the explicit formulas above constitute an actual algorithm for calculating Steenrod squares.

Assuming that the face operators are evaluated in constant time, a measure for the computational complexity of these formulas is given by the number of the face operators involved in the definition of $Sq^i(c)$, which becomes $O((i+j)^3)$.

It is necessary to discuss several facts on the computation of these formulas. First of all, whenever X presents a finite number of non-degenerated simplices at each degree, our method can be considered as an actual algorithm. For instance, if the number of non-degenerated simplices in every X_ℓ is $O(\ell^2)$ then, assuming that each face operator of X is an elementary operation, the complexity of our algorithm is $O((i+j)^5)$.

Nevertheless, the most interesting examples appearing in Algebraic Topology show, in general, a high complexity in the number of non-degenerated simplices at each degree, so that our method will only be useful in low dimension. Perhaps, appropriately combining these combinatorial formulas with classical properties of Steenrod squares (see [11, 14]) and well-known studies for calculating cocycles could allow us to do a substantial improvement of the method.

This is a general fact in the way we are leading our researches: we start from some algorithms in Algebraic Topology and Homological Algebra and improve them by applying refined perturbation techniques. Although the complexity of these algorithms often bears exponential costs, in many cases we are able to reduce this exponential nature, by considering some particular properties.

A complementary work to do is the real implementation of the algorithms designed.

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