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PROBLEMS OF STABILITY OF DYNAMIC SYSTEMS AND COMPUTER ALGEBRA

Abstract. This paper presents examples of some problems of stability of motion, for solving which computer algebra systems (CAS) have been used. We have experience of developing and applying problem-oriented systems of symbolic computations and applied software packages in solving problems of dynamics of multi-body systems [1, 2]. The algorithms under consideration are implemented completely or partially with the aid of state-of-the-art CAS. They are intended for inclusion in the package of symbolic computation "Stability" [2].

1. Constructing Lyapunov functions and first integrals

Consider the differential equations describing the motion of mechanical systems. The equations have the form:

$$M\ddot{x} + D\dot{x} + 2G\dot{x} + Px + Kx = Q(x, \dot{x}) \tag{1.1}$$

where $M = M^T > 0$ $D = D^T$ $G = -G^T$ $P = -P^T$ $K = K^T$ are constant $n \times n$ -matrices, $\ddot{x},\ \dot{x},\ x$ are the matrices $1 \times n$ of generalized coordinates, velocities and accelerations (in deviations from steady motion); D K are the matrices of dissipative and potential linear forces; G P are the matrices of gyroscopic and nonconservative linear forces; $Q(x, \dot{x})$ is matrix nonlinear with respect to $x,\ \dot{x}$ forces; $Q(0, 0) = 0$. If the system is a controlled one, $Q(x, \dot{x})$ can be linear.

For a sufficiently wide class of nonlinear forces, programs using CAS for decomposed of forces have been developed. The position forces are decomposition into potential $Q^p = \partial U(x)/\partial x$ and nonconservative $R(x)$ satisfying the condition $R^T x = 0$. The forces dependent on the velocities of forces, are resolved into dissipative $D = \partial \tilde{F}(x, \dot{x})/\partial \dot{x}$ (with a positive or negative dissipation) and gyroscopic Γ , for which $\Gamma^T \dot{x} = 0$ [3].

Consider the problem of investigation of stability or stabilization of an unstable equilibrium position $x = \dot{x} = 0$ of the system (1.1). If the system (1.1) is not critical in Lyapunov's sense, the stability problem is solved completely by the linear part with respect to the roots of the characteristic equation. In critical cases, only the Lyapunov's second method is used. But then there arises the problem of the proximity of sufficient stability conditions to necessary conditions, i.e. of a "good" choice of the Lyapunov function.

Definition. *The relation* $df(\dot{x}, x)/dt = F_1(\dot{x}, x)$ *is said to be the differential consequence of equations* (1.1) *if it holds for all solutions of the system.*

In the theorems of the Lyapunov's second method, we shall use differential consequences of equations (1.1). One of the algorithms for constructing differential consequences is described in [2] and involves the following procedure: we multiply (1.1) on the left by matrices of the form $x^T L_1$, $\dot{x}^T L_2$, $\ddot{x}^T L_3$, and transform the final expressions by selecting the symmetric parts. The matrices L_i are equal to those of the input equation or their combinations.

Example 1.1. Let $L_3 = GM^{-1}$, $L_2 = KM^{-1}$, and $L_1 = GM^{-1}$. After selecting the full derivative, we obtain three consequences. We combine them into one bundle:

$$\frac{d}{dt}(\dot{x}^T(4G^T M^{-1}G + K)\dot{x} + x^T(KM^{-1}K - 2GM^{-1}G)x+$$

$$+2x^T(KM^{-1}G + PM^{-1}G - G)\dot{x}) =$$

$$= -2\dot{x}^T(GM^{-1}G^T M^{-1}D + KM^{-1}D + G^T M^{-1}P)\dot{x}-$$

$$-2x^T(G^T M^{-1}K + G^T M^{-1}P)x + 2x^T(2KM^{-1}G^T M^{-1}D-$$

$$-2PM^{-1}GM^{-1}D - G^T M^{-1}D - P^T M^{-1}K)\dot{x} \ . \tag{1.2}$$

Now, by W we denote the form in (1.2) on the right. If in (1.2) the form under the sign of derivative has the fixed sign and the form W has the fixed sign with an opposite sign, in accordance with Lyapunov's theorem the trivial solution of the system is asymptotically stable irrespective of nonlinear forces. Consequently, theorem 1.1 holds.

Theorem 1.1. *If*

$$4G^T M^{-1}G + K - (KM^{-1}G + PM^{-1}G - G)^T \, *$$

$$* \, (KM^{-1}K - 2GM^{-1}G)^{-1}(KM^{-1}G + PM^{-1}G - G) > 0$$

$$A = (G^T M^{-1}K + G^T M^{-1}P + KM^{-1}G + PM^{-1}G) > 0$$

$$2(GM^{-1}G^T M^{-1}D + KM^{-1}D + G^T M^{-1}P) -$$

$$-(2KM^{-1}G^TM^{-1}D - 2PM^{-1}GM^{-1}D - G^TM^{-1}D - P^TM^{-1}K)^T *$$
$$* A^{-1}(2KM^{-1}G^TM^{-1}D-$$
$$-2PM^{-1}GM^{-1}D - G^TM^{-1}D - P^TM^{-1}K) > 0$$

then the solution $\dot{x} = x = 0$ *of the system* (1.2) *is asymptotically stable with respect to* $\dot{x}$ x *for any nonlinear forces.*

Noteworthy, then the characteristic equation of the system has all roots with the negative real part. The Routh–Hurwitz's conditions are satisfied. If the form W in (1.2) is of constant signs (the sign opposite to that of the form on the left), and the set $\mathcal{L} = \{ x, \dot{x} : W = 0 \}$ does not contain the full trajectories except for $x = \dot{x} = 0$, the trivial solution of the linear system (1.1) is asymptotically stable. In [2], some algorithms for verifying the presence or absence of the full trajectories on the set are described.

As investigations show, in critical cases sufficiently soft stability conditions are obtained when the first integral of the differential equations of motion is taken as the Lyapunov function.

Consider the description of another algorithm for constructing differential consequences and, in particular, the first integrals for some variants of the system (1.1). We select a function

$$V = \dot{x}^TN\dot{x}+x^TLx+x^TB^T\dot{x}+\dot{x}^TBx+F(x,\dot{x}) \quad L = L^T \quad N = N^T \quad (1.3)$$

and calculate from it the derivative by equations (1.1):

$$\frac{dV}{dt} = \dot{x}(-NM^{-1}(D+2G) - (D-2G)M^{-1}N + B + B^T)\dot{x}+$$
$$+x^T((P-K)M^{-1}B - B^TM^{-1}(P+K))x + \dot{x}(L - NM^{-1}(K+P)-$$
$$(D-2G)M^{-1}B)x + x^T((L + (P-K)M^{-1}N - B^TM^{-1}(D+2G))\dot{x}+$$
$$\dot{x}^T(\frac{\partial F}{\partial x} + +2NM^{-1}Q - (D-2G)M^{-1}\frac{\partial F}{\partial \dot{x}})+$$
$$x^T(2B^TM^{-1}Q - (K-P)M^{-1}\frac{\partial F}{\partial \dot{x}}) + Q^TM^{-1}\frac{\partial F}{\partial \dot{x}} \; . \quad (1.4)$$

Further manipulations with (1.3) and (1.4) depend on the problem chosen. The three procedures are possible.

1). Differential consequences of equations (1.1) are constructed by putting N L B equal to a unit matrix or matrices of the initial system (or their combinations), $F = 0$.

2). The first integral of the quadratic type for the system (1.1) for $Q = 0$ is find. Put $F = 0$ in (1.4). By equating the matrix of all quadratic and bilinear forms in (1.4) to zero, we obtain the system of matrix equations for matrices N and B of dimension $n \times n$:

$$NM^{-1}(D+2G) + (D-2G)M^{-1}N - B - B^T = 0\ ; \qquad (1.5)$$

$$(K-P)M^{-1}B = -B^T M^{-1}(K+P)\ ; \qquad (1.6)$$

$$(K-P)M^{-1}N - NM^{-1}(K+P) =$$

$$= (D-2G)M^{-1}B - B^T M^{-1}(D+2G). \qquad (1.7)$$

The matrices N and B obtained are used to construct the matrix

$$L = 1/2((K-P)M^{-1}N + NM^{-1}(K+P)+$$

$$+(D-2G)M^{-1}B + B^T M^{-1}(D+2G))$$

will be constructed.

The system (1.5)–(1.7) has the trivial solution $N = 0$ $B = 0$ ($B^T = 0$). It is an overdetermined system of homogeneous linear equations and can therefore have a unique trivial solution or a set of solutions. Conditions of existence of a nontrivial solution and conditions of compatibility will impose restrictions on the matrices of the initial system, for example, on the matrices D and G if the problem of stabilization of the system with position forces by dissipative and gyroscopic forces is solved.

Example 1.2. Let $D = 0$ $P = 0$ $Q = 0$. Then we have a system with potential and gyroscopic forces. Equations (1.5) - (1.7) allow to obtain:

solution 1: $B = 0$ $N = E$ which allow the integral of energy $\dot{x}^T M\dot{x} + x^T Kx = \text{const}$ for the linear system (1.1);

solution 2: $N = K - 4GM^{-1}G$ $B = 2G^T K$ which defines the first integral $\dot{x}^T K\dot{x} + (Kx+2G\dot{x})^T M^{-1}(Kx+2G\dot{x}) = \text{const}$;

solution 3: let $K = \beta M$, then $B = \alpha G$, $N = M^{-1}$ (α and β being constant); the linear system (1.1) has the integral $\dot{x}^T M^{-1}\dot{x} + x^T(\beta E + 2\alpha GG)x + 2\alpha(x^T GG\dot{x} + \dot{x}^T GGx) = \text{const}$.

In the general case, it becomes necessary to examine obvious variants which lead to matrix equations of the form:

$$AX = XA^T \ ; \tag{1.8}$$

$$A_1 X = -XA_1^T . \tag{1.9}$$

For equations (1.8), a symmetric solution for X always exists; the skew-symmetric solution exists only if A has divisible eigenvalues, to which there correspond simple elementary divisors and/or various Jordan blocks. For (1.9), the nontrivial solution exists if the roots of the characteristic polynomial $f(\lambda)$ of the matrix A_1 include zero roots and/or $f(\lambda)$ contains only even degrees of λ.

Variant 2.1). $B = 0$. Then N must be the solution of equations (1.8)–(1.9) for $A = (K - P)M^{-1}$ and $A_1 = (D - 2G)M^{-1}$.

Example 1.3). If $D = 0$ $G = 0$, then for a linear system with position forces there exist $n_1 \geq n$ quadratic integrals

Variant 2.2). $N = 0$. Then from (4) we have $B = -B^T$, and this skew-symmetric matrix should be the solution of equations (1.8) with the matrix $A = (K - P)M^{-1}$ and (1.9) with the matrix $A_1 = (D-2G)M^{-1}$. If for (1.8) we have found a skew-symmetric solution then, after substituting it into (1.9), we obtain the conditions of compatibility, and if these are satisfied, the first integral $V = x^T B\dot{x}$ exists. We solve the conditions of compatibility for the elements of the matrices D or G .

Variant 2.3). $B = -B^T$. $B \neq 0$ $N \neq 0$. The matrix $N \neq 0$ if the characteristic polynomial $f(\lambda)$ of the matrix $A_1 = (D - 2G)M^{-1}$ contains only even degrees of λ and/or does not contain a constant term. Therefore, first we compute and analyze the characteristic polynomial of the matrix $(D - 2G)M^{-1}$, determine the skew-symmetric solution of the system (1.6) and check (1.7) for compatibility.

Example 1.4. Let $K = 0$ $P = 0$ $Q = 0$. Equations (1.5)–(1.6) have the solution $B = G$ $N = M$. If the condition $DM^{-1}G + GM^{-1}D = 0$ is satisfied, then the system (1.1) has the first integral $\dot{x}^T M\dot{x} - 2x^T GM^{-1}Gx + 2\dot{x}Gx = \text{const}$.

Remark 1. If some matrices in (1.1) are absent, it makes sense, first, to turn to the matrix equations (1.5)–(1.7) before calculating the elements of the matrices N and B.

Remark 2. When investigating stability, there is no need to determine integrals of fixed sign only. To construct a fixed sign bundle of two integrals having alternating signs, it is possible to Finsler's theorem. Furthermore, it is often sufficient to find conditions of existence of such a bundle.

The integral of a linear system can also be used as the Lyapunov function for a nonlinear system. The value of its derivative due to the nonlinear system can be determined by (1.4).

3). The first integral of the form (1.3) for the nonlinear system (1.1) is to be found.

To the set of equations (1.5)–(1.7) it is necessary to add:

$$\frac{\partial F}{\partial x} + 2NM^{-1}Q - (D-2G)M^{-1}\frac{\partial F}{\partial \dot{x}} = 0\ ; \qquad (1.10)$$

$$2B^T M^{-1}Q - (K-P)M^{-1}\frac{\partial F}{\partial \dot{x}} = 0\ ; \qquad (1.11)$$

$$Q^T M^{-1}\frac{\partial F}{\partial \dot{x}} = 0. \qquad (1.12)$$

These equations serve to construct $F(x,\dot{x})$, if any, or to determine Q, for which (1.3) is an integral. The function $F(x,\dot{x})$ must satisfy the conditions:

$$\frac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_j} = \frac{\partial^2 F}{\partial \dot{x}_j \partial \dot{x}_i}\ ; \qquad \frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i}\ ; \qquad \frac{\partial^2 F}{\partial \dot{x} \partial x} = \frac{\partial^2 F}{\partial x \partial \dot{x}}\ ;$$

$$(\ i,\ j\ = 1, \ldots, n\ ). \qquad (1.13)$$

Variant 3.1). If $F = F(x)$, then from (1.11) it follows that $Q^T M^{-1} B = 0$, equation (1.12) is satisfied, and the conditions (1.13) lead to the requirement:

$$NM^{-1}\frac{\partial Q}{\partial x} = (\frac{\partial Q}{\partial x})^T M^{-1}N. \qquad (1.14)$$

If the condition (1.14) holds for a given Q, then

$$F(x) = -2\int Q^T M^{-1} N dx \qquad (F(0)=0)$$

Variant 3.1.1). Let $B = 0$. The matrix N will be determined under the conditions of variant 2.1 from the system (1.5)–(1.7) and (1.14).

Variant 3.1.2). Let $B \neq 0$. The matrix N will be determined from the system of equations (1.5) - (1.7), (1.14) and $Q^T M^{-1} B = 0$.

Variant 3.2). Let $F = F(x, \dot{x})$.

Variant 3.2.1). If $det(K+P) \neq 0$ $det(D-2G) \neq 0$, then from (1.11) and (1.10) we have

$$M^{-1} \frac{\partial F}{\partial \dot{x}} = 2(K-P)^{-1} B^T M^{-1} Q \ ;$$

$$\frac{\partial F}{\partial x} = -2NM^{-1}Q + 2(D-2G)(K-P)^{-1}B^T M^{-1} Q.$$

The condition (1.12) hold due to (1.6). From (1.10) and (1.13) we have

$$B^T M^{-1} \frac{\partial Q}{\partial \dot{x}}(K+P) = (K-P)(\frac{\partial Q}{\partial \dot{x}})^T M^{-1} B \ ;$$

$$(N+(D-2G)(K-P)^{-1}B^T)\frac{\partial Q}{\partial x} = (\frac{\partial Q}{\partial x})^T (N + B^T (K+P)^{-1}(D+2G)) \ ;$$

$$(N+(D-2G)(K-P)^{-1}B^T)\frac{\partial Q}{\partial \dot{x}} = 2(K-P)^{-1}B^T \frac{\partial Q}{\partial x}.$$

When these conditions hold, we have

$$F(x,\dot{x}) = \int (-2NM^{-1}Q + 2(D-2G)(K-P)^{-1}B^T M^{-1}Q)^T dx +$$

$$+2((K-P)^{-1}B^T M^{-1} Q)^T d\dot{x} \qquad (F(0,0)=0).$$

When $det(K+P) \neq 0$ we select the solution $B = 0$ then $M^{-1}\partial F \partial \dot{x} = 0$; we obtain variant 3.1.1.

Variant 3.2.2). If $\det(K+P) = 0$, then the condition of compatibility (1.10)–(1.12) is either $Q^T M^{-1} B = 0$ or $\operatorname{rank}(K+P) = \operatorname{rank}(K+P, -2Q^T M^{-1} B)$.

Variant 3.2.3). If $det(D-2G) = 0$, then either $\partial F/\partial x = -2NM^{-1}Q$ or $\operatorname{rank}(D-2G) = \operatorname{rank}(D-2G \ \ 2NQ + \partial F/\partial x)$. In addition, (1.11)–(1.13) must be satisfied.

The calculated first integrals (or their bundles) can be used to construct Lyapunov functions in proving the stability of the trivial solution for the system (1.1) if Sylvester's inequalities of sing-definiteness of the quadratic part of the integral (1.3) with respect to all variables of the problem are satisfied.

Example 1.5. Let $D = G = 0$ $Q = Q(x)$ $(M = E)$, i.e. let only position forces act in the system.

Proposition. *Let* N *satisfy* (1.9), *and let the conditions* (1.13) *be satisfied when* $B = 0$ $\partial F/\partial \dot{x} = 0$. *Then the system allows for the first integral* V *of the form* (1.3). *Its property of having fixed sign depends on the quadratic part of* V. *The conditions* $N > 0$ (< 0) *and* $L = 1/2((K - P)N + N(K + P)) > 0$ (< 0) *are sufficient conditions of stability with respect to* x, $\dot{x}$ *of the trivial solution of both the linear and nonlinear system* (1.1) *when selecting the nonlinearity* Q, *determined by the conditions* (1.14). *If also in* (1.1) $Q = Q(\dot{x})$ *when* $\dot{x}^T LQ < 0$ *the trivial solution is stable with respect to* x *and asymptotically stable with respect to* $\dot{x}$.

Having investigated for this case equations (1.5)–(1.7), (1.10)–(1.11), we obtain the theorem:

Theorem 1.2. *If the linear system* (1.1) $(D = G = 0)$ *allows for a quadratic first integral with a diagonal matrix for* $\dot{x}$, *then the requirements of the property of sing-definiteness of this integral with respect to* $\dot{x}$, x *are the sufficient conditions of stability of the trivial solution of the nonlinear system* (1.1), *in which the nonlinearities* $Q = \partial U/\partial x$, *where* $U(x) = \sum_{i=1}^{n} U_i(x_i)$.

2. Problem with perturbations

Let the objects of the investigation be described by ordinary differential equations of the form

$$\dot{x} = X(x) \quad x \in U \subset R^n,$$

with smooth right-hand sides.

For such equations, we shall consider the problem of conservation of the first integrals of a definite structure in the case of perturbations on the right-hand sides:

$$\dot{x} = X(x) + f(x), \quad x \in R^n.$$

In this case, the perturbations are usually restricted by different conditions: potentiality, algebraicity, linearity, etc. They can be given specifically or can belong to a definite class of functions. The solution of such a problem makes it possible, in particular, to define new, completely integrable cases similar to known ones and obtain the Lyapunov functions in the form of first integrals.

In the general form, the problem under consideration leads to the need to solve a certain system of partial differential equations. When the structure of perturbations of the equations and the first integral is given, the problem can be reduced to the algebraic one. Typical problems arising in this case will be illustrated on a example (Kovalevskya's top). Let us introduce perturbations into the equations of motion of a rigid body with one fixed point $(A = B = 2C\ ;\ z_0 = y_0 = 0)$ in the Euler–Poisson form:

$$2\dot{p} = qr + f_1\ ;\quad 2\dot{q} = -rp + x_0\gamma_3 + f_2\ ;\quad \dot{r} = -x_0\gamma_2 - f_3\ ;$$
$$\dot{\gamma}_1 = r\gamma_2 - q\gamma_3\ ;\quad \dot{\gamma}_2 = p\gamma_3 - r\gamma_1\ ;\quad \dot{\gamma}_3 = q\gamma_1 - p\gamma_2. \qquad (2.1)$$

Here $p,\ q,\ r$ stand for the projection of the angular velocity of a the body onto the principal axes of inertia, bounding with this body, $\gamma_1,\ \gamma_2,\ \gamma_3$ are the direction cosines of the vertical in the above axes, $x_0,\ y_0,\ z_0$ are the coordinates of the mass center of the body (normalized), and $f_1,\ f_2,\ f_3$ are the perturbations of the moments of forces which effect on the body. We shall formulate the problem of conservation of Kovalevskya's integral in the case of corresponding perturbations. The structure of the integral will be assigned as follows:

$$V = (p^2 - q^2 - x_0\gamma_1 + \Phi_1)^2 + (2pq - x_0\gamma_2 + \Phi_2)^2 = k^2. \qquad (2.2)$$

If no restrictions are imposed on $f_1,\ f_2,\ f_3,\ \Phi_1,\ \Phi_2$, then in order to find them we employ the following partial differential equation:

$$\frac{dV}{dt} = 2\{[-(p^2 - q^2 - x_0\gamma_1)\Phi_2 +$$
$$+(2pq - x_0\gamma_2)\Phi_1]r + [(p^2 - q^2 - x_0\gamma_1)p + (2pq - x_0\gamma_2)q]\ f_1 +$$
$$+[(2pq - x_0\gamma_2)p - (p^2 - q^2 - x_0\gamma_1)q]f_2 + \Phi_1(pf_1 - qf_2) + \Phi_2(qf_1 + pf_2) +$$
$$+(p^2 - q^2 - x_0\gamma_1)\frac{d\Phi_1}{dt} + (2pq - x_0\gamma_2)\frac{d\Phi_2}{dt} + \Phi_1\frac{d\Phi_1}{dt} + \Phi_2\frac{d\Phi_2}{dt}\} = 0.$$

The left-hand side of the equation is obtained by differentiation of (2.2) due to the differential equations (2.1); $d\Phi_1/dt$ and $d\Phi_2/dt$ denote derivatives of Φ_1 and Φ_2 due to equations (2.1). We limit ourselves to a linear dependence on $\gamma_1,\ \gamma_2,\ \gamma_3$ for unknown functions:

$$F_1 = a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3\ ;\quad f_2 = b_1\gamma_1 + b_2\gamma_2 + b_3\gamma_3\ ;\quad f_3 = c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3\ ;$$
$$\Phi_1 = m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3\ ;\quad \Phi_2 = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3\ ; \qquad (2.3)$$

and we require that the field of forces, which effect on the body, be a potential one. We substitute the expressions (2.3) into equation (2.2), equate the coefficients of all monomials with variables for the problem $(p,\ q,\ r,\ \gamma_1,\ \gamma_2,\ \gamma_3)$ to zero, and obtain a system of linear algebraic equations for determining the nine unknown parameters. Obviously, the operation of substitution and selection of the above equations is quite bulky even in the above problem, and without using computer algebra systems it would require much time and attention. We do not reproduce above equations here because of their complexity but have to note that one of the solutions of this system (on account of the requirement that forces must be potential) has the form:

$$f_1 = -y_0\gamma_3\ ;\quad f_2 = 0\ ;\quad f_3 = y_0\gamma_1\ ;\quad \Phi_1 = y_0\gamma_2\ ;\quad \Phi_2 = y_0\gamma_1 .$$

Here a constant parameter y_0 can be interpreted as a coordinate of the mass center of the body on the axis y (normalized). When deriving the solutions in problems with several integrals by such a technique, it is always necessary to verify whether the result obtained is trivial. For example, whether the new equations and the new first integral obtained are the "old" modified ones due to a modification (transformation) of the reference frame. For identifying such situations, it is necessary to take into consideration the presence of symmetries in the initial problem. In our case such a symmetry is present due to the equality of inertia moments of the body with respect to the axes x and y, which corresponds to the invariance of the mechanics problem with respect to a group of rotation about the axis z. The existence of such an invariance can be verified with the aid of CAS. In fact, the perturbed dynamic equations and integrals obtained

$$2\dot{p} = qr - y_0\gamma_3\ ;\quad 2\dot{q} = -rp + x_0\gamma_3\ ;\quad \dot{r} = -x_0\gamma_2 + y_0\gamma_1\ ;$$

$$2H = 2(p^2+q^2)+r^2+2x_0\gamma_1+2y_0\gamma_2 = 2h\ ;\ \ V_1 = 2(p\gamma_1+q\gamma_2)+r\gamma_3 = m\ ;$$

$$V_2 = (p^2+q^2-x_0\gamma_1+y_0\gamma_2)^2 + (2pq - x_0\gamma_2 - y_0\gamma_1)^2 = k^2\ ;$$

$$V_3 = {\gamma_1}^2 + {\gamma_2}^2 + {\gamma_3}^2 = 1\ ;$$

can be treated as the equations and integrals of the initial mechanics problem after a suitable rotation of the coordinate axes. Noteworthy, the derivation of equations and integrals by transformations allowed by symmetries of the physical problem, can be used to obtain new systems with the first integrals, if, for example, a "decomposition" of the initial force field into parts is possible. In the above example, it is possible "to split" the existing moment of gravity into two moments: the moment of gravity, and the "magnetic moment". After that, x_0 is understood, as before, the coordinate of the barycenter of the body, and $y_0 = b$ can be treated as the magnetic strength component on the axis y of the body, and the magnetic field vector with respect to the axes $x,\ y,\ z$ will be determined by the direction cosines ($\delta_1,\ \delta_2,\ \delta_3$). As a result, we obtain the equations of motion of the body with one fixed point in gravitational and magnetic fields

$$2\dot{p} = qr - b\delta_3\ ;\quad 2\dot{q} = -rp + x_0\gamma_3\ ;\quad \dot{r} = -x_0\gamma_2 - b\delta_1\ ;$$

$$\dot{\gamma}_1 = r\gamma_2 - q\gamma_3\ ;\quad \dot{\gamma}_2 = p\gamma_3 - r\gamma_1\ ;\quad \dot{\gamma}_3 = q\gamma_1 - p\gamma_2\ ;$$

$$\dot{\delta}_1 = r\delta_2 - q\delta_3\ ;\quad \dot{\delta}_2 = p\delta_3 - r\delta_1\ ;\quad \dot{\delta}_3 = q\delta_1 - p\delta_2.$$

They allow for the first integrals [4]

$$V = (p^2 - q^2 - x_0\gamma_1 + b\delta_2)^2 + (2pq - x_0\gamma_2 - b\delta_1)^2 = k^2\ ;$$

$$2H = 2(p^2 + q^2) + r^2 + 2x_0\gamma_1 + 2b\delta_2 = 2h \quad (\text{ integral of energy }).$$

Systems with the first integrals, different from the initial ones, can thus be obtained. For such systems, it is possible to select sets of invariant manifolds of steady motions (IMSM). For this purpose, solutions of stationarity conditions for some or all independent integrals from algebra of first integrals of the problem are investigated. Stability conditions of selected IMSM can be obtained on the basis of the Lyapunov's second method. In our example, the stationarity conditions of the first integral V :

$$\frac{\partial V}{\partial p} = 4\{(p^2 - q^2 - x_0\gamma_1 + b\delta_2)p + (2pq - x_0\gamma_2 - b\delta_1)q\} = 0\ ;$$

$$\frac{\partial V}{\partial q} = 4\{-(p^2 - q^2 - x_0\gamma_1 + b\delta_2)q + (2pq - x_0\gamma_2 - b\delta_1)p\} = 0\ ;$$

$$\frac{\partial V}{\partial \gamma_1} = -2\ x_0(p^2 - q^2 - x_0\gamma_1 + b\delta_2) = 0\ ;$$

$$\frac{\partial V}{\partial \gamma_2} = -2\ x_0(2pq - x_0\gamma_2 - b\delta_1) = 0\ ;$$

$$\frac{\partial V}{\partial \delta_1} = -2b(2pq - x_0\gamma_2 - b\delta_1) = 0\ ;\quad \frac{\partial V}{\partial \delta_2} = -2b(p^2 - q^2 - x_0\gamma_1 + b\delta_2) = 0$$

determine the invariant manifold

$$(p^2 - q^2 - x_0\gamma_1 + b\delta_2) = 0\ ;\quad (2pq - x_0\gamma_2 - b\delta_1) = 0 \tag{2.4}$$

similar to invariant Delaunay's manifold in Kovalevskya's gyroscope. This invariant manifold is stable. In fact, the integral V in the neighbourhood of this manifold has the form:

$$\Delta V = u_1^2 + u_2^2.$$

(Here

$$U_1 = (p^2 - q^2 - x_0\gamma_1 + b\delta_2)\ ;\quad u_2 = (2pq - x_0\gamma_2 - b\delta_1)\ ).$$

It is a positive definite form. In accordance with Zubov's theorem of stability of a set, this is sufficient for stability of the manifold (2.4).

3. Graphical solution of systems of algebraic inequalities

When investigating problems of stability and stabilization of nonlinear or linear models of mechanical systems, there arises the problem of "parametric analysis" of algebraic inequalities obtained. These can be, for example, Sylvester's, Lineard–Chipart's and Routh–Hurwitz's conditions, and the like. In parametric analysis, it is important to be able to estimate the range of values of parameters, for which the required state (property) of the system is provided.

In the framework of software package "Stability" [2], it was possible to solve:

Problem 1. *Finding the domains, which satisfy the inequalities of one-parameter polynomials with numerical coefficients*

$$f_i(p) = a_n^{(i)}\, p^n + a_{n-1}^{(i)}\, p^{n-1} + \ldots + a_0^{(i)} \geq 0 \quad (\leq 0) \quad (\ i = 1, \ldots, n_1\ ) \tag{3.1}$$

The solution (the domain obtained) is represented as a union of disjoint intervals of the numerical axis.

Problem 2. *Search and graphical constructing of two-dimensional domains corresponding to a system of algebraic inequalities*

$$f_i\ (p_1\ \ p_2) \geq 0 \qquad (\leq 0) \qquad (\ i = 1, \ldots, n_2\ ) \tag{3.2}$$

where p_1 is the parameter reckoned on the abscissa axis, and p_2 is the parameter reckoned on the axis of ordinates. The boundaries of variation of the parameters p_1 and p_2 for constructing $2D$ – domains are determined as the solutions of problem 1.

CAS "Mathematica" [5] is used as the tool for solving both problems 1 and 2 and problems of the package "Stability". A description of the algorithms developed for solving the systems of algebraic inequalities (3.1) and (3.2) is given in [3].

Example 3.1. To illustrate the performance of the program for graphical construction of 2D-domains, we conduct parametric analysis of asymptotic stability conditions of the equilibrium position of a satellite with a controlled gravitational stabilizer on the circular orbit [6] (without taking into account the gyrowheel and the elasticity of the rod). The linear equations of motion are subdivided into two subsystems:

with respect to the pitch angle (θ)

$$C\ \ddot{\theta} + F\ \ddot{\alpha} + 3\omega^2\ [\ (B - A)\ \theta + F\ \alpha\ ]\ =\ 0$$

$$F\ \ddot{\theta} + G\ \ddot{\alpha} + 3\omega^2 F\ (\ \theta + \alpha\ )\ =\ Q_\alpha$$

and with respect to the yaw and bank angles ($\psi\ \ \varphi$)

$$A\ \ddot{\psi} + \omega\ \dot{\varphi}\ (C - B - A) + \omega^2\ (C - B)\ \psi\ =\ 0$$

$$B\ \ddot{\varphi} + F\ \ddot{\beta} - \omega\ \dot{\psi}\ (C - B - A) + 4\omega^2\ [\ (C - A)\ \varphi + F\ \beta\ ]\ =\ 0$$

$$F\ \ddot{\varphi} + G\ \ddot{\beta} + 4\omega^2 F\ \varphi + \omega^2\ (3F + G)\ \beta\ =\ Q_\beta$$

where ω is the modulus of orbital angular velocity;

$$B = J_x + m\ r\ (r + d) + \frac{1}{3}\ m\ d^{\ 2} + M\ (r + d)^2\ ;$$

$$C = J_z + m\ r\ (r + d) + \frac{1}{3}\ m\ d^{\ 2} + M\ (r + d)^2\ ;$$

$$A = J_y\ ;\quad G = (\frac{1}{3}\ m + M\ )\ d^{\ 2}\ ;\quad F = \frac{1}{2}\ m\ r\ d + \frac{1}{3}\ m\ d^{\ 2} + M\ d\ (r + d)\ ;$$

m and M are, respectively, the masses of the rod and the weight at its end; $d > 0$ is the length of the rod; $r > 0$ is the distance from the mass center of the system to the point of fastening of the rod; $J_x\ J_y\ J_z$ are the principal moments of inertia of the satellite; and $\alpha\ \beta$ are the rotation angles of the rod relative to the satellite body.

If $Q_\alpha = Q_\beta = 0$, then potential and gyroscopic forces influence the system. They are determined by forces of gravitational attraction and by the motion along the orbit. Such a system cannot be asymptotically stable. Indeed, the differential equations allow for the integral of energy (example 1.2):

$$C\ \dot{\theta}^2 + G\ (\dot{\alpha}^2 + \dot{\beta}^2) + 2F\ (\dot{\theta}\dot{\alpha} + \dot{\beta}\dot{\varphi}) + B\ \dot{\psi}^2 + A\ \dot{\psi}^2 +$$

$$+\ 3\omega^2(B - A)\ \theta^2 + 3\omega^2 F\alpha^2 + 6\omega^2 F\alpha\ \theta + \omega^2(C - B)\ \psi^2 +$$

$$+ 4\omega^2(C-A)\ \varphi^2 + \omega^2(3F+G)\ \beta^2 + 8\omega^2 F\varphi\ \beta \ = \ \text{const}\,.$$

The requirements of definite positiveness of this quadratic form give conditions for secular stability of the system:

$$C > B > A + F\ ; \qquad (\ 3F + G\ )\ (\ C - A\ ) > 4F^2. \tag{3.3}$$

Actually, at the fixed point of the rod to the body elastic $(-k_\alpha\ \alpha\ -k_\beta\ \beta\ )$ and dissipative forces $(-k'_\alpha\ \dot\alpha\ -k'_\beta\ \dot\beta\ )$ act, which do not ensure complete dissipation. For provision of asymptotic stability with high quality of the transient process, it is possible to introduce a complete dissipation and nonconservative forces in terms of $Q_\alpha\ Q_\beta$. Therefore, we consider the following actions:

$$Q_\alpha = k_\theta\ \theta + k'_\theta\ \dot\theta k_\alpha\ \alpha - k'_\alpha\ \dot\alpha\ ; \quad Q_\beta = k_\varphi\ \varphi + k'_\varphi\ \dot\varphi - k_\beta\ \beta - k'_\beta\ \dot\beta.$$

With such forces, the system is under the conditions of example 1.1. Therefore, it is possible to apply theorem 1.1 or of Lyapunov's stability theorem in the first approximation to the definition $k_i\ k'_i\ \ (\ i = \alpha,\ \beta,\ \varphi,\ \theta\ )$. The software package "Stability" was used to obtain the characteristic equation of the system and Lineard-Chipart's conditions which ensure that all its roots have negative real parts. With respect to the pitch channel, Lineard-Chipart's conditions have the form:

$$a_0 = 3\omega^2\ [\ (B-A)\ k_\alpha + F\ k_\theta + 3\omega^2 F\ (B-A-F)\ ]\ > 0\ ;$$

$$a_1 = 3\omega^2\ [\ F\ k_\theta{}' + k_\alpha{}'\ (B-A)\ ]\ > 0\ ;$$

$$a_2 = C\ k_\alpha + F\ k_\theta + 3\omega^2\ [\ F\ (C-2F) + G\ (B-A)\ ]\ > 0\ ;$$

$$a_3 = F\ k_\theta{}' + C\ k_\alpha{}'\ > 0\ ; \quad a_4 = C\ G - F^2\ > 0\ ;$$

$$\Delta \ = \ 3\omega^2\ F\ (C-B+A)\ \{\ (k_\theta{}')^2\ F\ [\ k_\alpha + 3\omega^2\ (\ F-G\ )\ ] +$$

$$+\ k_\theta{}'\ k_\alpha{}'\ [\ C\ k_\alpha - F\ k_\theta + 3\omega^2\ (\ C\ F + G\ (A-B)\ )\ ] +$$

$$+\ (k_\alpha{}')^2\ [\ 3\omega^2 F\ (C-B+A) - C\ k_\theta\ ]\ \}\ > 0. \tag{3.4}$$

Here $a_i\ \ i = 0, \ldots, 4$ are the coefficients of a characteristic polynomial; the conditions $a_4 > 0\ \ C - B + A = J_z - J_x + J_y > 0\ \ C\ F + G\ (A-B) > 0$
$F\ (C-2F) + G\ (B-A) > 0$ are satisfied always due to, respectively, definite positiveness of kinetic energy and the condition for moments of inertia in an absolutely rigid body.

In (3.4) $\Delta = \Delta(k_\theta{}', \ k_\alpha{}')$ is a quadratic form with coefficients

$$d_{11} = F \ (\ k_\alpha + 3\omega^2 \ (F - G) \ ) \ ; \quad d_{22} = 3\omega^2 F \ (C - B + A) - C \ k_\theta \ ;$$

$$d_{12} = d_{21} = \frac{1}{2} \ [\ C \ k_\alpha - F \ k_\theta + 3\omega^2 \ (\ CF + G \ (A - B) \ ) \ ].$$

Sylvester's determinant for it has the form:

$$D \ = \ D \ (\ k_\theta \ k_\alpha \ ) \ = \ \det \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \ = \ - \ F^2 \ k_\theta^2 \ +$$

$$+ \ 2F \ [\ 3\omega^2 \ (\ G \ (C - B + A) - C \ (F - G) \ ) - C \ k_\alpha \ ] \ k_\theta \ +$$

$$+ \ [\ 9\omega^4 \ (\ 4F^2 \ (F - G) \ (C - B + A) - (\ CF + (A - B)G \ )^2 \ ) \ +$$

$$+ \ 6\omega^2 \ (\ 2F^2 \ (C - B + A) + CG \ (B - A) - C^2 \ F \ ) \ k_\alpha \ - \ C^2 \ k_\alpha^2 \ ]. \quad (3.5)$$

It will be shown that Δ cannot be a positive definite function, i.e., when $d_{11} > 0$ we necessarily have $D < 0$. Indeed, let us compose the equation $D = 0$; it is quadratic with respect to k_θ with the negative coefficient at a higher degree of polynomial. The sign of the discriminant of this equation ($-48\omega^2 \ F^2 \ (C - B + A) \ (CG - F^2) \ d_{11}$) is opposite to the sign of d_{11} . Therefore, for all the values of k_α satisfying the condition $d_{11} > 0$ we have $D = D(k_\theta) < 0$. When $d_{11} < 0$ the function $D(k_\theta)$ from (3.5) is singalternating. The domain of $k_\theta \ k_\alpha$, in which $d_{11} < 0$ and $D(k_\theta) > 0$, must be eliminated since the function Δ from (3.4) is negative definite in it. With any signs of d_{11} and $D = D(k_\theta) < 0$, the function Δ is alternating, and there exists the domain of $k_\alpha{}' \ k_\theta{}'$, in which $\Delta > 0$.

Thus, the system of inequalities, defining the domain for selecting the parameters $k_\theta \ k_\alpha$ has the form: $a_0 > 0 \ a_2 > 0 \ D < 0$. Note, when $D < 0$ the second-order curve $\Delta = \Delta(k_\theta{}', \ k_\alpha{}')$ describes two real straight lines intersecting at zero.

In parametric analysis of the inequalities (3.4), we specify the following mass distribution in the system:

$d = 12$ m ; $r = 2.5$ m ; $M = 1.531$ kg ; $m = 0.0143$ kg ;

$J_y = 28$ kgm^2 ; $J_x = 75$ kgm^2 ; $J_z = 76$ kgm^2 ; $\omega = 0.0011$ radian/sec.

It should be noted that for these values the conditions of secular stability of the whole system (3.3) are satisfied.

```
D = -0.000770376 - 10.4571*ka - 39803.2*ka^2
    - 5.8182*kt -  53313.7*ka*kt - 17852.5*kt^2 ;
```

```
a2 = 0.000414981 + 1.0*ka + 0.669716*kt ;
a0 = 0.00026947 + 1.0*ka + 0.722205*kt
```

From now on, k_θ k_α $k_\theta{}'$ k_α' Δ are denoted by kt, ka, kt', ka', Delta in the figures. The boundaries of variation of values of the parameters kt, ka, kt', ka' are chosen arbitrarily from the interval $]-\infty,\ \infty[$. The result of operation of the program written in the programming language CAS "Mathematica":

```
Region [  D < 0, a0 > 0, a2 > 0 ,
          kt, -0.002, 0.004 ,  ka, -0.003, 0.002  ]
```

is shown in Figure 1.

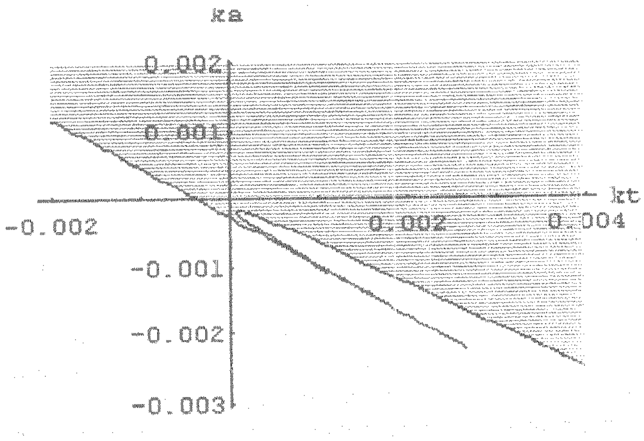

Fig. 1.

The domain for selecting the parameters $k_\theta{}'$ k_α' will be determined from the system

$a_1 > 0$ $a_3 > 0$ $\Delta > 0$ at fixed values of the parameters k_θ k_α from the shaded area in Figure 1 (i.e., sections of the space of parameters are constructed). Upon carrying out the numerical – symbolic analysis, it has been noticed that the most extensive domain for the parameters $k_\theta{}'$ k_α' takes place for the values of k_θ k_α lying on the straight line $F\ k_\alpha + (B-A)\ k_\theta = 0$. For example, when $k_\alpha = A - B$ $k_\theta = F$ we have

```
Delta = 1.0*(ka')^2  + 1.39192*ka'*kt'
        + 0.483672*(kt')^2 ;
a3 = 1.0*ka' + 0.669716*kt';
a1 = 1.0*ka' + 0.722205*kt'.
```

The domain corresponding to

```
Region [ Delta > 0, a1 > 0, a3 > 0 ,
          kt', -1, 1 ,  ka', -1, 2 ]
```

is shown in Figure 2.

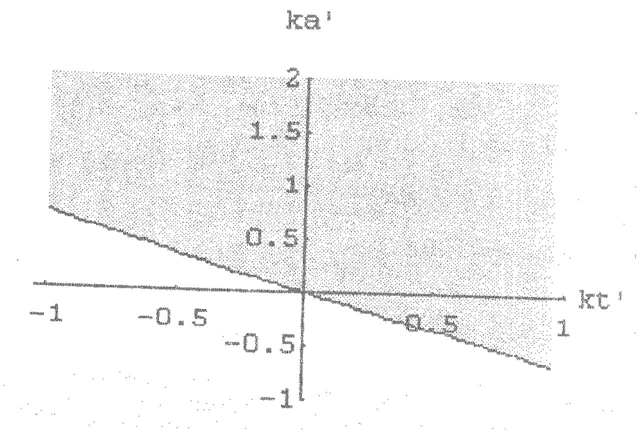

Fig. 2..

For the sections, where k_θ k_α do not belong to the straight line $F\ k_\alpha + (B-A)\ k_\theta = 0$, the required domain for the parameters $k_\theta{}'$ $k_\alpha{}'$ is always located in the shaded area in Figure 2.

Thus, the addition of nonconservative positional (parameter k_θ) or potential (k_α) forces to a stable potential system restricts the domain of asymptotic stability for the parameters $k_\theta{}'$ $k_\alpha{}'$ with respect to the pitch channel (i.e. in the example under consideration, for the mass distribution specified above some destabilizing effect takes place).

A similar parametric analysis with the aid of the software described above has also been carried out for the "bank and yaw" subsystem.

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