Записки научных
семинаров ПОМИ
Том 258, 1999 г.

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SYMBOLIC COMPUTATION OF THE AVERAGED HAMILTONIAN, DESCRIBING THE MOTION OF ASTEROIDS UNDER THE INFLUENCE OF MAJOR PLANETS

ABSTRACT. The problem of developing a semianalytical theory of motion of minor planets is considered. The theory is constructed by means of Lie–series method using the Le Verrier expansions of planetary disturbing function. The averaging procedure is realized symbolically by means of computer algebra system MAPLE. The averaged Hamiltonian is used for symbolic computation of right parts of canonical differential equations of motion. These equations are integrated numerically by means of traditional numerical software.

One of the fascinating problems of modern dynamical astronomy is the study of chaotical behaviour of asteroids in the case of mean motion resonances with major planets. In order to predict the motion of asteroids on a large interval of time sometimes it is not very efficient to use the direct numerical integration of equations of motion in Cartesian coordinates. It is connected with the divergence of initially closed trajectories because of chaos, and therefore to get the qualitative picture of the dynamic behaviour one should integrate the system with the plenty nearly situated initial conditions. The corresponding numerical estimations are given in [1]. Therefore for investigation of behaviour of resonant asteroids it is preferable to use the semianalytical approach which is based on the numerical integration of the averaged equations of motion. The averaging procedure is connected with the elimination of short–periodic terms (which are not of interest for the dynamical evolution) from the initial Hamiltonian describing the motion of an asteroid under the influence of major planets. Such an averaging procedure can be realized, in particular, by means of Lie–series method [2, 3], and, as a result, we obtain the averaged Hamiltonian which contains only long–periodic, secular and resonant terms. It gives a possibility to increase substantially the step of integration. Moreover, the averaged Hamiltonian contains substantially less terms as the initial Hamiltonian because the number of short–periodic terms is much more as long–periodic, secular and resonant terms.

In the present work we average symbolically the Hamiltonian describing the motion of asteroids taking into consideration the terms of the first and the second order with respect to the mass of disturbing body. Partial derivatives of this Hamiltonian with respect to orbit elements correspond to the right parts of canonical equations of motion which can be integrated numerically for studying the evolution of asteroids.

Let us consider the classical Le Verrier expansion of planetary disturbing function [4]. Using the approach by Yuasa [5] we can express this disturbing function via orbit elements of a minor planet and a disturbing body. Hamiltonian of our problem can be written then in the following form:

$$H = H_0 + m' \sum_{i,j} L_i^{(j)} C_i \cos\left(i_1 l + i_2 l' + i_3 g + i_4 g' + i_5 h + i_6 h'\right), \quad (1)$$

where H_0 is the Hamiltonian of unperturbed motion, m' is the mass of disturbing body, C_i are some polynomials with respect to eccentricities and inclinations of the asteroid and the major planet, l, g, h, l', g', h' are the mean anomaly, argument of pericenter and longitude of the node of asteroid and disturbing body, respectively. Coefficients $L_i^{(j)}$, called Le Verrier coefficients, are defined as some functions of parameter $z = a/a'$, where a and a' are the semi-major axis of asteroid and disturbing planet, respectively:

$$L_i^{(j)} = L_i^{(j)}(z)$$

and can be calculated via Laplace coefficients [6]. We notice that Laplace coefficients themselves are determined by means of relation:

$$b_n^{(k)}(z) = \frac{\left(\frac{n}{2}, k\right)}{(1, k)} z^k F\left(\frac{n}{2}, \frac{n}{2} + k, k+1; z^2\right)$$

Here $\left(\frac{n}{2}, k\right)$ is the Appel symbol, F is the hypergeometric function.

Long–periodic and secular terms appear in Hamiltonian (1) in the case:

$$i_1 = i_2 = 0. \quad (2)$$

Resonant terms appear, if there are such integers i_1 and i_2 that

$$|i_1 \bar{n} + i_2 \bar{n}'| < \varepsilon, \quad (3)$$

where $\bar{n}$, $\bar{n}'$ is the mean motion of asteroid and major planet, respectively, ε is a small parameter.

We construct the averaged Hamiltonian in the following form:

$$F = H_0 + m' F_1^* + {m'}^2 F_2^* + \dots ,$$

where H_0 corresponds to the case of unperturbed motion, F_1^* contains secular, long–periodic and resonant terms corresponding to the first order with respect to the mass of disturbing body, F_2^* contains also only secular, long–periodic and resonant terms, but of the second order with respect to m'.

Unperturbed Hamiltonian H_0 is determined as follows:

$$H_0 = \frac{\mu^2}{2L^2},$$

where μ is the gravitational constant for the asteroid under consideration, $L = \sqrt{\mu a}$ is a Delaunay variable.

In order to find the function F_1^* we separate from the initial Hamiltonian all secular, long–periodic and resonant terms, satisfying to conditions (2) and (3). The sum of such terms gives us the function $m' F_1^*$.

The function ${m'}^2 F_2^*$ is defined as the result of calculation of Poisson brackets:

$$\frac{1}{2}\{\mathcal{H}, S_1\} = \frac{1}{2}\left(\frac{\partial \mathcal{H}}{\partial L}\frac{\partial S_1}{\partial l} + \frac{\partial \mathcal{H}}{\partial G}\frac{\partial S_1}{\partial g} + \frac{\partial \mathcal{H}}{\partial \Theta}\frac{\partial S_1}{\partial h} - \right.$$
$$\left. - \frac{\partial \mathcal{H}}{\partial l}\frac{\partial S_1}{\partial L} - \frac{\partial \mathcal{H}}{\partial g}\frac{\partial S_1}{\partial G} - \frac{\partial \mathcal{H}}{\partial h}\frac{\partial S_1}{\partial \Theta} \right), \tag{4}$$

from which we separate all secular, long–periodic and resonant terms. Here $\mathcal{H} = H - H_0 + m' F_1^*$ and L, G, Θ, l, g, h are Delaunay variables, S_1 is the generating function of canonical transformation of Lie–series method [2]. This generating function can be found as the solution of Hamilton–Jacobi equation:

$$-\bar{n}\frac{\partial S_1}{\partial l} - \frac{\partial S_1}{\partial t} + F_s = 0, \tag{5}$$

where

$$F_s = H - H_0 - m' F_1^* ,$$

t is the time.

For solving Hamilton–Jacobi equation (4) we use the method of variation of arbitrary constants. The algorithm is as follows.

At first, we represent function F_s in the form:

$$F_s = m' \sum_{i,j} L_i^{(j)} C_i \cos\big(k_1\lambda + k_2\lambda' + k_3\omega + k_4\omega' + k_5\Omega + k_6\Omega'\big) \ , \quad (6)$$

where

$$\lambda = l + g + h, \qquad \lambda' = l' + g' + h',$$

$$\omega = g + h, \qquad \omega' = g' + h',$$

$$\Omega = h, \qquad \Omega' = h'.$$

After this, we introduce new functions p', q', r', v' using the following relations:

$$\begin{aligned} p' &= e' \cos\omega' \ , \qquad & q' &= e' \sin\omega' \ , \\ r' &= s' \cos\Omega' \ , \qquad & v' &= s' \sin\Omega' \ , \end{aligned} \quad (7)$$

where e' is the eccentricity of disturbing planet, s=sin$i$ and i is the inclination of disturbing planet.

Functions p', q', r', v' describe the inequalities in motion of the disturbing body and can be presented in the form of fragments of Fourier series with respect to time. The theories of motion of such major planets as, for example, Jupiter and Saturn, show that in the frames of accuracy of semianalytical theory of the second order with respect to m' it is enough to take into consideration only two harmonics for description of inequalities in motion of disturbing planets [5,7]:

$$\begin{aligned} p' &= \nu_1 \cos(g_1 t + \beta_1) + \nu_2 \cos(g_2 t + \beta_2) \,, \\ q' &= \nu_1 \sin(g_1 t + \beta_1) + \nu_2 \sin(g_2 t + \beta_2) \,, \\ r' &= \mu_1 \cos(f_1 t + \gamma_1) + \mu_2 \cos(f_2 t + \gamma_2) \,, \\ v' &= \mu_1 \sin(f_1 t + \gamma_1) + \mu_2 \sin(f_2 t + \gamma_2) \,. \end{aligned} \quad (8)$$

Numerical values of constants ν_1, ν_2, μ_1, μ_2, g_1, g_2, f_1, f_2, β_1, β_2, γ_1, γ_2 can be taken, in particular, from the theory of Brouwer and van Woerkom [7].

Using relations (6) and (7) we rewrite function (5) in the form

$$F_s = A(t)\cos(k_1\lambda) + B(t)\sin(k_1\lambda) \,, \quad (9)$$

where k_1 is a constant. Putting now $\lambda' = \lambda_0{}' + \bar{n}'t$, where $\lambda_0{}'$ is the value of λ' at t=0, we compute symbolically the following integrals:

$$A_s = \int \left[A\left(t\right) \cos\left(k_1 \bar{n} t\right) + B\left(t\right) \sin\left(k_1 \bar{n} t\right) \right] dt \ ,$$

$$B_s = \int \left[B\left(t\right) \cos\left(k_1 \bar{n} t\right) - A\left(t\right) \sin\left(k_1 \bar{n} t\right) \right] dt \ , \tag{10}$$

Finally, we find the following expression for generating function S_1:

$$S_1 = A_s \cos\left(k_1 \left(\lambda - \bar{n} t\right)\right) + B_s \sin\left(k_1 \left(\lambda - \bar{n} t\right)\right) \ . \tag{11}$$

We notice that quite generally the Hamilton–Jacobi equation (4) can be solved in the case of conditions (6) and (7) symbolically by means of the standard MAPLE–function `pdesolve`. But in this case we obtain the result in very cumbersome form and need to use a lot of computer memory and computing time. Our computer experiments show that using our algorithm we are able to solve Hamilton–Jacobi equation (4) under conditions (6)–(7) much more faster and more efficiently from the point of view of economy of computer memory. Moreover, the solution is obtained in substantially more compact form.

The computational scheme described above to compute the averaged Hamiltonian in symbolic form is realized as a package of computer programs designed in computer algebra system MAPLE V [8]. This package contains the following programs:

1) program for computation of Le Verrier expansion of the disturbing function via orbit elements;

2) program for evaluation of Le Verrier and Laplace coefficients and their derivatives (here we have used relations given in [4,6]);

3) program for selection of secular, long–periodic and resonant terms in the initial Hamiltonian in order to obtain the function $m'F_1^*$;

4) program for symbolic solution of Hamilton–Jacobi equation (4) using the method of variation of arbitrary constants (the corresponding formulae are presented above);

5) program for symbolic computation of Poisson brackets $\{\mathcal{H}, S_1\}$;

6) program for determination of new secular, long–periodic and resonant terms of the second order with respect to the mass of disturbing body;

7) program for translation of the analytical expression of the averaged Hamiltonian (and its derivatives with respect to orbit elements) into

Fortran for numerical integration of the averaged canonical equations of motion by means of traditional numerical software.

The test computer experiments have been done in MAPLE V (Release 4) on SUN–Workstation under UNIX operating system. These experiments have shown the high efficiency of our algorithms and programs and the possibility to use them for developing an efficient semianalytical theory of motion of minor planets.

The analytical expression of averaged Hamiltonian is used in order to obtain symbolically the right parts of canonical equations of motion which we integrate numerically by means of Bulirsch–Stoer method [9]. We suppose to apply our semianalytical theory to study the evolution of asteroids in the belt between 4:1 and 5:1 mean motion resonance with Jupiter and, in particular, to study the effects of so called mixed resonances, when there are such integers i_1, i_2, i_3, that

$$|i_1\bar{n}_A + i_2\bar{n}_J + i_3\bar{n}_S| < \varepsilon,$$

where $\bar{n}_A$, $\bar{n}_J$, $\bar{n}_S$ are mean motions of asteroid, Jupiter and Saturn, respectively, ε is a small parameter. In the future we suppose also to use our theory for studying the fine structure of main resonances in the asteroid belt.

The authors are very grateful to Prof. M. Soffel and Dr. S. Klioner for the help in our research and for useful discussions. A. Vakhidov thanks also the German Academician Exchange Service (DAAD) for the financial support of this work and the possibility to realize it in Germany in the frames of a one–year schoolarship.

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Поступило 10 июня 1999