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CENTRAL CONFIGURATIONS OF THE 5-BODY PROBLEM WITH EQUAL MASSES IN THREE-DIMENSIONAL SPACE

ABSTRACT. We enumerate central configurations with axial symmetry in the 5-body problem with equal masses in three-dimensional space. It is thus shown, that only two of these configurations, have a unique symmetry axis. The use of Symbolic Computation techniques is necessary because of the very tedious computations involved in the proof.

1. INTRODUCTION

Inspired by the results proved in [2], and [11] we undertook a systematic study of the central configurations in the newtonian 5-body problem with equal masses in three-dimensional space. The starting point is the characterization of the central configurations as the critical values of the newtonian potential energy function, as it appears in [7]. The algebraic equations that arise, are inaccessible to calculations by hand and necessitate sophisticated Computer Algebra techniques for their resolution.

The fundamental tools used in our approach are a Linear Algebra scheme, proposed by the second author, for properly performing the elimination of the variables and two advanced algorithms in Gröbner bases theory. These algorithms (see [10] and [9]) have been implemented by J.-C. Faugère in two very powerful Gröbner bases engines, namely GB and FGB (see [8]). All Gröbner bases computations have been performed using this software.

After a brief exposition of the spatial 5-body problem, we introduce the equations of the central configurations. We consider five cases with respect to the symmetries involved. In each case, we apply the techniques mentioned above in order to solve the equations. We prove that there are four configurations having an axis of symmetry, i.e. invariant by some rotation.

2. THE SPATIAL 5 - BODY PROBLEM

We shall briefly describe the general setting of the problem. Consider 5 particles of masses $m_i, i = 1, \ldots, 5$ moving under their mutual gravitational attraction. Choose a point as the origin and denote by $r_i, i = 1, \ldots, 5$ the position vector of the i-th particle. Define the newtonian potential energy function as:

$$U = \sum_{1 \leq i < j \leq 5} \frac{m_i m_j}{|| r_i - r_j ||}$$

where $|| \cdot ||$ is the euclidean norm. In order to describe the motion of this system, we have the following newtonian formulation:

$$m_i \ddot{r_i} = \frac{\partial U}{\partial r_i}, \quad i = 1, \ldots, 5. \tag{1}$$

Note that on the left-hand side we take the derivative of the position vector w.r.t. time, that is the acceleration of the i-th particle, and that the right-hand side represents the total force exercised on the i-th particle by the other four.

The center of mass of the system is defined as:

$$r_c = \frac{m_1 r_1 + \cdots + m_5 r_5}{m_1 + \cdots + m_5} \ .$$

The symmetry of the function U implies that $\ddot{r_c} = 0$, which means that r_c is a linear function of time. This is the principle of the conservation of linear momentum, which says that the center of mass moves in a straight line with uniform velocity. This allows us to shift the center of mass at the origin, without loss of generality and thus it remains to determine the motion relative to the center of mass. Using this and other similar considerations, like the conservation of energy and angular momentum, the order of the system is considerably reduced, but it is still quite difficult to study it.

3. Central Configurations

When dealing with such a difficult problem, one seeks first to identify special solutions, that is solutions which exhibit certain nice properties. A classical approach initiated by such prominent scientists as Euler, Lagrange and Laplace, consists in studying special classes of solutions of (1) which help us to gain further insight for the general situation.

Let the center of mass of the system be at the origin of the coordinate axes. This means that we have $r_c = 0$. In this case, we have the following definition:

Definition 1. *A configuration of the n particles is called a central configuration when the acceleration vector of each particle is a common scalar multiple of his position vector. In mathematical terms this means that there exists a scalar λ such that we have $\ddot{r_i} = \lambda r_i$ for $i = 1, \dots, 5$.*

In spite of the fact that this kind of solutions of the n-body problem are rather rare, astronomers study them for at least two reasons. Firstly because they can be used to analyze actual configurations of celestial bodies in the universe. A well-known example is the Sun-Jupiter-Trojan asteroids configuration. Secondly because they are of capital importance in the analysis of collision orbits of expanding gravitational systems. Indeed, the asymptotic or bifurcation behaviour of such systems are described in terms of central configurations. A thorough introduction as well as historical remarks on the subject can be found for example in [12], [14], [15] and any other classical treatise of Celestial Mechanics.

4. The equations of central configurations

In this paragraph we will be interested in the central configurations of the 5-body problem in a Euclidean space of dimension 3. We will give a characterization of the central configurations by a system of algebraic equations, proposed by O. Dziobek [7]. The innovative idea of Dziobek was to use the mutual distances between the bodies as the coordinates to express the equations of motion. The resulting algebraic system is tractable by symbolic computation techniques after some preprocessing using linear algebra.

Conserving the notations of § 2, we let $r_{ij} = \| r_i - r_j \|$ for $1 \le i < j \le 5$ and $s_{ij} = r_{ij}^2$. So s_{ij} is the square of the euclidean distance of the ith particle from the jth particle. There are ten such mutual distances. Finally, let Δ_1, Δ_2, Δ_3, Δ_4, Δ_5 be the oriented volumes of the five tetrahedra formed by the points $(2,3,4,5)$, $(3,4,5,1)$, $(4,5,1,2)$, $(5,1,2,3)$, $(1,2,3,4)$ respectively. They verify the relation:

$$\sum_{i=1}^{5} \Delta_i = 0. \tag{2}$$

Central configurations are then characterized by the following two groups of equations:

$$(A) \qquad \sum_{j \ne i} \Delta_j s_{ij} = X, \quad \text{for some } X \text{ independent of } i$$

(B) there are two real numbers x and y such that

$$s_{ij}^{-3/2} = x + y\frac{\Delta_i \Delta_j}{m_i m_j}$$

The group (A) equations are due to A. Albouy (see [1]). To prove the group (A) equations we just have to remark that the expression

$$\sum_{i=1}^{5} \Delta_i \parallel r_i - r \parallel^2$$

is independent of r. We put $t_j = \sum_{i=1}^{5} \Delta_i \parallel r_i - r_j \parallel^2 = \sum_{i=1}^{5} \Delta_i s_{ij}$ for $j = 1, \ldots, 5$ and we immediately have

$$t_1 = t_2 = t_3 = t_4 = t_5$$

which is a more symmetric way of writing the group (A) equations.

The analog of group (B) equations for the central configurations of the planar 4-body problem with arbitrary masses appears in [7], where it is stated that the proof is based on Analytic Geometry considerations.

The general principles developed in [3] for the N-body problem in general, allow to prove both groups of equations using a construction in Linear Algebra.

The exponent $-3/2$ reflects the fact that we are working with the newtonian potential. The law of mutual attraction is the inverse square law. It is interesting to note, that this formulation of the equations is valid for any potential function as well.

We are interested in the case of equal masses, that is $m_1 = \ldots = m_5$. Without loss of generality, we can assume that the common value of the masses is one, by taking as unknowns $\dfrac{\Delta_i}{m_i}$ instead of Δ_i. This is justified by the fact that the equations of group (A) are homogeneous of degree 1 in Δ_i. In the sequel we will work under this assumption without any other explicit mention.
In order to introduce linear algebra techniques, we need some preliminary definitions:

Definition 2. *For every $1 \leq i < j \leq 5$, let $S_{ij} = s_{ij}^{-3/2}$.*

Definition 3. *Let $M_{(10)}$ be the 10×3 matrix with rows $(S_{ij}, 1, \Delta_i \Delta_j)$ for $1 \leq i < j \leq 5$.*

It is necessary to parametrize the definition of the matrix M by the number of lines, because this number varies according to the number of different mutual distances that are needed to describe a given symmetry condition, as will be seen later. One of the fundamental remarks in our approach is that the group (B) of equations can be related to a condition about the rank of the matrix $M_{(10)}$.

Theorem 1. *The group (B) equations imply that the matrix $M_{(10)}$ must have rank $r \leq 2$.*

Proof. The matrix $M_{(10)}$ is the augmented matrix of the system of 10 linear equations in the unknowns x and y, specified by the group (B) equations. Since this linear system must have a solution in x and y, it follows that its augmented matrix must have rank $r \leq 2$. □

A partial inverse to the previous theorem, is furnished by:

Theorem 2. *If the matrix $M_{(10)}$ has rank $r \leq 2$, if relation (sommeDi) holds and if the Δ_i are not all simultaneously zero, then the group (B) equations hold.*

Proof. Suppose that the matrix $M_{(10)}$ has rank $r \leq 2$. This means that its three columns are linearly dependent, that is they are related by a nontrivial linear combination. Then there are real numbers x, y, z not all simultaneously zero, such that we have

$$zS_{ij} = x + y\ \Delta_i\ \Delta_j$$

for every $1 \leq i < j \leq 5$. We distinguish two cases for z:

a) If $z \neq 0$ then we divide by z throughout, and we obtain the group (B) equations.

b) If $z = 0$ then we distinguish two cases for x:

b1) If $x = 0$, since y must be non-zero, we must have $\Delta_i \Delta_j = 0$ for all $1 \leq i < j \leq 5$. This and condition (2) imply that $\Delta_i = 0$ for every $1 \leq i \leq 5$ and this is absurd by the hypothesis.

b2) If $x \neq 0$, since y cannot be zero, $\Delta_i\ \Delta_j$ must be a non zero constant for every $1 \leq i < j \leq 5$ and so Δ_i must be a constant for all $1 \leq i \leq 5$. This means that all of the Δ_i are of the same sign and therefore condition (2) implies a contradiction. □

5. Symmetric Solutions of Symmetric Equations

The equations governing the 5-body problem, present a certain symmetry, in their initial form (groups (A) and (B)). It is a natural idea to look for symmetric solutions of symmetric equations. We start our analysis by considering the most simple cases of symmetric solutions and we continue with progressively more difficult cases.

It seems that the best way to proceed is to distinguish different cases for the Δ_i and there are some good reasons which justify this choice. Firstly, any symmetry hypothesis on the configuration of the five points in space, can be expressed as an equality between some Δ_i or corresponding pairs of equalities between the s_{ij}. Secondly, any equality between some Δ_i, can be translated immediately in terms of equalities between the s_{ij}, using the group (B) equations. The converse is only partially true, as it can be easily seen by the following formula for the volume V of a tetrahedron $ABCD$ as a Cayley determinant of the lengths of its edges:

$$V^2 = \frac{1}{288} \begin{vmatrix} 0 & c^2 & b^2 & p^2 & 1 \\ c^2 & 0 & a^2 & q^2 & 1 \\ b^2 & a^2 & 0 & r^2 & 1 \\ p^2 & q^2 & r^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

where

$$a = BC,\ b = CA,\ c = AB,\ p = DA,\ q = DB,\ r = DC.$$

This formula is applicable here, due to the fact that the Δ_i represent tetrahedra volumes.

We will compute all the solutions which have an axis of symmetry, i.e. all the solutions which are invariant under some rotation. As we suppose that the five points are not coplanar ($V \neq 0$), only rotations of angle $\pi/2, 2\pi/3, \pi$ are possible. The group (B) equations show that such an invariance implies some equalities between the Δ_i. For example, a plane symmetry, implies equality of two Δ_i. Since we are working in three-dimensional Euclidean space, there is an interesting geometric interpretation attached to each algebraic solution. We will also describe the solutions from this geometric viewpoint.

5.1. $\Delta_1 = 0$. If the five points are invariant by a rotation of angle $\pi/2$, four of them should be coplanar. This implies that some Δ_i is 0. Conversely, we will show that this condition implies that the (unique) solution is invariant by a rotation of $\pi/2$. Let us suppose that $\Delta_1 = 0$. This is of course a generic hypothesis, in the sense that by properly renumbering the five points, nullity of any Δ_i can be replaced by the condition $\Delta_1 = 0$.

By recalling that Δ_1 represents the oriented volume of the tetrahedron $(2,3,4,5)$ we see that the four points $2,3,4,5$ must be coplanar.

From group (B) equations and definition (2) we have that

$$S_{12} = S_{13} = S_{14} = S_{15} \ (= x) \ \text{ and so } \ s_{12} = s_{13} = s_{14} = s_{15}.$$

Geometrically speaking, the distances of the point 1 from the other four points are equal and this means that the point 1 is the summit of a pyramid with basis in the plane defined by $2,3,4,5$. We can easily see that these four points must also lie on a circle. Moreover, the group (A) and (B) equations imply that the points $2,3,4,5$ must form a central configuration solution of the planar 4-body problem, with equal masses. It is known (see [2]) that the only planar convex central configuration of four equal masses is the square. Recall that a configuration is called convex if no point is situated inside the convex envelope of the others. This shows that the point 1 is the summit of an orthogonal square pyramid.

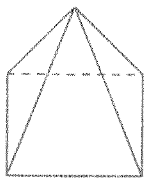

Fig. 1. The square pyramid solution.

If we normalize to 1 the value of the sides of the square formed by the points $2,3,4,5$, by examining the Gröbner basis of the system we find that $s_{24} = s_{35} = 2$ and that s_{12} is a root of the degree six polynomial:

$$49\ s_{12}^6 - 576\ s_{12}^3 + 1024.$$

This polynomial has only two real roots, but only one of these gives positive values for all S_{ij}. We conclude that

$$s_{12} = \frac{2}{7}\ \sqrt{[3]}252 - 112\ \sqrt{2} = \sqrt{[3]}\frac{32}{49}(2\sqrt{2}-1)^2 \ \simeq \ 1.297288007.$$

The height h of the pyramid, satisfies the relation

$$s_{12} = r_{12}^2 = h^2 + \left(\frac{\sqrt{2}}{2}\right)^2$$

from which we find $h = 0.8929098536$. This shows that the pyramid is a quite fat one (neither flat nor sharp).

5.2. $\Delta_1 \neq 0$ and $\Delta_2 = \Delta_3 = \Delta_4 = \Delta_5$. We will see in this section that this condition is equivalent to the condition that the five points are invariant under the symmetry of a regular tetrahedron.

¿From group (B) equations we find that there are two groups of equal mutual distances: Firstly, the distances of point 1 from the others (denoted a) are all equal. Secondly, all the other mutual distances (denoted f). This means that the four points $2, 3, 4, 5$ form a regular tetrahedron and that the fifth point rests on the tetrahedron's barycenter. We recall the well-known fact that the distances from the barycenter of a regular tetrahedron to each summit are all equal.

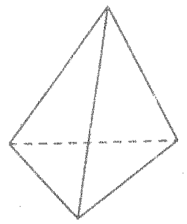

Fig. 2. The regular tetrahedron solution.

In order to find the numerical values of the distances, we put $A = a^{-3/2}$, $F = f^{-3/2}$, we take $y = -1$ by virtue of the homogeneity of equations (A) and we compute the Gröbner basis of the system of equations:

$$\Delta_1 + 4\ \Delta_2,\ a^3\ A^2 - 1,\ f^3\ F^2 - 1, 4\ \Delta_2\ a - \Delta_1\ a -$$
$$-3\ \Delta_2\ f, A - x + \Delta_1\ \Delta_2,\ F - x + \Delta_2^2$$

after taking $a = A = 1$, since we are interested in classes of homothetic solutions. The result is:

$$\big\{5x - 4F - 1, \Delta_1 + 4\Delta_2, 5\Delta_2^2 + F - 1, \Delta_2(3f - 8),$$
$$512F^2 + 27f^3 - 539, (3f - 8)(F - 1), 3f^4 - 8f^3 - 3f + 8\big\}.$$

Since f must be real and the last polynomial of the Gröbner basis is factored as

$$(3f - 8)(f - 1)(f^2 + f + 1),$$

we have only two cases to consider.

5.2.1. The case $f = 1$. We have that $a = f = 1$, i.e. five equidistant points in three-dimensional space, and this solution does not have a physical meaning.

5.2.2. The case $f = 8/3$. We find the following solution, which corresponds to the regular tetrahedron configuration:

$$a = 1,\ f = \frac{8}{3},\ A = 1,\ F = \frac{3}{32}\sqrt{6},$$
$$\Delta_1 = \mp\frac{\sqrt{320 - 30\sqrt{6}}}{10},\ \Delta_2 = \pm\frac{\sqrt{320 - 30\sqrt{6}}}{40}.$$

Clearly this configuration is invariant under various rotations of angles $2\pi/3$ and π.

5.3. $\Delta_1 = \Delta_2$ and $\Delta_3 = \Delta_4 = \Delta_5$. In this section we will show that the solution consists of an equilateral triangle and two points symetrically situated on the axis passing from the triangle's barycenter and perpendicular to the plane defined by the triangle.

From group (B) equations we see that only three mutual distances need to be retained. For brevity we put

$$s_{12} = a,\ s_{34} = f,\ s_{13} = b,\ S_{12} = A,\ S_{34} = F,\ S_{13} = B$$

Equation (2) becomes

$$2\ \Delta_1 + 3\ \Delta_3 = 0.$$

The group (A) equations reduce to the following equation

$$\Delta_1\ a + 3\ \Delta_3\ b - 2\ \Delta_1\ b - 2\ \Delta_3\ f = 0$$

Using the homogeneity of the equations with respect to the Δ_i we can take $y = -1$. Moreover, we are allowed to put $b = B = 1$ since we are interested in classes of homothetic solutions. We can add the equation

$$a = \alpha^2$$

in order to calculate directly the value of the mutual distance between the points 1 and 2 and thus obtaining a polynomial with small coefficients. Moreover, we add the equation

$$a\ \alpha\ A = 1$$

in order to avoid parasite solutions for α. It is an easy task to calculate the Gröbner basis of this system for a lexicographical ordering of the variables, in GB. It has 13 elements one of which is a univariate polynomial of degree 15 in α which factorizes as follows:

$$(\alpha - 1)\ (\alpha^2 + \alpha + 1)\ P_{12}(\alpha)$$

where

$$P_{12}(\alpha) = 3\,\alpha^{12} - 36\,\alpha^{10} + 12\,\alpha^{9} + 144\,\alpha^{8} - \\ - 144\,\alpha^{7} - 116\,\alpha^{6} + 576\,\alpha^{5} - 144\,\alpha^{4} - 768\,\alpha^{3} + 576\,\alpha^{2} - 768$$

Since α must be a positive real number, and the second factor has only complex roots, there are only two cases to consider.

5.3.1. The case $\alpha = 1$. This is the easiest case to handle. By direct substitution into the Gröbner basis we obtain the values of the other unknowns and thus the following trivial solution of our system:

$$a = 1,\ b = 1,\ f = 1,\ \Delta_1 = 0,\ \Delta_3 = 0,\ A = 1,\ B = 1,\ F = 1, x = 1.$$

Since all Δ_i are equal to zero, and all the mutual distances are equal to 1, this is an algebraic solution of the system which does not correspond to any physical configuration.

5.3.2. The case of $P_{12}(\alpha)$. The polynomial $P_{12}(\alpha)$ has two real roots, namely

$$\alpha_1 = -1.016545061\ \alpha_2 = 1.428905140$$

Only the second value is of course admissible. An exact calculation using the evala function of MAPLE shows that if we denote by p a generic root of $P_{12}(\alpha)$ then A, F, x can be expressed as polynomials of degree 11 in p. In order to minimize numerical errors due to the evaluation of three polynomials of degree 11, we examine the structure of the Gröbner basis and we see that the following simple relations hold:

$$f = -\frac{3}{4}\,\alpha^2 + 3, \quad F = \frac{2\,A + 1}{3} \quad \Delta_3^2 = \frac{4}{15}(1 - A) \quad x = \frac{2\,A + 3}{5}$$

These relations are of course compatible with the formulae involving the degree 11 polynomials. In this way we need to evaluate only one polynomial of degree 11 for calculating the values of A, F, x, f, Δ_3. After performing all necessary operations we obtain one solution to the system, corresponding to the admissible value α_2:

a	f	A	F
2.0417698	1.4686725	0.3427597	0.5618398

Δ_1	Δ_3	x
−0.6279682	.4186455	0.7371038

which in turn gives the values $r_{12} = \alpha_2,\ r_{34} = 1.211888021$ for the mutual distances.

5.4. $\Delta_1 \neq \Delta_2,\ \Delta_3 = \Delta_4 = \Delta_5$**.** The group (B) equations imply that the points $3, 4, 5$ form an equilateral triangle and that the points 1 and 2 lie on the axis passing form the triangle's axis of symmetry. In other words, these solutions are those which are invariant by a rotation of $2\pi/3$.

By inspection of the equations of group (B), this symmetry condition implies that we need to retain only four different mutual distances. For simplicity we set

$$a = s_{12}, \quad b = s_{13}, \quad d = s_{23}, \quad f = s_{34}$$

and accordingly

$$A = s_{12}^{-3/2},\ B = s_{13}^{-3/2},\ D = s_{23}^{-3/2},\ F = s_{34}^{-3/2}. \tag{3}$$

So we have the defining relations:

$$A^2a^3 - 1 = 0 \quad B^2b^3 - 1 = 0 \quad D^2d^3 - 1 = 0 \quad F^2f^3 - 1 = 0 \tag{4}$$

Finally, equation (2) becomes:

$$\Delta_1 + \Delta_2 + 3\ \Delta_3 = 0. \tag{5}$$

Generalizing a proposition due to A. Albouy which also appears in [11], for the newtonian planar 4-body problem with equal masses, we demonstrate the following:

Theorem 3. *The equations* (5), (A) *and* (B) *are equivalent to the following:*

$$\begin{aligned} &3\ (b - d)\ (B - D) + B + D - 6\ F + 4 = 0 \\ &3\ (b - d)\ (B + D - 2\ F) + 5\ (B - D) = 0 \\ &3\ (b - d)^2\ - 6\ (b + d) + 4\ f + 3 = 0. \end{aligned} \tag{6}$$

Proof. From theorem 1 we have that the following matrix must have rank $r \leq 2$.

$$M_{(4)} = \begin{bmatrix} A & 1 & \Delta_1\ \Delta_2 \\ B & 1 & \Delta_1\ \Delta_3 \\ D & 1 & \Delta_2\ \Delta_3 \\ F & 1 & \Delta_3^2 \end{bmatrix}. \tag{7}$$

We put $\Delta_M = \Delta_1 - \Delta_2$ and using relation (5) we eliminate Δ_1, Δ_2.

$$M_{(4)} = \begin{bmatrix} A & 1 & \frac{9}{4}\,\Delta_3^2 - \frac{1}{4}\,\Delta_M^2 \\ B & 1 & -\frac{3}{2}\,\Delta_3^2 + \frac{1}{2}\,\Delta_3\Delta_M \\ D & 1 & -\frac{3}{2}\,\Delta_3^2 - \frac{1}{2}\,\Delta_3\,\Delta_M \\ F & 1 & \Delta_3^2 \end{bmatrix}$$

By elementary operations on the rows of the matrix $M_{(4)}$ we obtain:

$$M_{(4)} \sim \begin{bmatrix} B+D-6F+4A & 0 & -\Delta_M^2 \\ B-D & 0 & \Delta_3\Delta_M \\ 3F+B+D & 5 & 0 \\ 2F-B-D & 0 & 5\Delta_3^2 \end{bmatrix}$$

In the sequel we will refer to the above matrix as $M_{(4)}$. This matrix has of course the same rank as the initial matrix $M_{(4)}$ because elementary row operations do not change the rank of a matrix.

Claim.

$$\text{rank } M_{(4)} \leq 2 \iff$$

$$\iff \begin{cases} (B+D-6F+4A)\Delta_3 + (B-D)\Delta_M & = & 0 \\ 5\,(B-D)\Delta_3 + (B+D-2F)\Delta_M & = & 0 \end{cases} \tag{8}$$

Proof of the claim.. Suppose first that $M_{(4)}$ has rank $r \leq 2$. This means that all its 3×3 sub-determinants must vanish. If we denote $D_{(i,j,k)}$ the 3×3 sub-determinant formed by rows i, j, k, in which common constant factors have been cancelled out, then we see that $D_{(1,2,3)}, D_{(2,3,4)}$ are exactly equations (8). Moreover, $D_{(1,2,4)}$ is identically zero and $D_{(1,3,4)}$ is a linear combination of $D_{(2,3,4)}, D_{(1,2,3)}$, namely

$$D_{(1,3,4)} = -D_{(2,3,4)}\,\Delta_M - 5D_{(1,2,3)}\,\Delta_3$$

Conversely, all the 3×3 sub-determinants of the matrix $M_{(4)}$ are in the ideal generated by equations (8). This means that all these 3×3 sub-determinants are equal to 0 and so we have rank $M_{(4)} \leq 2$. This completes the proof of the claim.

If we consider the group (A) equations plus equation (5) as equations in the unknowns $\Delta_1,\ \Delta_2,\ \Delta_3$, then we obtain the following matrix:

$$m = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & a & 3b & 1 \\ a & 0 & 3d & 1 \\ b & d & 2f & 1 \end{bmatrix} \tag{9}$$

By elementary operations on the rows of the matrix m we obtain:

$$m \sim m' = \begin{bmatrix} 1 & 1 & 3 & 0 \\ -a & a & 3b-3d & 0 \\ 3b+d-a & 3d+b-a & 4f & 0 \\ b & d & 2f & 1 \end{bmatrix}$$

Passing on to the new variables $\Delta_3,\ \Delta_M$ and after some elementary row operations we have:

$$m' \begin{bmatrix} \frac{\Delta_M - 3\Delta_3}{2} \\ \frac{-\Delta_M - 3\Delta_3}{2} \\ \Delta_3 \\ X \end{bmatrix} \sim \begin{bmatrix} 0 \\ (3b-3d)\Delta_3 - a\Delta_M \\ (-6b-6d+3a+4f)\Delta_3 + (b-d)\Delta_M \\ (3b+3d-3a)\Delta_3 + 2X \end{bmatrix}$$

The above matrix gives two more equations in the unknowns Δ_3, Δ_M:

$$(3b-3d)\Delta_3 - a\Delta_M = 0 \tag{9}$$

$$(-6b-6d+3a+4f)\Delta_3 + (b-d)\Delta_M = 0 \tag{10}$$

We take $a = A = 1$ in order to eliminate Δ_M from equations (8), (11) using equation (10), and this proves the theorem. □

Remark 1. It is straightforward to generalize the above theorem for the n-body problem in a space of dimension $n-2$. One just has to introduce into the equations the parameter $k = n-2$, representing the dimension of the space.

Remark 2. It is important to note that in the course of the elimination, we have taken special care to preserve the symmetries between b, d, B, D. That is why the symmetry appears in the final equations.

Remark 3. The third equation (6) is the determinant of the matrix (9). This fact is natural because it guarantees that the system may have other solutions than the trivial one $\Delta_1 = \Delta_2 = \Delta_3 = 0$.

5.4.1. Resolution of equations (6). First we homogenize equations (6) by making the substitutions:

$$b = \beta^2,\ B = \frac{1}{\beta^3},\ d = \delta^2,\ D = \frac{1}{\delta^3},\ f = \phi^2,\ F = \frac{1}{\phi^3},$$

The homogenization of the variables f, F introduces some parasite solutions, but divides also in half, the time of the final Gröbner basis computation. After clearing out denominators and factorizing, we get the following equations:

$$\begin{aligned}
&3\,\phi^3\beta^5 - 3\,\phi^3\beta^3\delta^2 - 3\,\phi^3\delta^3\beta^2 + 3\,\phi^3\delta^5 - \beta^3\phi^3 - \\
&-4\,\delta^3\beta^3\phi^3 + 6\,\delta^3\beta^3 - \delta^3\phi^3 - 3\,\delta^4\phi^3 + 6\,\delta^4\beta^3 - 3\,\delta^3\beta\,\phi^3 + 6\,\delta^3\beta^4 + \\
&+5\,\phi^3\delta^2 - 3\,\delta\,\beta^3\phi^3 + 5\,\delta\,\beta\,\phi^3 - 3\,\beta^4\phi^3 + 5\,\phi^3\beta^2 3\,\beta^4 - 6\,\beta^2\delta^2 + \\
&+3\,\delta^4 - 6\,\beta^2 - 6\,\delta^2 + 3 + 4\,\phi^2
\end{aligned} \tag{12}$$

The case $\beta = \delta$, which implies $b = d$, $B = D$ and consequently $\Delta_1 = \Delta_2$, has been treated in the previous paragraph.

We remark that (12) are symmetric polynomials in the variables β, δ. By the fundamental theorem in the theory of symmetric polynomials, they can be written as functions of the elementary symmetric polynomials $s = \beta + \delta$ and $p = \beta\ \delta$. Expressing the previous equations in these news variables we find:

$$\begin{aligned}
-4\,p^3\phi^3 + 6\,p^3 + 12\,p^2 s\phi^3 - 15\,ps^3\phi^3 + 3\,ps\phi^3 - s^3\phi^3 + 3\,s^5\phi^3 \\
-5\,p\phi^3 + 6\,p^3 s + 9\,ps^2\phi^3 + 5\,s^2\phi^3 - 3\,s^4\phi^3 \\
3 + 4\,\phi^2 + 12\,p - 12\,ps^2 - 6\,s^2 + 3\,s^4
\end{aligned}$$

Equations (13) constitute a system of 3 equations in 3 variables. The Gröbner basis for a lexicographical ordering of the variables is computed with GB using the FGLM algorithm.

Remark 4. This computation is done in approximately 220 seconds on a two-processor Pentium Pro machine. It is interesting to note that if we brutally try to calculate the lexicographical Gröbner basis of the initial system of equations (6) coupled with the defining relations (4), then the computation on the same machine, takes approximately 13 hours.

We obtain a univariate polynomial of degree 56 in ϕ, which has the following factorization:

$$P = \phi^7\ (3 + 4\phi^2)^2\ (3\phi^2 - 8)\ P_{43}(\phi) \tag{14}$$

where $P_{43}(\phi) \in \mathbb{Z}[\phi]$ is the following polynomial of degree 43 in ϕ:

$$P_{43}(\phi) = 31104\,\phi^{43} - 2238678\,\phi^{41} - 3172608\,\phi^{40} + 4299696\,\phi^{39} +$$

$$+82401057\,\phi^{38} + 147278268\,\phi^{37} - 309919608\,\phi^{36} - 865276476\,\phi^{35} -$$

$$-3800850210\,\phi^{34} + 7126315778\,\phi^{33} + 733665114\,\phi^{32} + 50333677740\,\phi^{31}$$
$$-84536396175\,\phi^{30} + 54761689638\,\phi^{29} - 407185692120\,\phi^{28}+$$
$$+632999590560\,\phi^{27} - 588645975519\,\phi^{26} + 2353657296636\,\phi^{25}$$
$$-3242763707526\,\phi^{24} + 3529599532956\,\phi^{23} - 9834864198930\,\phi^{22}+$$
$$+10874183827806\,\phi^{21} - 12929906294532\,\phi^{20} + 25468176715884\,\phi^{19}-$$
$$20799000437175\,\phi^{18} + 26462275638408\,\phi^{17} - 32404595385312\,\phi^{16}+$$
$$+17924890161384\,\phi^{15} - 24267247513272\,\phi^{14} + 11128176042048\,\phi^{13}-$$
$$-2859261720048\,\phi^{12} + 1566352284768\,\phi^{11} + 8812742163408\,\phi^{10}$$
$$-8998593854976\,\phi^{9} + 9865121279616\,\phi^{8} - 9903015936000\,\phi^{7}+$$
$$+8795401003008\,\phi^{6} - 4873412542464\,\phi^{5} + 6091765678080\,\phi^{4}-$$
$$-949399584768\,\phi^{3} + 2724364025856\,\phi^{2} + 495338913792$$

By the Sturm algorithm we see that this polynomial has seven real roots. A calculation with ten significant digits of precision in MAPLE shows that these roots are:

$$-7.784812685,\ -3.441201939,\ -0.8879733014,$$
$$1.086766801,\ 1.447239258,\ 1.661288262,\ 8.796140284$$

Only the four positive roots may be admissible values of ϕ, but we will show that only one of them gives positive values for β and δ. From (14), we see that the only other admissible value is $\phi^2 = 8/3$ which corresponds to the solution of 5.2.

5.4.2. Uniqueness of the solution (6). In this section we will show that in fact only one value of ϕ gives an admissible solution to our problem.

We start by homogenizing equations (6) but only in the variables β, δ. After clearing out denominators and a factor of the form $(\beta - \delta)$, which leads to a solution with $\Delta_1 = \Delta_2$, we use the fundamental theorem on symmetric polynomials to express the equations as functions of $s = \beta + \delta$ and $p = \beta\,\delta$. We obtain the three equations:

$$-4\,p^3 + 6\,p^3 F + 12\,p^2 s - 15\,ps^3 + 3\,ps - s^3 + 3\,s^5 \tag{15}$$
$$5\,p - 6\,p^3 sF - 9\,ps^2 - 5\,s^2 + 3\,s^4 \tag{16}$$
$$3 + 4\,f + 12\,p - 12\,ps^2 - 6\,s^2 + 3\,s^4 \tag{17}$$

The resultant of equations (15) and (16) w.r.t. F is

$$6\,p^3\left(-4\,p^3s+12\,p^2s^2-15\,ps^4-6\,ps^2+2\,s^4+3\,s^6+5\,p-5\,s^2\right) \quad (18)$$

Solving equation (16) for F and equation (17) for f and using the last equation (4) we obtain another equation in p, s of total degree 20. We compute the resultant w.r.t. s and p successively, of this last equation and the big factor of equation (18). Here are the results:

$$\begin{gathered}\mathcal{R}_s = p^{20}\,(3\;p^2-8)\;Q_{43}(p) \quad \text{and} \\ \mathcal{R}_p = s^9\;(s-1)^6\;(s+1)^6\;(-3\;s^2+6\;s+5)\;S_{43}(s)\end{gathered} \quad (19)$$

where $Q_{43}(p) \in \mathbb{Z}[p]$ and $S_{43}(s) \in \mathbb{Z}[s]$ are two irreducible polynomials of degree 43. The polynomial $Q_{43}(p)$ has seven real roots which are all positive. The polynomial $S_{43}(s)$ has seven real roots only two of which are positive. We are interested of course in the positive roots, since s and p must be positive. It is easy to see which roots of $Q_{43}(p)$ correspond to the two positive real roots of $S_{43}(s)$. We have the following two couples:

$$(s = 1.6471406536,\; p = 0.5816954601)$$

$$(s = 3.0079280622,\; p = 2.1262221358)$$

Remark 5. The quadratic equation which has as solutions β and δ, is $x^2 - s\;x + p = 0$. The discriminant $s^2 - 4\;p$ is positive for both couples above. So it is not possible to eliminate one of them at this stage.

We replace these values of s and p in equations (15), (16), (17) and we see that only the second couple gives a positive value of F. This proves that only one solution of the system has a physical meaning. Here is this solution:

s	p	β	δ
3.007928	2.126221	1.135608	1.872319

b	d	f	B	D	F
1.289607	3.505579	2.759878	0.682832	0.152356	0.218104

Remark 6. The above solution has been found in [2] and [11] using different methods. The fact that these methods were not directly generalizable to the more difficult case that will be considered in the next paragraph, led us to seek for another approach to the problem.

5.5. $\Delta_1,\ \Delta_2 = \Delta_3,\ \Delta_4 = \Delta_5$**.** It follows from group (B) equations that the solutions satisfying these conditions are exactly those which are invariant by a rotation of π. By inspection of the equations of group (B), this symmetry condition implies that we need to retain only five different mutual distances. For simplicity and in order to keep up with standard notation, we set

$$a = s_{12}, \quad c = s_{14}, \quad d = s_{23}, \quad e = s_{24}, \quad l = s_{45}$$

and accordingly

$$A = s_{12}^{-3/2},\ C = s_{14}^{-3/2},\ D = s_{23}^{-3/2},\ E = s_{24}^{-3/2},\ L = s_{45}^{-3/2}$$

So we have the defining relations:

$$\begin{gathered} A^2a^3 - 1 = 0 \quad C^2c^3 - 1 = 0 \quad D^2d^3 - 1 = 0 \\ E^2e^3 - 1 = 0 \quad L^2l^3 - 1 = 0 \end{gathered} \tag{20}$$

Finally, equation (2) becomes:

$$\Delta_1 + 2\ \Delta_2 + 2\ \Delta_4 = 0. \tag{21}$$

Theorem 4. *The equations* (21), (A) *and* (B) *are equivalent to the linear equation*

$$A - C + 2\ D - 2\ L = 0 \tag{22}$$

and the fact that the matrix below must have rank $r < 2$.

$$\begin{bmatrix} -5A + 5C & -8A - 8C + 4L + 4D + 8E \\ D + L - 2E & -A + C \\ -8a - 8c + d + l + 4e & -2d + 2l + 8a - 8c \\ -4c + 4a - d + l & -8e + 2d + 2l \end{bmatrix}. \tag{23}$$

Proof. By theorem (1) the following matrix must have rank $r \leq 2$.

$$M_{(5)} = \begin{bmatrix} A & 1 & \Delta_1\ \Delta_2 \\ C & 1 & \Delta_1\ \Delta_4 \\ D & 1 & \Delta_2^2 \\ L & 1 & \Delta_4^2 \\ E & 1 & \Delta_2\ \Delta_4 \end{bmatrix} \tag{24}$$

We put $\Delta_M = \Delta_2 - \Delta_4$ and by virtue of equation (21) we eliminate Δ_2, Δ_4:

$$M_{(5)} = \begin{bmatrix} A & 1 & -\frac{1}{4}\Delta_1^2 + \frac{1}{2}\Delta_1\ \Delta_M \\ C & 1 & -\frac{1}{4}\Delta_1^2 - \frac{1}{2}\Delta_1\ \Delta_M \\ D & 1 & \frac{1}{16}\Delta_1^2 - \frac{1}{4}\Delta_1\ \Delta_M + \frac{1}{4}\Delta_M^2 \\ L & 1 & \frac{1}{16}\Delta_1^2 + \frac{1}{4}\Delta_1\ \Delta_M + \frac{1}{4}\Delta_M^2 \\ E & 1 & \frac{1}{16}\Delta_1^2 - \frac{1}{4}\Delta_M^2 \end{bmatrix}$$

By elementary operations on the rows of the matrix $M_{(5)}$ we obtain:

$$M_{(5)} \sim \begin{bmatrix} A - C & 0 & \Delta_1\ \Delta_M \\ 4\ D + 4\ L - 8\ A - 8\ C + 8\ E & 0 & 5\ \Delta_1^2 \\ A + C + 2D + 2L + 4E & 10 & 0 \\ D + L - 2E & 0 & \Delta_M^2 \\ A - C + 2\ D - 2\ L & 0 & 0 \end{bmatrix}$$

In the sequel we will refer to the above matrix as $M_{(5)}$. This matrix has of course the same rank as the initial matrix $M_{(5)}$ because elementary row operations do not change the rank of a matrix.

Claim.

$$\text{rank } M_{(5)} \leq 2 \iff$$

$$\iff \begin{cases} A - C + 2\ D - 2\ L & = 0 \\ (D + L - 2E)\Delta_1 + (-A + C)\Delta_M & = 0 \\ (-5A + 5C)\Delta_1 + (4D + 4L - 8A - 8C + 8E)\Delta_M & = 0 \end{cases} \tag{25}$$

Proof of the claim. The last line of the matrix $M_{(5)}$ shows that the linear equation (22) holds. The matrix $M_{(5)}$ must be of rank ≤ 2 and this means that all its 3×3 sub-determinants must vanish. Using the hypotheses that $\Delta_1 \neq 0$ and $\Delta_M \neq 0$ this gives the following equations:

$$(D + L - 2E)\Delta_1 + (-A + C)\Delta_M = 0 \tag{26}$$

$$(-5A + 5C)\Delta_1 + (4D + 4L - 8A - 8C + 8E)\Delta_M = 0 \tag{27}$$

Conversely, all the 3×3 sub-determinants of the matrix $M_{(5)}$ are in the ideal generated by equations (25). This means that all these 3×3 sub-determinants are equal to 0 and so we have rank $M_{(5)} \leq 2$. This completes the proof of the claim.

If we consider the group (B) equations plus equation (21) as equations in the unknowns $\Delta_1,\ \Delta_2,\ \Delta_4$, then we obtain the following matrix:

$$m = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 2a & 2c & 1 \\ a & d & 2e & 1 \\ c & 2e & l & 1 \end{bmatrix} \tag{28}$$

By elementary operations on the rows of the matrix m we obtain:

$$m \sim m' = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 2a & 2c & 1 \\ a-c & d-2e & 2e-l & 0 \\ a+c & d+2e-2a & l+2e-2c & 1 \end{bmatrix}$$

Passing on to the new variables Δ_1, Δ_M and after some elementary row operations we have:

$$m' \begin{bmatrix} \Delta_1 \\ -\frac{1}{4}\ \Delta_1 + \frac{1}{2}\ \Delta_M \\ -\frac{1}{4}\ \Delta_1 - \frac{1}{2}\ \Delta_M \\ X \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 0 \\ (d+l-8a-8c+4e)\ \Delta_1 + (8a-8c+2l-2d)\ \Delta_M \\ (l-d+4a-4c)\ \Delta_1 + (2d+2l-8e)\ \Delta_M \\ (6a+6c-d-l-4e)\ \Delta_1 + (2d-2l+4c-4a)\ \Delta_M + 4X \end{bmatrix}$$

Putting together equations (26), (27) and the appropriate lines of the above matrix, we obtain a matrix which must also be of rank smaller than 2 in order to have a non trivial solution in Δ_1, Δ_M. (in order to simplify the matrix, we have divided the second column by 2 and multiplied the second line by 2 as well)

$$\begin{bmatrix} -5A+5C & 2D+2L-4A-4C+4E \\ 2D+2L-4E & -A+C \\ 4e+d+l-8a-8c & 4a-4c+l-d \\ 4a-4c+l-d & d+l-4e \end{bmatrix}$$

Remark 7. As before, we took care to preserve the symmetries during the proof of the theorem. the condition $\Delta_2 = \Delta_3, \Delta_4 = \Delta_5$ shows that the system is invariant by exchanging the points $2,3$ and $4,5$, i.e. by making the following simultaneous change of variables:

$$A \longleftrightarrow C, a \longleftrightarrow c, D \longleftrightarrow L, d \longleftrightarrow l$$

It is immediate that this change of variables changes the sign of (22) as well as the sign of the equations resulting from taking the sub-determinants of matrix (23).

It is easy to see that the kind of symmetry already exhibited in the case $\Delta_3 = \Delta_4 = \Delta_5$ appears twice here. Indeed, each one of the packets of variables A, C, a, c and D, L, d, l presents the symmetry of B, D, b, d in the case $\Delta_3 = \Delta_4 = \Delta_5$. This fact will be verified again, during the resolution of the system.

Remark 8. The equation obtained by the 2×2 sub-determinant formed by the last two rows of the matrix (23) is equal to 8 times the determinant of the matrix (28). This fact is natural because it guarantees that the system may have solutions other than the trivial solution $\Delta_1 = \Delta_2 = \Delta_4 = 0$.

5.5.1. Resolution of the system. We have transformed the original equations, by eliminating the Δ_i, and showing that in the case where $\Delta_1 \neq 0$ and $\Delta_M \neq 0$, they are equivalent to the fact that the matrix (23) has rank $r < 2$. We may take the following 2×2 determinants of this matrix equal to zero:

$$D_{(1,2)},\ D_{(1,3)},\ D_{(1,4)},\ D_{(2,3)},\ D_{(2,4)},\ D_{(3,4)} \tag{29}$$

where $D_{(i,j)}$ denotes the 2×2 determinant obtained by extracting the lines i and j. In order to solve the equations, we will once more exploit the symmetry. It seems that the most judicious choice for doing that, is to define the following transformation:

$$\begin{cases} S_1 = \sqrt{[3]}A + \sqrt{[3]}C, \quad S_2 = \sqrt{[3]}D + \sqrt{[3]}L, \\ D_1 = \sqrt{[3]}C - \sqrt{[3]}A, \quad D_2 = \sqrt{[3]}D - \sqrt{[3]}L, \\ s_1 = \sqrt{a} + \sqrt{c}, s_2 = \sqrt{d} + \sqrt{l}, \\ d_1 = \sqrt{a} - \sqrt{c}, \quad d_2 = \sqrt{l} - \sqrt{d} \end{cases} \tag{30}$$

The geometric motivation behind this definition is that s_i and d_i are the sums and the differences of the mutual distances which are exchanged by the symmetry; similarly, S_i and D_i are the sums and the differences of the inverses of these distances. The algebraic motivation is that all these quantities appear naturally in matrix (23) and equation (22). Finally, after some experimentations we see that taking the square roots of the lower-case letters and the cubic roots of the upper-case letters, is the best way to obtain low degree equations from the defining relations (20) and at the same time maintain an overall low degree.

Transformation (30) is invertible, and we can easily see that its inverse transformation is given by:

$$\left\{\begin{array}{l} A = \left(\frac{S_1-D_1}{2}\right)^3, \quad C = \left(\frac{S_1+D_1}{2}\right)^3, \quad D = \left(\frac{S_2+D_2}{2}\right)^3, \\ L = \left(\frac{S_2-D_2}{2}\right)^3, \quad a = \left(\frac{s_1+d_1}{2}\right)^2, \quad c = \left(\frac{s_1-d_1}{2}\right)^2, \\ d = \left(\frac{s_2-d_2}{2}\right)^2, \quad l = \left(\frac{s_2+d_2}{2}\right)^2. \end{array}\right. \tag{31}$$

We apply now transformation (31) to equations (29) and equation (22) and it remains to determine its effect on relations (20). Using the fact that upper-case letters are inverses of cubes of mutual distances and lower-case letters are squares of mutual distances, we rewrite equations (20) as:

$$\sqrt{}[3]A\sqrt{a} = 1, \ \sqrt{}[3]C\sqrt{c} = 1, \ \sqrt{}[3]D\sqrt{d} = 1, \ \sqrt{}[3]L\sqrt{l} = 1.$$

We obtain the following equations:

$$(s_1 + d_1)(S_1 - D_1) = 4, \ (s_1 - d_1)(S_1 + D_1) = 4,$$
$$(s_2 - d_2)(S_2 + D_2) = 4, \ (s_2 + d_2)(S_2 - D_2) = 4$$

Coupling these four equations with equation (22), and the six equations resulting from the 2×2 sub-determinants of the matrix (23) we obtain a system of eleven equations. By homogeneity, we may take $e = E = 1$, reducing the number of unknowns to eight. The Gröbner basis for a lexicographical ordering of the variables is computed with FGB in approximately 16 minutes[1,2].

The result is given using a variant of the Rational Univariate Representation (RUR) in order to have smaller coefficients. Firstly, we compute the Gröbner basis with s_1 as the lexicographically smallest variable, in order to obtain a univariate polynomial in s_1. Secondly, we compute the Gröbner basis with s_2 as the lexicographically smallest variable, in order to obtain a univariate polynomial in s_2. Indeed, we obtain two polynomials of degree 228, which have the following factorizations:

$$P_{228}(s_1) = C_1 \ P_{12}(s_1) \ P_{216}(s_1)$$

[1] The computation of the degree basis is accessible in GB, but the change of basis is not.

[2] This is the only formulation of the equations, all accessible in FGB, which is partially accessible in GB.

$$Q_{228}(s_2) = C_2\ Q_{12}(s_2)\ Q_{216}(s_2)$$

Where C_1 and C_2 are constants and $P_{12}(s_1), P_{216}(s_1)$ belong to $\mathbb{Z}[s_1]$, $Q_{12}(s_2), Q_{216}(s_2)$ belong to $\mathbb{Z}[s_2]$ and subscripts denote degrees of polynomials. These factorizations were obtained using MAGMA and some C procedures developed by V. Shoup. The polynomials $P_{12}(s_1)$, $Q_{12}(s_2)$ correspond to a solution where $\Delta_1 = \Delta_4$ that has been already described. Here is the structure of the Gröbner basis, after the application of the lextriangular algorithm, for a lexicographical ordering of the variables with s_2 as the lexicographically smallest variable:

$$\begin{aligned}\mathcal{G} = \{&Q_{228}(s_2), s_1 + P_{225}(s_2), S_1 + P_{227}(s_2), S_2 + Q_{227}(s_2),\\ &d_2^2 + R_{227}(s_2), D_1 + d_2 S_{227}(s_2), D_2 + d_2 T_{227}(s_2), d1 + d_2 U_{227}(s_2)\}\end{aligned} \tag{32}$$

where all univariate polynomials in s_2 have rational coefficients.
It is important to mention that equations (32) are to be interpreted using the RUR (see [13]). Using the above representation of the Gröbner basis we can establish an one-to-one correspondence between the real roots of $P_{216}(s_1), Q_{216}(s_2)$.

The polynomial $P_{216}(s_1)$ has 10 real roots, namely

−1.05339335	2.40622346	−0.97442849
0.80410546	1.30286029	1.15012152

−1.23527834	−0.87870124
−.83669652	−1.46014747

The polynomial $Q_{216}(s_2)$ has 10 real roots, namely

−3.96240457	−3.18414039	−1.42834425
−0.49280548	−0.13502004	1.79582587

−0.97834829	−0.92981485
1.85034717	3.939108866

The fact that each of the polynomials $P_{216}(s_1), Q_{216}(s_2)$ has only 10 real roots, has been proved using the Uspensky algorithm. This algorithm yields isolating intervals with rational extremities for every real root of a univariate polynomial with integer coefficients. In practice, it is more efficient to work with this algorithm, than the Sturm algorithm. In fact it is impossible to compute the Sturm series of polynomials, for any of the two polynomials $P_{216}(s_1),\ Q_{216}(s_2)$. Calculating isolating intervals with rational extremities for the roots of a univariate polynomial,has the advantage that one can easily provide a supplementary verification by making sure that the polynomial changes sign in every interval. It is important to note that in both cases, there are only exact calculations involved. In this way we achieve delaying the use of numerical analysis, as much as possible. For a detailed explanation of the Uspensky algorithm, with references to other original papers and several implementation issues, one can consult [13].

Due to the fact that s_1, s_2 represent sums of positive numbers, only one couple of values, namely ($s_1 = 1.15012152,\ s_2 = 1.79582587$), is acceptable.

From the Gröbner basis in its RUR form, we compute now the value of d_2^2 for this couple of values. We obtain a negative value for d_2^2. This means that d_2 is an imaginary number and so none of the ten solutions has a physical meaning.

The computations have been done using three different programs:

1. RealSolving, [13]. Application of the Uspensky algorithm on $P_{216}(s_1)$, $Q_{216}(s_2)$. (computations done by Fabrice Rouillier).
2. MPS [4] and Maple [6]. Calculation of numerical values of their real roots of $P_{216}(s_1), Q_{216}(s_2)$ and manipulation of the FGB result to establish the correspondence between these roots. (computations done by Ilias Kotsireas and Jean-Charles Faugère).
3. Magma, [5]. Calculation of numerical values of the real roots of $P_{216}(s_1), Q_{216}(s_2)$. (computations done by Allan Steel).

All numerical calculations and verifications have been performed with an accuracy of 300 decimal digits.

5.6. Computer Algebra Algorithms and Tools.

In this section we will mention the Symbolic Computation algorithms that were used in the proofs, as well as some information about their implementations. All software used in this paper, is publicly available, with one exception that will be rectified in the future.

5.6.1. The Rational Univariate Representation (RUR).

The notion of the Rational Univariate Representation (RUR) of the solutions of a zero-dimensional system of polynomial equations, has been introduced by F. Rouillier in [13]. It has been subsequently integrated in the FGB program, because its use leads to smaller coefficients in the representation of the solution.

Let K be an ordered field with algebraic closure $\mathcal{C}$. Let $\mathcal{S} = \{f_1, \dots, f_s\}$ be a finite set of polynomials in $K[X_1, \dots, X_n]$. Let $\mathcal{I} \subset K[X_1, \dots, X_n]$ be the ideal generated by $\mathcal{S}$. The system $\mathcal{S}$ is called zero-dimensional, if it has a finite number of zeros in $\mathcal{C}^n$. The corresponding ideal $\mathcal{I}$ is also called zero-dimensional. Let $V_{\mathcal{C}}(\mathcal{I})$ be the variety defined by the ideal $\mathcal{I}$, that is the set of all common zeros on $\mathcal{C}^n$, of the elements of $\mathcal{S}$. The RUR manages to bring the study of a zero-dimensional ideal of $K[X_1, \dots, X_n]$ to the study of a family of rational fractions in $K[T]$ where T is another variable.

5.6.2. The GB **software.**

The GB program was developed by J.-C. Faugère in 1994. It is a program of resolution of systems of polynomial equations. Three basic algorithms of Gröbner bases theory have been implemented in GB.

1. The Buchberger algorithm with sugar strategy and modular trace.
2. The FGLM change of base algorithm.
3. The lextriangular algorithm.

The user can specify different representations for the polynomials and different orderings for the monomials, including degree and lexicographical orderings. Other capabilities are Hilbert Function and Normal Form computations.

5.6.3 The FGB **software.** The FGB (see [9]) program was developed by J.-C. Faugère in 1997. It is the successor of GB in terms of both efficiency due to programming improvements and algorithmic progress. The new algorithm (F4) has been implemented in FGB. The considerable speed-up in Gröbner bases computations with FGB is demonstrated by an on-line database of examples.

5.6.4. The MAPLE **Computer Algebra System.** The graphic interface of the Computer Algebra system MAPLE (see [6]) was of particular use in the proofs of theorems (6) and (23) in order to determine the best way to perform the elementary row operations without destroying the symmetry. MAPLE was also used for processing the results of the computations produced by GB and FGB and for some numerical verifications.

5.6.5. The MAGMA **Computer Algebra System.** We used MAGMA (see [5]) to find the roots of the polynomials P_{216} and Q_{216}. The algorithm used in MAGMA is due to A. Schönhage and it was implemented by X. Gourdon. An extensive internet correspondence with Allan Steel permitted to obtain the correct solutions.

5.6.6. The RealSolving **software.** It is a program developed by F. Rouillier (see [13]) which computes isolating intervals with rational extremities for real roots of a univariate polynomial with rational coefficients. It uses the Uspensky algorithm which is especially efficient for use with very big polynomials.

5.6.7. The MPS **package.** It is a package developed by D. Bini and G. Fiorentino (see [4]) within the European Community project FRISCO. It is interfaced with MAPLE and uses Aberth's method.

Conclusion. We have proved that there are exactly 4 central configurations of 5 non-planar points in 3-dimensional space, which are invariant under some rotations. Completing the enumeration of central configurations, would require to treat the case with only one planar symmetry ($\Delta_1 = \Delta_2$) and the case with no symmetry at all ($\Delta_i \neq \Delta_j \forall i \neq j$). It is conjectured that there are no such central configurations, i.e. that our list of central configurations is complete. The most important fact that can lead to easier proofs, seems to be a deeper understanding of symmetry.

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