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Gerd Baumann

SYMMETRY ANALYSIS OF DIFFERENTIAL EQUATIONS USING *MATHLIE*

ABSTRACT. This article discusses a general procedure to solve ordinary differential equations of arbitrary order. The method used is based on symmetries of differential equation. The known symmetries allow the derivation of first integrals of the equation. The knowledge of at least r symmetries of an ordinary differential equation of order n with $r \geq n$ is the basis for deriving the solution. Our aim is to show that Lie's theory is instrumental for solving an ordinary differential equation of higher-order.

1. INTRODUCTION

The subject of this article is the discussion of a general procedure allowing the solution of an ordinary differential equation (ODE) of arbitrary order. We discuss the solution of first order, second order and higher order ODEs. The procedure presented extends the capabilities of *Mathematica,* especially of DSolve[]. The technique of first integrals is one of several methods to integrate an ODE provided the symmetries of the equation are known.

The basis of our discussion is a theory conceived by S. Lie (1842–1899). Lie developed a general theory dealing with symmetries and group properties of differential equations. The theory of Lie is a valuable tool for solving ODEs and partial differential equations. In the current text we restrict our considerations to ordinary differential equations. The discussion of partial differential equations can be found in standard books by Olver [1]; Bluman and Kumei [2]; Ibragimov [3]; and Baumann [4]. The impact of Lie's theory is not restricted to differential equations but is omnipresent in the field of mathematics. Lie deals with fundamental areas of modern mathematics combining group theory and analysis in a nut shell. One century after Lie, continuous groups are today the central and most important part of modern mathematics.

Our aim is to show that Lie's theory is instrumental for solving an ordinary differential equation. We will show you a method capable of deriving solutions of an ODE of arbitrary order. The basis of these considerations are the symmetries of the equation.

This paper is organized as follows. In section two, we discuss the term symmetry in connection with ordinary differential equations. In section three, we recall some results for first order ODEs. Section four deals with second order equations. The fifth section gives a generalization of the method to third and higher order equations. The last section summarizes our results.

2. Symmetries

Before we apply Lie's method to ordinary differential equations, we will briefly discuss symmetries in connection with functions. This section introduces the main concepts of the theory.

One of the remarkable aspects of Lie's theory of symmetry transformations is the invariance of a function under the change of coordinates. When dealing with differential equations, oftentimes the equation is simplified by an appropriate change of variables. This transformation generally involves both the independent and the dependent variables. In *Mathematica,* we can represent such transformations by the following list of rules

$$\begin{aligned} x &= X(x, u) \\ u &= U(x, u) \end{aligned} \tag{1}$$

This kind of transformation involving the original independent and dependent variables (x, u) is called a point transformation. A point (x, u) of the manifold $\mathfrak{M}$ is transformed into another point (X, U). Point transformations only take into account changes of the coordinates. The point transformations considered by Lie are transformations depending on at least one parameter ϵ. As we will see, the parameter ϵ is the parameter of the related group. Thus we call ϵ the group parameter of the transformation. A one-parameter transformation is thus given by

$$\begin{aligned} x &= X(x, u; \epsilon) \\ u &= U(x, u; \epsilon) \end{aligned} \tag{2}$$

Such transformations have the following properties: They are invertible if a corresponding Jacobi determinant exists. Their repeated applications yield transformations of the same kind. An identity of the transformation for $\epsilon = 0$ is given. These three properties are the basis of the definition of a Lie group. The three properties are summarized in the definition of a symmetry transformation.

Definition Symmetry Transformation. *A set $\mathfrak{G}$ of transformations given, for example by* (2)*, is a one-parameter group if it contains the identical transformation $I = T_0$ and includes the inverse T_ϵ^{-1} and the composition $T_\epsilon \otimes T_\beta \in \mathfrak{G}$. By choosing the group parameter to be ϵ, the main group property $T_\epsilon \otimes T_\beta \in \mathfrak{G}$ can be written as*

$$T_\epsilon \otimes T_\beta = T_{\epsilon+\beta} \tag{3}$$

that is

$$X(X(x,u,\epsilon), U(x,u,e), \beta) = X(x,u,\epsilon+\beta) \tag{4}$$

$$U(X(x,u,\epsilon), U(x,u,\epsilon), \beta) = U(x,u,\epsilon+\beta). \tag{5}$$

In special applications, conditions (4) *and* (5) *only hold for sufficiently small values of ϵ and β. In such cases, we arrive at what is called a local one parameter group $\mathfrak{G}$ or an infinitesimal group.* •

These properties ensure that the transformations given in (2) form a one-parameter group of point transformations.

2.1. Infinitesimal transformations. One of the most important aspects of Lie's theory is the discovery that a symmetry transformation as given above can be represented in a much simpler form. Lie called this simplified representation *an infinitesimal transformation*. The main idea behind this simplification is the existence of an identity transformation. An identity transformation denotes the lowest approximation of a transformation around the identity. Correspondingly, the finite transformation can be expanded into a Taylor series around the identity transformation. In *Mathematica* we have direct access to such an expansion when applying the function Series[] to the global representation. The following is an example for a general point transformation in the (x, u) plane. We derive the infinitesimal representation by expanding the one-parameter transformation around $\epsilon = 0$ up to the first order in ϵ

```
infinitesimalTrafo=
Series[{X[x, u, ε], U[x, u, ε]}, {ε, 0, 1}]
```

$$\{X[x,u,0] + X^{(0,0,1)}[x,u,0]\epsilon + O[\epsilon]^2,$$

$$U[x,u,0] + (U^{(0,0,1)}[x,u,0]\epsilon + O[\epsilon]^2\}$$

If we use the group properties of the transformation, especially the identity $X(x, u, \epsilon = 0) = x$ and $U(x, u, \epsilon = 0) = u$, we can reduce the expression to

```
infinitesimalTrafo=MatrixForm[
infinitesimalTrafo/.({∂_ϵX[x, u, ϵ] → ξ[x, u],
∂_ϵU[x, u, ϵ] → φ[x, u],
X[x, u, 0] → x,
U[x, u, 0] → u}/.ϵ → 0)]
```

$$\begin{pmatrix} x + \xi[x,u]\epsilon + O[\epsilon]^2 \\ u + \phi[x,u]\epsilon + O[\epsilon]^2 \end{pmatrix}$$

The result shows that the transformation is represented by the original variables plus linear terms in ϵ. The coefficients of ϵ are called the infinitesimals of the transformation and are denoted by ξ and ϕ. Both expressions are the expansion coefficients of the Taylor expansion $\xi = \partial_\epsilon X|_{\epsilon=0}$, and $\phi = \partial_\epsilon U|_{\epsilon=0}$, respectively.

The above is summarized in Lie's first theorem. The theorem considers the inverse problem of an infinitesimal representation. Since, in this case the infinitesimals ξ and ϕ are known, we are looking for a global transformation.

Theorem: Lie's first theorem. *Let us assume that there exists a parameterization of a transformation such that the global transformation like rotation is equivalent to the solution of the initial value problem for the system of first order differential equations*

$$\frac{dX}{d\epsilon} = \xi(X(\epsilon), U(\epsilon)) \tag{6}$$

$$\frac{dU}{d\epsilon} = \phi(X(\epsilon), U(\epsilon)) \tag{7}$$

with the initial conditions

$$X(0) = x \quad \text{and} \quad U(0) = u. \quad \bullet \tag{8}$$

Lie's theorem states that the finite or global transformation represented by X and U is derived from an initial value problem. By integrating these equations with respect to ϵ and eliminating the group parameter, we end up with a global transformation.

So far we have demonstrated that a global transformation is equivalent to its infinitesimal representation. However, the question of how symmetries and transformations are connected remains. Their link is the subject of the following subsection.

2.2 Group invariants. We know that a one-parameter group of transformations is represented by infinitesimal transformations. As the transformation is applied to a point of the manifold $\mathfrak{M}$ it moves along a path in such a way that the path maps into itself. This basic concept of invariance needs a mathematical formulation. For the moment we restrict our considerations to curves in the plane.

Group invariants are the basic quantities of a symmetry analysis. A group invariant is defined as follows

Definition: Group invariant. *A function $F(x,u)$ is an invariant of the group of transformations, if*

$$F(X(x,u,\epsilon), U(x,u,\epsilon)) = F(x,u) \tag{9}$$

is satisfied identically in x, u for all values of the group parameter ϵ. •

This definition contains the basic information for an invariant in symmetry analysis in a nut shell. The definition states that a function is an invariant if it has the same representation in the original and in the new coordinates. The definition also shows us a way how the invariant F may be calculated.

We know that any curve in the (x,u)-plane can be represented by a function $F(x,u) = \text{const}$. Following our definition this curve is considered to be invariant under a transformation if the curve remains the same in both coordinate systems. As already mentioned, global transformations need only an infinitesimal representation like

infini=$\{X \to$ **Function**$[\{x,u,\epsilon\}, x+\epsilon\xi[x,u]]$,
$U \to$ **Function**$[\{x,u,\epsilon\}, u+\epsilon\phi[x,u]]\}$

$$\{X \to \text{Function}[\{x,u,\epsilon\}, x+\epsilon\xi[x,u]],$$
$$U \to \text{Function}[\{x,u,\epsilon\}, u+\epsilon\phi[x,u]]\}$$

where ξ and ϕ are the infinitesimals. The analytical expression for the invariance is simply stated by the relation

invar=$F[x,u] == F[X[x,u,\epsilon], U[x,u,\epsilon]]$

$$F[x,u] == F[X[x,u,\epsilon], U[x,u,\epsilon]]$$

showing that the form of the curve is the same in the old and new coordinates. Applying the transformation to this expression delivers the invariant condition in an implicit form

```
ivarI=invar/.infini
```

$$F[x,u] == F[x+\epsilon\xi[x,u], u+\epsilon\phi[x,u]]$$

The explicit representation of the invariance condition is obtained by a Taylor expansion of the equation of invariance with respect to parameter ϵ.

```
Thread[Series[ivarI,{ε,0,1}],Equal]
```

$$F[x,u] ==$$
$$F[x,u] + (\phi[x,u]F^{(0,1)}[x,u] + \xi[x,u]F^{(1,0)}[x,u])\epsilon + O[\epsilon]^2$$

The invariance condition is satisfied once all terms containing the infinitesimal parameter ϵ have vanished. Introducing Lie's *Symbol* $\overrightarrow{\nu} = \xi\partial_x + \phi\partial_u$, today known as tangent vector field or group generator, we can write the invariance condition as

$$F(x,u) = F(x,u) + \overrightarrow{\nu}F(x,u)\epsilon + o\big(\epsilon^2\big). \tag{10}$$

This calculation shows that a sufficient condition of invariance is $\overrightarrow{\nu}F = 0$. This condition is accessible by an algorithmic procedure and thus applicable to any function F.

Assuming that the tangent vector field applied to the curve F has to vanish, we can immediately conclude that all higher terms in (10) containing multiple applications of the vector field $\overrightarrow{\nu}$ also have to vanish. As a result, we have a sufficient condition of invariance. This condition is central to the determination of symmetries and can be generalized to a manifold of q independent variables $x = (x_1, x_2, \ldots, x_q)$ and p dependent variables $u = \big(u^1, u^2, \ldots, u^p\big)$. In the $q \times p$-dimensional space, the invariance condition reads

$$\sum_{i=1}^{q} \xi_i \frac{\partial F(x,u)}{\partial x_i} + \sum_{\alpha=1}^{p} \phi_\alpha \frac{\partial F(x,u)}{\partial u^\alpha} = 0 \tag{11}$$

where ξ_i and ϕ_α are the unknown infinitesimals of a point transformation.

The term of invariance boils down to the point that each one-parameter group of point transformations in the plane has one independent invariant which can be taken to be the left-hand side of any first integral $J(x,u) = C$ of the characteristic equation (11) of the tangent vector field.

2.3. Symmetries of ordinary differential equations. So far we have introduced symmetry transformations of curves. The essential difference between curves and ODEs is that the manifold $\mathfrak{M}$ is extended or prolonged by the derivatives. The manifold of an ODE does not only contain the independent and dependent variable but also the derivative $u' = du/dx = p$. Thus we have in the simplest case a three-dimensional space with coordinates (x, u, p). We are now looking for the symmetries of this kind of manifold. Again we can define the invariance of a differential equation in a similar way as for functions. Let us state the definition for the most general case of an nth order ODE.

Definition: Symmetry of a differential equation. *A group $\mathfrak{G}$ of point transformations for variables in the manifold $\mathfrak{M}$ is a symmetry group of a nth-order ordinary differential equation*

$$\Delta\left(x, u(x), \frac{du}{dx}, \ldots, \frac{d^n u}{dx^n}\right) = 0 \tag{12}$$

admitting $\mathfrak{G}$ if the nth extended manifold $\mathfrak{M}^n$ is invariant with respect to the nth prolongation $\mathfrak{G}_n$ of the group $\mathfrak{G}$. •

This definition contains the special case for first order ordinary differential equations. A first-order differential equation

$$\Delta(x, u(x), u'(x)) = 0 \tag{13}$$

admits a group $\mathfrak{G}$ if the once extended manifold $\mathfrak{M}^1$, the surface in the space x, u, u', is invariant with respect to the first prolongation $\mathfrak{G}_1$ of $\mathfrak{G}$. This means that equation (13) is invariant under the coordinate transformations $X = X(x, u, \epsilon)$ and $U = U(x, u, \epsilon)$. The invariance condition can be formulated as

$$\Delta(X, U, U') = \Delta(x, u, u'). \tag{14}$$

This kind of relation is also true for the general case of a nth order ODE. Consequently by knowing the transformations X and U, we also know the symmetries of the equation. Lie demonstrated that the knowledge of the infinitesimal transformations is sufficient to solve the equations. We will state here only the fact that the knowledge of the infinitesimals is sufficient to solve the equations. We do not show how the infinitesimals can be calculated with *MathLie* [4]. Once we know the symmetries, it is quite easy to derive the solutions of the equations. The following sections will illustrate solutions for first-order, second-order, and higher-order equations.

3. First Order ODEs

The general form of a first order ordinary differential equation is given by

$$F\big(x, u(x), \frac{du}{dx}\big) = 0 \tag{15}$$

where F is an arbitrary function combining the independent and the dependent variables x, u, and the derivatives du/dx in a general way. Our aim is to use point symmetries for the solution. Knowing the infinitesimal transformation of equation (15) allows two basic strategies for a solution. The first strategy is based on canonical variables extensively discussed in [3, 2], and [1]. The second strategy originally invented by Lie in 1874 [5] uses the well-known procedure of an integrating factor. In this part of the series, we restrict our considerations to the second strategy.

The common belief in literature is that the method involving an integrating factor is only useful in connection with a first order ordinary differential equation. In contrast Lie already developed a generalization of the procedure in the late 1870.

In the previous section, we demonstrated that a curve is invariant under a symmetry transformation if the tangent vector applied to the curve vanishes. Let us here assume that the curves discussed are integral curves or solutions of a first order differential equation. In this case, we can state the following theorem

Theorem: Integrating Factor. *If the first order ordinary differential equation*

$$\frac{du}{dx} = \omega(x, u) \tag{16}$$

allows a one-parameter group given by the vector field $\vec{\nu} = \xi\partial_x + \phi\partial_u$, *then and only then the following function*

$$\mu = \frac{1}{\xi\omega - \phi} \tag{17}$$

is an integrating factor with $\xi\omega - \phi \neq 0$. *If this is the case, the original equation is solved by a quadrature*

$$\int (\omega dx - du)\mu = \text{const}\,. \quad \bullet$$

The theorem states that the knowledge of the infinitesimals ξ and ϕ allows the derivation of a solution just by integration. According to Lie, relation (18) can be symbolically simplified if we introduce two determinants of the form

$$d\Delta = \det \begin{pmatrix} dx & du \\ 1 & \omega(x,u) \end{pmatrix} \tag{19}$$

and

$$\Delta = \det \begin{pmatrix} \xi & \phi \\ 1 & \omega(x,u) \end{pmatrix} = \frac{1}{\mu}. \tag{20}$$

Then relation (18) reduces to the simple form

$$\int \left(\frac{d\Delta}{\Delta} \right) = \text{const}\,. \tag{21}$$

Equation (21) combines all information necessary for our solution in a nut shell. All we need to know are the infinitesimals and the right hand side of equation (16). Integration over the manifold provides us with the solution.

As a first example, we discuss the Riccati equation used by Ibragimov [6] to demonstrate the procedure of canonical transformations. The equation under consideration is

```
riccati = ∂x u[x] + u[x]^2 - 2/x^2 == 0;riccati//LTF
```

$$u^2 - \frac{2}{x^2} + u_x == 0$$

The infinitesimal symmetries of this equation are scaling symmetries for the coordinates given by $\xi = x$ and $\phi = -u$. These two expressions and the right hand side of the equation are the main components for deriving the Δ matrix of equation (21). With MathLie [7], we are able to create the Lie-matrix by using the independent and the dependent variable, the right hand side of the equation ω, and the infinitesimals ξ and ϕ.

```
Δ matrix=DeltaMatrix[x, u, -u[x]^2 + 2/x^2, 1, {{{x}, {-u[x]}}}];
TableForm[Δ matrix]
```

$$\begin{array}{cc} x & -u[x] \\ 1 & \frac{2}{x^2} - u[x]^2 \end{array}$$

The above matrix is the basis for deriving a first integral via (21). The integral is the result of a series of integrations along a path in the (x,u)-plane. FirstIntegral[] of *MathLie* carries out the calculation

```
int1 = FirstIntegral [x,u,Δ matrix,1]== c1
```

$$\mathrm{Log}[x] + \frac{1}{3}\mathrm{Log}[-2 + xu[x]] - \frac{1}{3}\mathrm{Log}[1 + xu[x]] == c1$$

The input parameters for function FirstIntegral[] are the independent and dependent variables, the Lie matrix, and the number of the first integral. The number of first integrals is given by the order of the ODE. In case of first-order ODEs there exists only one integral. For second-order ODEs there exist two integrals and for nth-order ODEs there may exist n first integrals. If we solve the above expression with respect to the unknown function u, we end up with the solution of the problem.

```
sriccati =Solve[int1,u[x]]
```

$$\{\{u[x] \to -\frac{e^{3c1} + 2x^3}{x(e^{3c1} - x^3)}\}\}$$

Here $c1$ is an arbitrary constant of integration. The following section gives a generalization of the integrating factor method for a second order ODE.

4. Second Order ODEs

Second order ordinary differential equations are very important for applications in physics and engineering. All equations based on Newton's second law are second order equations. Thus mechanics, for example, is mainly based on second order ordinary differential equations. The integration theory of second-order ODEs was developed by Lie during the years 1871-1874 and published in [8]. Scheffers a student of Lie, assembled all of Lie's work on ordinary differential equations in a beginners book. In this book Lie described the problem of integrating a differential equation as follows:

I noticed that the majority of ordinary differential equations which were integrable by the old methods were left invariant under certain transformations, and that these integration methods consisted in using that property. Once I had thus represented many old integration methods from a common viewpoint, I set myself the natural problem: to develop a general theory of integration for all ordinary differential equations admitting finite or infinitesimal transformations [9].

We will exemplify Lie's line of thought below. The most general form of a second-order ODE is given by

$$F(x, u, u', u'') = 0. \tag{22}$$

For our purposes, we assume that equation (22) is solvable with respect to the second-order derivative. Thus we consider equations in the form

$$u'' = \omega(x, u, u') \tag{23}$$

where ω is a given function of x, u, and u'.

According to Lie we can now state: If a second-order equation admits a finite symmetry of dimension $r \geq 2$, it can be integrated by a group theoretic quadrature method. This can be done in various ways, one of which is given by the following Algorithm:

1. compute the Lie algebra L_r. A basis of L_r is the set $\overrightarrow{\nu}_1, \overrightarrow{\nu}_2, \ldots, \overrightarrow{\nu}_r$. The tangent vector fields $\overrightarrow{\nu}_i$ follow by appropriately specifying the group constants.
2. If $r = 2$, go to the next step;
   if $r > 2$, distinguish any two-dimensional subalgebra L_2 of L_r.
   If $r = 1$, the order of the equation may be lowered;
   if $r = 0$, the group method is not useful.
3. Calculate the Lie determinants $d\Delta_i$ and Δ and determine the two first integrals by integration. The Lie determinants are defined by

$$d\Delta_1 = \det\begin{pmatrix} dx & du & du' \\ \xi_2 & \phi_2 & \phi_2' \\ 1 & u' & \omega \end{pmatrix}, \quad d\Delta_2 = \det\begin{pmatrix} \xi_1 & \phi_1 & \phi_1' \\ dx & du & du' \\ 1 & u' & \omega \end{pmatrix} \tag{24}$$

   and

$$\Delta = \det\begin{pmatrix} \xi_1 & \phi_1 & \phi_1' \\ \xi_2 & \phi_2 & \phi_2' \\ 1 & u' & \omega \end{pmatrix} \tag{25}$$

   where ξ_i and ϕ_i are the infinitesimals corresponding to the vector field $\overrightarrow{\nu}_i$, and ϕ_i' denote the first extensions of the infinitesimals ϕ_i. The first integrals ψ_i related to the Lie determinants $d\Delta_i$ are given by

$$\psi_i = \int \frac{d\Delta_i}{\Delta} = c_i, \quad i = 1, 2. \tag{26}$$

   The constants c_i denote the integration constants of the ODE.
4. Solve one of the two integrals with respect to the first order derivative and substitute the result into the remaining relation.
5. If we can solve the resulting relation with respect to the unknown function u, we end up with an explicit solution. Otherwise, we get the solution in an implicit form.

These five steps are completely algorithmic and are implemented among others in the package *MathLie*. The functions of *MathLie* carry out the needed calculations completely automatically. To show you how the solution procedure works, we demonstrate the application of the algorithm to an example.

Let us consider a nonlinear second order equation in which the nonlinearity is given by the square of the first derivative. This nonlinear term is multiplied by a real constant α. Our example equation is a generalization of the equations No. 6.1 and 6.2 by Kamke [10] with $\alpha = 1$ and $\alpha = 6$, respectively. The generalized equation reads

firstExample =$\partial_{\{x,2\}}u[x] - \alpha\partial_x u[x])^2$; **firstExample//LTF**

$$-\alpha u_x^2 + u_{x,x} == 0$$

The infinitesimal symmetries are derived by the function Infinitesimals[] which is part of the package *MathLie*.

infi = Infinitesimals[firstExample,
u,x, $\{\alpha\}$, **SubstitutionRules**$\rightarrow \{\partial_{\{x,2\}}u[x]\}$];
infi //LTF

$$\phi_1 == \frac{-e^{-u\alpha}k6 + e^{u\alpha}k7 + e^{u\alpha}k1x + k8\alpha + k2x\alpha}{\alpha}$$
$$\xi_1 == k4 + (k5 + e^{-u\alpha}k6)x - \frac{e^{-u\alpha}k3}{\alpha} - k2x^2\alpha$$

The result is a symmetry group with 8 group parameters ki, $i = 1, 2, \ldots 8$. The group constants ki determine the symmetries of the equation. We note that this equation allows a transformation which reduces the original equation to the simple form $y'' = 0$. This reduction is always possible if a second order equation allows a symmetry group of order eight [8].

To solve the nonlinear second order ODE, we select a solvable subgroup of order two from the eight-dimensional group. The last argument of the function DeltaMatrix[] contains the groups related to the solvable algebra. We choose subgroups $k8 = \beta$ and $k2 = 1$.

inf1 =$\{\{\xi[1][x,u]\},\{\phi[1][x,u]\}\}$\ .**infi**\.
$\{k1 \rightarrow 0, k2 \rightarrow 0, k3 \rightarrow 0, k4 \rightarrow 0, k5 \rightarrow 0, k6 \rightarrow 0, k7 \rightarrow 0, k8 \rightarrow \beta\}$ \ .
$u \rightarrow u[x]$;
inf2 =$\{\{\xi[1][x,u]\},\{\phi[1][x,u]\}\}$\.**infi**\.

```
k1 → 0, k2 → 1, k3 → 0, k4 → 0, k5 → 0, k6 → 0, k7 → 0, k8 → 0} \ .
u → u[x];
```

Lie's matrix replacing the integrating factor is calculated by using the function DeltaMatrix[]

```
Δ matrix=DeltaMatrix[x, u, α(∂_x u[x])^2, 2, {inf1,inf2}];
TableForm[Δ matrix]
```

$$\begin{array}{ccc} 0 & \beta & 0 \\ -x^2\alpha & x & 1+2x\alpha u'[x] \\ 1 & u'[x] & \alpha u'[x]^2 \end{array}$$

The result is a 3×3 matrix containing the infinitesimals and the first prolongation of the two subgroups and the right hand side of the equation. The determinant of this matrix

```
Det [Δ matrix]
```

$$\beta + 2x\alpha\beta u'[x] + x^2\alpha^2\beta u'[x]^2$$

is a non-vanishing expression containing a second order polynomial in the first derivatives. The coefficients of this polynomial depends on the independent variable x and the parameters α and β. Inserting the Lie matrix into equation (26), we are able to calculate the first integrals of the nonlinear ODE

```
integrals =
Thread[FirstIntegral[x,u,Δ matrix,{1, 2}]=={c1,c2}]
```

$$\left\{-\frac{\log[1+x\alpha u'[x]]}{\alpha\beta} + \frac{u[x]}{\beta} == c1,\ \frac{1}{x\alpha} - \frac{1}{x\alpha(1+x\alpha u'[x])} == c2\right\}$$

The result contains two expressions for the integrals combining the variable u and its first order derivative in an algebraic way. These two first integrals define two manifolds in the space (x, u, u'). The projection of the intersection onto the (x, u)-plane of these two manifolds defines the solution we are looking for. The following figure represents a case with fixed values $c1$ and $c2$.

Since we know two first integrals, we are able to eliminate the derivatives from the two integrals. Let us solve the first relation with respect to u'

```
sol1 =Solve[integrals[[1]],u'[x]]/
```

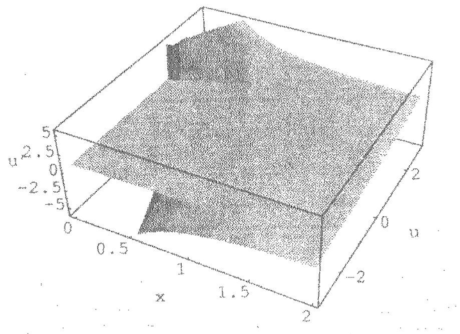

Fig. 1. Figure x shows the intersection of the two integrals for $c1 = c2 = 1$. The parameters α and β take the values $\alpha = 1/10$ and $\beta = 1$. The intersecting line represents the solution of the equation if we project the intersection to the (x, u)-plane. .

```
/ Simplify
```

$$\left\{\left\{u'[x] \to \frac{-1 + e^{\alpha(-c1\beta + u[x])}}{x\alpha}\right\}\right\}$$

Substituting this result into both expressions into *integrals* gives us

```
int21 = integrals /. sol1[[1,1]]  / /PowerExpand//Simplify
```

$$\left\{True, \frac{1 - e^{c1\alpha\beta - \alpha u[x]}}{x\alpha} == c2\right\}$$

The resulting list contains *True* and an implicit representation of the solution. The explicit solution follows from the second relation if we solve it with respect to u.

```
solution = Solve[int21[[2]],u[x]]
```

$$\left\{\left\{u[x] \to -\frac{-c1\alpha\beta + \mathrm{Log}\left[x\left(-c2 + \frac{1}{x\alpha}\right)\alpha\right]}{\alpha}\right\}\right\}$$

The constants $c1$ and $c2$ denote constants of integration. α and β are the model constant and the group parameter, respectively. The occurrence of the group parameter β as a multiplier reminds us that the additive integration constant is related to the group of translations.

The example above demonstrates that the technique of integrating factors can be generalized to second order ODEs. The following section will discuss the extension of the integrating factor technique to a general nth order ODE. Lie called this procedure the *method of generalized multipliers.*

5. Higher Order ODEs

Differential equations of higher order arise naturally in physics. For example, third order ODEs come up in fluid dynamics and fourth order ODEs in elasticity. For general higher order equations there exist hardly any techniques for obtaining explicit symbolic solutions. Higher order ODEs are thus not well studied in the literature. In the following, we will present a symbolic technique for producing explicit solutions independent of the order of the equation. The method described in the preceding sections can, without essential changes, be applied to the solution of differential equations of higher order. The essential change in the method is the extension of the Lie matrix to higher prolongations. Extending the Lie matrix to higher prolongations is the key step for generalizing the procedure of integrating factors. Lie called this extension the determination of the multiplier of the differential equation.

The five steps of integration discussed for second order equations remain the same for higher order equations. However, the dimension of the Lie matrix changes from a 3×3 matrix to a $(n+1) \times (n+1)$ matrix. Before we describe the algorithm to solve higher order ODEs let us first state the general settings for higher order equations.

The most general form of a nth-order ODE is given by

$$F\big(x, u, u', u'', \dots, u^{(n)}\big) = 0. \tag{27}$$

where $u^{(n)=d^n u/dx^n}$ denotes the nth derivative of u with respect to x. In the following, we assume that equation (24) can be solved with respect to the nth derivative. The actual equation under consideration is thus

$$u^(n) = \omega\big(x, u, u', \dots, u^{(n-1)}\big)$$

where ω is a given function of $x, u, u', \dots, u^{(n-1)}$.

If an nth-order equation admits a finite symmetry of dimension $r \geq n$, then the equation can be integrated by the group theoretic quadrature method. This again can be done in various ways. One of the group theoretic algorithms is based on first integrals. The algorithm is summarized as follows:

1. compute the Lie algebra L_r. A basis for L_r is the set $\vec{\nu}_1, \vec{\nu}_2, \dots, \vec{\nu}_r$. The tangent vector fields $\vec{\nu}_i$ follow from appropriately specifying the group constants.
2. If $r = n$, go to the next step;
   if $r > n$, distinguish any n-dimensional subalgebra L_n of L_r.
   If $r = n - 1$, the order of the equation may be lowered;
   if $r = 0$, the group method is not useful.
3. Calculate the Lie determinants $d\Delta_i$ and Δ and if possible determine the n first integrals by integration. The Lie determinants are defined by

$$d\Delta_1 = \det \begin{pmatrix} dx & du & du' & \dots & du^{(n-1)} \\ \xi_2 & \phi_2 & \phi_2' & \dots & \phi_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u' & u'' & \dots & \omega \end{pmatrix} \tag{29}$$

and

$$d\Delta_i = \det \begin{pmatrix} \xi_1 & \phi_1 & {\phi_1}' & \dots & \phi_1^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ dx & du & du' & \dots & du^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u' & u'' & \cdots & \omega \end{pmatrix} \tag{30}$$

where i denotes the row of the Lie matrix in which infinitesimals are replaced by differentials. $n \times n$ Lie determinant reads

$$\Delta = \det \begin{pmatrix} \xi_1 & \phi_1 & \phi_1' & \dots & \phi_1^{(n-1)} \\ \xi_2 & \phi_2 & \phi_2' & \dots & \phi_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u' & u'' & \dots & \omega \end{pmatrix} \tag{31}$$

where ξ_i and ϕ_i are the infinitesimals of the vector field $\vec{\nu}_i$ and ϕ_i' denote the first and $\phi_i^{(n-1)}$ the $(n-1)$th extension of the infinitesimals ϕ_i. The corresponding first integrals ψ_i are given by

$$\psi_i = \int \frac{d\Delta_i}{\Delta} = c_i, i = 1, 2, \dots, n. \tag{32}$$

The constants c_i denote the integration constants of the ODE.
4. Solve one of the n integrals with respect to the $(n-1)$th order derivative and substitute the result into the remaining relations. Repeat this procedure until no derivative remains in the relations.

5. If we can solve the resulting relation with respect to the unknown function u, we have found an explicit solution. Otherwise, our solution is implicit.

This procedure becomes very cumbersome with increasing orders of the equation if done by hand. In principle, the procedure can be applied to any kind of linear or nonlinear ODE. How the algorithm works in particular is demonstrated below.

An example to test the algorithm from above is a third-order equation listed by Kamke [10] as No. 7.13.

thirdOrderExample = $\partial_{x,x}u[x]\partial_{\{x,3\}}u[x] - \alpha\sqrt{1+\beta^2(\partial_{x,x}u[x])^2}$;
thirdOrderExample // LTF

$$-\alpha\sqrt{1+\beta^2 u_{x,x}^2} + u_{x,x}u_{x,x,x} == 0$$

The parameters α and β are real constants. According to Kamke this third-order ODE is solvable and the solution can only be represented in parametric form. We will show that an explicit solution of the equation is possible. First let us check if *Mathematica* can solve the third-order equation.

```
DSolve[thirdOrderExample==0,u,x]
```

$$DSolve\big[-\alpha\sqrt{1+\beta^2 u''[x]^2} + u''[x]u'''[x] == 0, u, x\big]$$

The above line shows that *Mathematica* is not capable of solving the equation. The question now is can we derive the necessary number of symmetries in order to integrate the equation? The derivation of the symmetries is the first step in the general algorithm. The calculation of symmetries is carried out by the *MathLie* function Infinitesimals[]

infi = Infinitesimals[thirdOrderExample,
u, x, $\{\alpha,\beta\}$**, SubstitutionRules→** $\{\partial_{\{x,3\}}u[x]\}$**];**
infi//LTF

$$\phi_1 == k2 + k3x$$
$$\xi_1 == k1$$

The result is a symmetry group of order three. The specific symmetries are denoted by the group constants $k1$, $k2$, and $k3$. Each of these parameters is related to a vector field $\overrightarrow{\nu}_i$, $i = 1, 2, 3$. Since the number of vector fields is equal to the order of equation, we can go to step three of the algorithm. In the third step we determine the Lie matrix by inserting the prolongations of the infinitesimals and the equation itself

```
inf1 ={{ξ[1][x,u]},{φ[1][x,u]}} / .infi/.
      {k1 → 1, k2 → 0, k3 → 0} /.
  u → u[x];
inf2 = {{ξ[1][x,u]},{φ[1][x,u]}} /. infi /.
      {k1 → 0, k2 → 1, k3 → 0} /.
  u → u[x];
inf3 = {{ξ[1][x,u]},{φ[1][x,u]}} /. infi /.
    {k1 → 0, k2 → 0, k3 → 1} /.
  u → u[x];
```

The Lie matrix is derived by

```
Δ matrix=
DeltaMatrix[x, u, α Sqrt[1+β^2(∂_{x,x}u[x])^2]/∂_{x,x}u[x], inf1,inf2,inf3}];
TableForm[Δ matrix]
```

$$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 1 & u'[x] & u''[x] & \frac{\alpha\sqrt{1+\beta^2 u''[x]^2}}{u''[x]} \end{array}$$

One of the three first integrals is

```
integ1 = FirstIntegral[x,u,Δ matrix,1]== c1/ /Simplify
```

$$x - \frac{\sqrt{1+\beta^2 u''[x]^2}}{\alpha\beta^2} == c1$$

The result depends on u'' and now allows us to rewrite all terms containing u'' in the Lie matrices. Next we solve the first integral with respect to u''. Since the integral depends quadratically on u'', we get two solutions

```
sol1 = Solve[integ1,∂_{x,x}u[x]]
```

$$\Big\{\Big\{u''[x] \to -\sqrt{-\frac{1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2}\Big\},$$
$$\Big\{u''[x] \to \sqrt{-\frac{1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2}\Big\}\Big\}$$

The first of the two solutions is used to replace u'' in the Lie matrix

```
dma = Δ matrix /. sol1[[1]]
```

$$\{\{1,0,0,0\},\{0,1,0,0\},\{0,x,1,0\},$$
$$\{1,u'[x],-\sqrt{-\frac{1}{\beta^2}+c1^2\alpha^2\beta^2-2c1x\alpha^2\beta^2+x^2\alpha^2\beta^2},$$
$$-(\alpha\surd(1+\beta^2(-\frac{1}{\beta^2}+c1^2\alpha^2\beta^2-2c1x\alpha^2\beta^2+x^2\alpha^2\beta^2)))/$$
$$\left(\sqrt{-\frac{1}{\beta^2}+c1^2\alpha^2\beta^2-2c1x\alpha^2\beta^2+x^2\alpha^2\beta^2}\right)\}\}$$

The simplified Lie matrix again is used to calculate the second first integral of the third-order equation.

```
integ2 = FirstIntegral[x,u,dmat,3]==
c2/ /Simplify
```

$$\Big(\sqrt{-\frac{1}{\beta^2}+c1^2\alpha^2\beta^2-2c1x\alpha^2\beta^2+x^2\alpha^2\beta^2}$$
$$(-1+c1^2\alpha^2\beta^4-2c1x\alpha^2\beta^4+x^2\alpha^2\beta^4)+$$
$$\alpha\beta^2\sqrt{(c1-x)^2\alpha^2\beta^4}u'[x]\Big)/\big(\alpha\beta^2\sqrt{(c1-x)^2\alpha^2\beta^4}\big)==c2$$

As expected we find the integral depending only on first derivatives of u. Since this integral is linear in u', it is uniquely solvable in u'

```
sol2 = Solve[integ2,u'[x]]/ /Simplify
```

$$\{\{u'[x]\to c2-(\surd(-\frac{1}{\beta^2}+c1^2\alpha^2\beta^2-2c1x\alpha^2\beta^2+x^2\alpha^2\beta^2)$$
$$(-1+c1^2\alpha^2\beta^4-2c1x\alpha^2\beta^4+x^2\alpha^2\beta^4))/\big(\alpha\beta^2\sqrt{(c1-x)^2\alpha^2\beta^4}\big)\}\}$$

The resulting expression contains radicals of quadratic polynomials in x. Inserting this result into the Lie matrix, we can eliminate the derevatives u'. We find

```
dmat1 = dmat /. sol2 [[1,1]]  / /Simplify
```

$$\{\{1,0,0,0\},\{0,1,0,0\},\{0,x,1,0\},$$

$$\{1, c2 - \left(\sqrt{-\frac{1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2}\right.$$

$$\left.(-1 + c1^2\alpha^2\beta^4 - 2c1x\alpha^2\beta^4 + x^2\alpha^2\beta^4)\right)/\left(\alpha\beta^2\sqrt{(c1-x)^2\alpha^2\beta^4}\right),$$

$$-\sqrt{-\frac{1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2},$$

$$-\frac{\alpha\sqrt{(c1-x)^2\alpha^2\beta^4}}{\sqrt{-\frac{1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2}}\}\}$$

In a last step, we insert the Lie matrices into the third first integral depending on u and x

```
integ3 = FirstIntegral[x,u,dmat1,2]== c3/ /Simplify
```

$$\Big(- c2x^3\alpha^4\beta^8 + c2\alpha\beta^2\sqrt{(c1-x)^2\alpha^2\beta^4} -$$

$$c2x^2\alpha^3\beta^6\sqrt{(c1-x)^2\alpha^2\beta^4} + \sqrt{-\frac{1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2} -$$

$$2x^2\alpha^2\beta^4\sqrt{\frac{-1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2} +$$

$$c1^4\alpha^4\beta^8\sqrt{\frac{-1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2} -$$

$$4c1^3x\alpha^4\beta^8\sqrt{\frac{-1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2} +$$

$$x^4\alpha^4\beta^8\sqrt{\frac{-1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2} + 2c1x\alpha^2\beta^4\big(c2x\alpha^2\beta^4 +$$

$$c2\alpha\beta^2\sqrt{(c1-x)^2\alpha^2\beta^4} + 2\surd\big(\frac{-1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2\big) -$$

$$2x^2\alpha^2\beta^4\surd\big(\frac{-1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2\big)\big) +$$

$$c1^2\alpha^2\beta^4\big(- c2x\alpha^2\beta^4c2\alpha\beta^2\sqrt{(c1-x)^2\alpha^2\beta^4} -$$

$$2\sqrt{(\frac{-1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2)}+$$

$$6x^2\alpha^2\beta^4\sqrt{(\frac{-1}{\beta^2} + c1^2\alpha^2\beta^2 - 2c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2)})+$$

$$(c1-x)^2\alpha^4\beta^8 u[x]\Big)/((c1-x)^2\alpha^4\beta^8) == \ c3$$

The derived integral contains u linearly. In turn, we end up with a unique solution for the Kamke equation 7.13 which is

```
solution = Solve[integ3,u[x]]/ /Simplify
```

$$\{\{u[x] \to \ c3 - (\ c1^4\alpha^4\beta^8\sqrt{(\frac{-1}{\beta^2} + \ c1^2\alpha^2\beta^2 - 2\ c1x\alpha^2\beta^2 + x^2\ \alpha^2\beta^2)}-$$

$$4\ c1^3x\alpha^4\beta^8\sqrt{(\frac{-1}{\beta^2} + \ c1^2\ \alpha^2\beta^2 - 2\ c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2)}+$$

$$\sqrt{(-\frac{1}{\beta^2} + \ c1^2\alpha^2\beta^2 - 2\ c1x\ \alpha^2\beta^2 + x^2\alpha^2\beta^2)}(-1+x^2\alpha^2\beta^4)^2-$$

$$c2\alpha\beta^2(x^3\alpha^3\beta^6 - \sqrt{(\ c1-x)^2\ \alpha^2\beta^4} + x^2\alpha^2\beta^4\sqrt{(\ c1-x)^2\alpha^2\ \beta^4})+$$

$$c1^2\alpha^2\beta^4(2\sqrt{(\frac{-1}{\beta^2} + \ c1^2\ \alpha^2\beta^2 - 2\ c1x\alpha^2\beta^2 + x^2\alpha^2\beta^2)}(-1+3x^2\alpha^2\beta^4)-$$

$$c2\alpha\beta^2(x\alpha\beta^2 + \sqrt{(\ c1-x)^2\ \alpha^2\beta^4})) + 2\ c1x\alpha^2\beta^4$$

$$(-2\sqrt{(\frac{-1}{\beta^2} + \ c1^2\alpha^2\beta^2 - 2\ c1\ x\alpha^2\beta^2 + x^2\alpha^2\beta^2)}(-1+x^2\alpha^2\beta^4)+$$

$$c2\alpha\beta^2(x\alpha\beta^2 + \sqrt{(\ c1-x)^2\ \alpha^2\beta^4})))/((\ c1-x)^2\alpha^4\beta^8)\}\}$$

The solution depends on three constants $c1$, $c2$, and $c3$, all of which are constants of integration. The parameters α and β are the parameters of the original equation. To get a feeling how the solution evolves, we plot the solution for different parameter sets $c1$, $c2$, $c3$ at fixed α and β.

We note that the solution is explicitly represented by a complicated expression containing radicals and polynomials. This result is new since Kamke only offers a parametric representation of the solution. The example shows that we are able to arrive at a solution for higher order ODEs with Lie's procedure. *Mathematica* is not yet able to tread such types of equations.

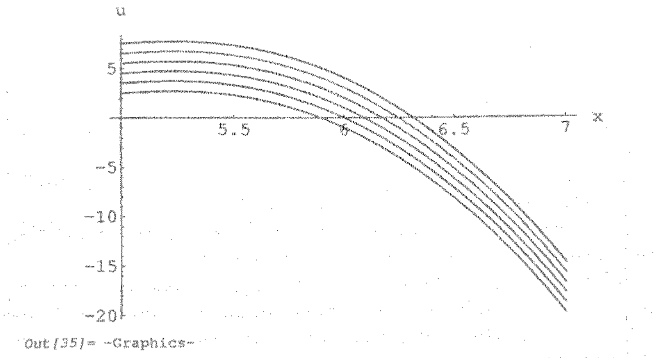

Fig. 2. Figure x represents the real valued solutions of the equation $\partial_{x,x}u[x]\partial_{\{x,3\}}u[x]-\alpha\sqrt{1+\beta^2(\partial_{x,x}u[x])^2}=0$. The different curves represent the solutions for values of $c3 \in \{0,1,2,3,4,5\}$ and fixed values for $c2 = 1/2, c1 = 1$. The parameters of the equation are $\alpha = 1$ and $\beta = 1/2$.

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6. Conclusions

We have demonstrated that the well known term of an integrating factor can be generalized by using Lie matrices. These matrices are the bases for the multiplier of differential equations. Knowing the multiplier, we are able to derive first integrals of motion by using integrations (quadratures). The derived integrals of motion allow to eliminate higher derivatives. This step is equivalent to a reduction of order in the derivatives. If we are able to construct n first integrals of an nth-order ODE, we are able to solve the equation implicitly. However, if we fail to determine all n first integrals, we can find a solution, if at least one first integral is known. Once a minimal number of symmetries of the equation is known, we are always able to arrive at a solution as demonstrated by examples for first-order, second-order, and third-order equations.

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Department of Mathematical
Physics University of Ulm
D-89069 Ulm, Germany
e-mail: Gerd.Baumann@physik.uni-ulm.de

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