Записки научных
семинаров ПОМИ
Том 259, 1999 г.

A. V. Ivanov and J. F. Rodrigues

EXISTENCE AND UNIQUENESS OF A WEAK SOLUTION TO THE INITIAL MIXT BOUNDARY VALUE PROBLEM FOR QUASILINEAR ELLIPTIC-PARABOLIC EQUATIONS

§1. Introduction

Let $Q_T = \Omega \times (0,T]$, $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$. In the cylinder $Q_T$we consider equation

$$Mu := \partial_t b(u) - \operatorname{div} a(b(u), \nabla u) = f(x,t). \tag{1.1}$$

Here $b(u)$ is a nondecreasing function continuous on $\mathbb{R}$ with $b(0)=0$ and $a(n,p)$ is a vector-valued function continuous on $\mathbb{R}\times\mathbb{R}^n$ and satisfying the growth condition

$$|a(w,p)|^{m'} \leqslant c(|p|^m + \Psi(w)+1), \quad m>1, \quad 1/m+1/m'=1, \tag{1.2}$$

where Ψ is the Legendre transform of the primitive of b, i.e.,

$$\Psi(w) = \sup_{z\in\mathbb{R}} \left\{ zw - \int\limits_0^z b(\zeta)d\zeta \right\}. \tag{1.3}$$

From (1.3) it follows that

$$\Psi(b(u)) = ub(u) - \int\limits_0^u b(\zeta)d\zeta. \tag{1.4}$$

Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geqslant 1$, with Lipschitz boundary $\partial\Omega$, $\gamma_D b$ be H^{n-1}-measurable with positive measure, $\gamma_s = \partial\Omega\backslash\gamma_D$, $\Gamma_D = \gamma_D \times (0,T]$, $\Gamma_s = \gamma_s \times (0,T]$. Define

$$Nu := a(b(u),\nabla u)\vec{\nu} \tag{1.5}$$

This work was partially supported by INTAS (grant No. 96-0835) and RFFI (grant No. 99-01-00108), Федеральная целевая программа "Интеграция", грант No. 326.53, "Научные школы", грант No. 96-15-96121.

with an outer unit vector $\vec{\nu}$ of $\partial\Omega$.

Under the strict monotonicity condition

$$(a(w,p)-a(w,q))\cdot(p-q)\geqslant c|p-q|^m, \quad m>1, \tag{1.6}$$

in [1] Alt–Luckhaus proved the solvability of the problem

$$\left.\begin{array}{l} Mu=f \quad \text{in} \quad Q_T, \\ u=g \text{ on } \Gamma_D,\ Nu=h \text{ on } \Gamma_s,\ b(u)=b_0 \text{ on } \Omega\times\{t=0\} \end{array}\right\} \tag{1.7}$$

(with $f=0$, $g\in W_m^{1,0}(Q_T)\cap L_\infty(Q_T)$, $\partial_t g\in L_1(0,T;L_\infty(\Omega)$, $h=0$, $b_0=b(u_0)$, $u_0=u_0(x)$ is measurable on Ω, $b_0\in L_1(\Omega)$, $\Psi(b_0)\in L_1(\Omega))$ in the natural class of weak solutions with finite energy

$$\sup_{t\in[0,T]}\int\limits_\Omega \Psi(b(u))dx+\iint\limits_{Q_T}|\nabla u|^m dx\,dt<+\infty. \tag{1.8}$$

The uniqueness of such weak solution was proved in the recent paper [2] by Otto; along with (1.2) and (1.6) in [2] it was assumed that

$$\begin{gathered}|a(b(u_1),p)-a(b(u_2),p)|^{m'}\leqslant \\ c|u_1-u_2|\{|p|^m+\Psi(b(u_1))+\Psi(b(u_2))+1\}.\end{gathered} \tag{1.9}$$

Also, under the same assumptions, Otto established an important comparison theorem for the problem (1.7).

Actually, the results of [1] and [2] mentioned above were obtained for a mixt problem with an additional term $g(b(u))$ on the left-hand side of (1.1) (under appropriate restrictions on the function $g(w)$). For simplicity of presentation, here we limit ourselves to considing the problem (1.7) only.

In the study of turbulent filtration of a fluid and a gas in porous media [3–6], or of unstationary saturated-unsaturated flows in porous media [7, 8], as well as in the theory of non-Newtonian fluids [9] and in mathematical glaciology [10, 11], the following equations arise

$$\mathcal{L}u:=\partial_t b(u)-\operatorname{div}\{|\nabla u+k(b(u))\vec{e}|^{m-2}(\nabla u+k(b(u))\vec{e})\}=f(x,t) \tag{1.10}$$

where $m>1$, b is a nondecreasing function continuous on $\mathbb{R}$, k is a function continuous on $b(\mathbb{R})$, and $|\vec{e}|=1$.

However, the results of [1, 2], and [12] do not apply to equation (1.10) with $m\in(1,2)$, because the strict monotonicity condition (1.7) is fulfilled for (1.10) only if $m \geqslant 2$. If $m \in (1,2)$, for (1.10) we have the following strict monotonicity condition:

$$[a(w,p)-a(w,q)](p-q) \geqslant c|p-q|^2(|p+k(w)\vec{e}|^m+|q+k(w)\vec{e}|^m)^{\frac{m-2}{m}}. \tag{1.11}$$

The extention of results of [1] and [2] to equation (1.11) with $m \in (1,2)$ is given in paper [12] in the case of the Dirichlet boundary condition only; moreover in [12] the obstacle problem for equations (1.10) with $m > 1$ is stadied.

The main aim in the present paper is to extend the results of [1], [2] and [12] on the mixt boundary value problem to equations (1.10) with any $m > 1$.

§2. Statements of the main results

Throughout, we assume that the following conditions are fulfilled:

1) Ω is a bounded open set in $\mathbb{R}^n$, $n \geqslant 1$ with Lipschitz boundary $\partial\Omega$, γ_D is H^{n-1}-measurable with positive measure, $\gamma_s = \partial\Omega \setminus \gamma_D$, $\Gamma_D = \gamma_D \times (0,T]$, $\Gamma_s = \gamma_s \times (0,T]$;

2) b is a monotone nondecreasing continuous functions on $\mathbb{R}$ such that $b(0) = 0$, and k is a function on $b(\mathbb{R})$;

3) $f = f(x,t)$, $g = g(x,t)$, and $h = h(x,t)$ are functions satisfying

$$f \in L_{m'}(Q_T), \quad 1/m + 1/m' = 1, \quad m > 1, \tag{2.1}$$

$$g \in L_m(0,T;W_m^1(\Omega)) \cap L_\infty(Q_T), \quad \partial_t g \in L_1(0,T;L_\infty(\Omega)), \tag{2.2}$$

$$h \in L_{m'}(\Gamma_s), \quad 1/m + 1/m' = 1, \quad m > 1; \tag{2.3}$$

4) $b_0 = b(u_0)$ is an initial function such that $b_0 = b(u_0)$ for some function $u_0 = u_0(x)$ measurable on Ω, and

$$b_0 \in L_1(\Omega), \quad \Psi(b_0) \in L_1(\Omega) \tag{2.4}$$

(see (1.3)).

In [1] it was shown that Ψ is superlinear in the following sense:

$$|w| \leqslant \delta\Psi(w) + C_\delta \text{ for any } \delta > 0.$$

Therefore, the first condition in (2.4) is in fact a consequence of the second.

Let $V = \{v \in W_m^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, and let V^* denote the space dual to the Banach space V.

Assume that the growth condition (1.2) is fulfilled. A function u is called a weak solution of the problem (1.7) if $u \in g + L_m(0,T;V)$ satisfies (1.8) and

$$\left.\begin{aligned} M\{u,\zeta\} := \iint\limits_{Q_T} [(b_0 - b(u))\partial_t\zeta + a(b(u),\nabla u)\nabla\zeta]dx\,dt = \\ = \iint\limits_{Q_T} f\zeta dx\,dt + \iint\limits_{\Gamma_s} h\zeta ds\,dt =: G(f,h,\zeta) \\ \text{for any } \zeta \in L_m(0,T;V)\cap W_1^1(0,T;L_\infty(\Omega)) \text{ with } \zeta(T)=0. \end{aligned}\right\} \tag{2.5}$$

Obviously, the identity in (2.5) implies that the derivative $\partial_t b(u)$ (in the sense of distributions) belongs to $L_{m'}(0,T;V^*)$, and

$$\left.\begin{aligned} \int\limits_0^T \langle\partial_t b(u),\zeta\rangle dt + \iint\limits_{Q_T} a(b(u),\nabla u)\nabla\zeta dx\,dt = G(f,h,\zeta) \\ \text{for any } \zeta \in L_m(0,T;V), \end{aligned}\right\} \tag{2.6}$$

where $\langle\cdot,\cdot\rangle$ denotes duality between V^* and V.

The following important statements were proved by Alt and Luckhaus.

Proposition 2.1 (Alt and Luckhaus, [1]). *Let*

$$u \in L_m(0,T;V),\ \ b(u) \in L_\infty(0,T;L_1(\Omega)),\ \ \partial_t b(u) \in L_{m'}(0,T;V^*).$$

Suppose that for all $\zeta \in L_m(0,T;V)$ *with* $\partial_t\zeta \in L_\infty(Q_T)$ *and* $\zeta(T) = 0$ *we have*

$$\int\limits_0^T \langle\partial_t b(u),\zeta\rangle dt = \iint\limits_{Q_T} (b_0 - b(u))\partial_t\zeta dx\,dt.$$

Then $\Psi(b(u))$ *belongs to* $L_\infty(0,T;L_1(\Omega))$, *and*

$$\int\limits_\Omega [\Psi(b(u(\tau))) - \Psi(b_0)]dx =$$

$$= \int\limits_0^\tau \langle\partial_t b(u),u\rangle dt - \int\limits_0^\tau\!\!\int\limits_\Omega (b(u) - b(u_0))\partial_t g\,dx\,dt + \int\limits_\Omega (b(u(t)) - b(u_0)g(t)dx$$

for almost all (a.a.) $\tau \in (0,T)$.

Proposition 2.2 (Alt and Luckhaus, [1]). *Suppose that u_ε converge weakly in $L_m(0,T;W^1_m(\Omega))$ to function u with the estimates*

$$\frac{1}{h}\int_0^{T-h}\int_\Omega (b(u_\varepsilon(t+h))-b(u_\varepsilon(t)))(u_\varepsilon(t+h)-u_\varepsilon(t))dt \leqslant c \tag{2.7}$$

and

$$\int_\Omega \Psi(b(u_\varepsilon(t)dx \leqslant c \quad \text{for} \quad 0<t<T.$$

Then $b(u_\varepsilon)\to b(u)$ strongly in $L_1(Q_T)$, and $\Psi(b(u_\varepsilon))\to\Psi(b(u))$ almost everywhere (a.e.) in Q_T.

With the help of Propositions 2.1 and 2.2 Alt and Luckhaus proved in [1] that under conditions (1.2) and (1.6) problem (1.7) (with $f=0$ in Q_T and $h=0$ on Γ_s) has at least one weak solution.

In order to state the results by Otto, first we give some definitions, assuming that condition (1.2) is fulfilled.

A function u is called as a subsolution of the problem (1.7) with initial function b_0 (satisfying condition 4) above) if $u\in g+L_m(0,T;V)$ satisfies (1.8) and for all functions ζ nonnegative on Q_T and possessing the same properties as in (2.5) we have

$$M\{u,\zeta\}\leqslant G(f,h,\zeta). \tag{2.8}$$

A function u is called as supersolution of the problem (1.7) with initial function b_0 (satisfying condition 4)) if $u\in g+L_m(0,T;V)$ satisfies (1.8) and for ζ as above we have

$$M\{u,\zeta\}\geqslant G(f,h,\zeta). \tag{2.9}$$

The results of [2] imply the following comparison theorem.

Proposition 2.3. *Suppose that conditions* (1.2), (1.6), *and* (1.9) *are fulfilled. Let u_1 be a subsolution of the problem* (1.7) *with right-hand part f_1 and initial function $b_0^{(1)}$ and let u_2 be a supersolution of the problem* (1.7) *with right-hand part f_2 and initial function $b_0^{(2)}$; we assume that f_1 and f_2 satisfy conditions* 3), *and that $b_0^{(1)}$ and $b_0^{(2)}$ satisfy* 4). *Then for* *any nonnegative function* $\gamma \in C_0^\infty(\mathbb{R}^n \times (-\infty, T])$ *we have*

$$\left.\begin{aligned}\iint\limits_{Q_T} \{[(b_0^{(1)} - b_0^{(2)})^+ - (b(u_1) - b(u_2))^+]\partial_t\gamma + \\ +\operatorname{sign}^+(u_1 - u_2)[a(b(u_1), \nabla u_1) - a(b(u_2), \nabla u_2)]\nabla\gamma\} dx\, dt \leqslant 0.\end{aligned}\right\} \tag{2.10}$$

In particular,

$$\int\limits_\Omega [b(u_1(t)) - b(u_2(t))]^+ dx \leqslant \int\limits_\Omega (b_0^{(1)} - b_0^{(2)})^+ dx \tag{2.11}$$

for a.e. $t \in (0, T)$.

Uniquence theorem of Otto [2] is a consequence of Proposition 2.3: under conditions (1.2), (1.6), (1.9) the problem (1.7) with $f(x, t) = 0$ in Q_T and $h(x, t) = 0$ on S_T has at most one weak solution.

Now we consider the problem

$$\left.\begin{aligned}&\mathcal{L}u = f \quad \text{in} \quad Q_T \\ &u = g \text{ on } \Gamma_D,\ Nu = h \text{ on } \Gamma_S,\ b(u) = b_0 \text{ on } \Omega \times \{t = 0\}\end{aligned}\right\} \tag{2.12}$$

where $\mathcal{L}u$ is the operator defined in (1.10); it is assumed that on the set $b(\mathbb{R})$ we have

$$|k(w)|^m \leqslant c(\Psi(w) + 1), \quad m > 1. \tag{2.13}$$

Since, by (2.13), the function

$$a(w, p) = |p + k(w)\vec{e}|^{m-2}(p + k(w)\vec{e}) \tag{2.14}$$

satisfies (1.2), the above definition of a weak solution applied to problem (2.12).

Let $\Phi_m(z)$ denote the following map acting from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$\Phi_m(z) := |z|^{m-2}z, \quad m > 1. \tag{2.15}$$

It is well known that

$$\left.\begin{aligned}[\Phi_m(z_1) - \Phi_m(z_2)](z_1 - z_2) \geqslant c|z_1 - z_2|^\varkappa(|z_1|^m + |z_2|^m)^{1-\frac{\varkappa}{m}} \\ \varkappa = m \text{ if } m \geqslant 2,\ \varkappa = 2 \text{ if } m \in (1, 2),\ c = c(m, n).\end{aligned}\right\} \tag{2.16}$$

Then for the function $a(w, p)$ defined by (2.14), for all $w \in \mathbb{R}$, $p, q \in \mathbb{R}^n$ we have

$$\left.\begin{aligned}[a(w, p) - a(w, q)](p - q) \geqslant |p - q|^\varkappa(|p + k(w)\vec{e}|^m + |q + k(w)\vec{e}|^m)^{\frac{m-\varkappa}{m}} \\ \varkappa = m \text{ if } m \geqslant 2,\ \varkappa = 2 \text{ if } m \in (1, 2) \quad c = c(n, m).\end{aligned}\right\} \tag{2.17}$$

Obviously, (2.17) is equivalent to (1.6) for $m \geqslant 2$ and to (1.11) for $m \in (1,2)$. Also, it is easy to check that (2.17), (2.13) imply the inequality

$$a(w,p)p \geqslant \nu_0|p|^m - c(\Psi(w)+1), \quad \nu_0>0, \quad c \geqslant 0, \quad m>1. \tag{2.18}$$

We note that for any $m>1$ inequality (2.18) is also a consequence of condition (1.6).

Now we introduce the following conditions:

$$|k(w_1)-k(w_2)| \leqslant c|w_1-w_2|^{1/m}, \quad w_1,w_2 \in b(\mathbb{R}) \tag{2.19}$$

and

$$\left.\begin{array}{l} |k(b(u_1))-k(b(u_2))| \leqslant c|u_1-u_2|^{\alpha}(\Psi(b(u_1))+\Psi(b(u_2))+1)^{1/m}, \\ \alpha = 1/m' \text{ if } m \geqslant 2, \ \alpha = 1/m \text{ if } m \in (1,2), \ u_1,u_2 \in \mathbb{R}. \end{array}\right\} \tag{2.20}$$

The results of the present paper imply the following statements.

Theorem 2.1. *Under conditions* (2.13) *and* (2.19), *for any* $m>1$ *problem* (2.12) *has at least one weak solution.*

Theorem 2.2. *Under conditions* (2.13) *and* (2.20), *for any* $m>1$ *problem* (2.12) *has at most one weak solution.*

Remark 2.1. Essentially for $m \geqslant 2$ Theorems 2.1 and 2.2 are consequences of the results of [1] and [2] respectively. Indeed, if $m \geqslant 2$, then (2.13), (2.17) imply (1.2) and (1.6) while from (2.13), (2.20) it follows that (1.9) is true. These implications were proved in [12], p. 79. The proof of Theorems 2.1 and 2.2 in the case $m \in (1,2)$ are given in the sections 3 and 4 of the present paper; in the case when in addition $\gamma_D = \Omega$ Theorems 2.1 and 2.2 for $m \in (1,2)$ were proved in [12]. The proof of existence of weak solution of problem (2.12) given in the present paper is different from one obtained in [12].

§3. The approach of Alt–Luckhaus

As in [12] the proof of existence of weak solution of the initial boundary value problem for equation (1.10) essentially used the ideas of paper [1] by Alt–Luckhaus. Here as approximating problems we use following [1] backward time differences instead of parabolic regularization like in paper [12].

First we consider problem (1.7) assuming that conditions (1.2) and (2.18) are fulfilled. Recall that (2.18) is a consequence of condition (1.6) for all $m > 1$. In the same way as in [1] we can use the following arguments.

We choose linearly independent functions $e_i \in V \cap L_\infty(\Omega)$ such that the subspace spanned by these functions is dense in V. We are looking for a function

$$u_{hl}(x,t) = g(x,t) + \sum_{i=1}^{l} \alpha_{hli}(t) e_i(x)$$

with $\alpha_{hli} \in L_\infty([0,T])$ such that for a.a $t \in (0,T)$ the equation

$$\left.\begin{aligned} \mathcal{L}_{hl}\{u_{hl},\zeta\} = \int_\Omega [\partial_t^{-h} b(u_{hl}(t)\zeta + a(b(u_{hl}(t), \nabla u_{hl}(t))\nabla\zeta] dx - \\ - \int_\Omega f_h \zeta dx - \int_{\gamma_s} h_h \zeta ds = 0 \end{aligned}\right\} \tag{3.1}$$

holds for all test functions $\zeta \in V_m := \operatorname{span}\{e_1, \dots, e_m\}$, where the initial data are given by

$$u_{hl} = u_0^h(t) \text{ for } -h < t < 0.$$

We choose the approximating initial, boundary, and right-hand part functions as follows:

$$\left.\begin{aligned} u_0^h &:= \min\left(1, \frac{1}{h|u_0|}\right) u_0, \\ g_h &:= \frac{1}{h} \int_{(k-1)h}^{kh} g(x,t)dt \text{ for } (k-1)h \leqslant t \leqslant kh, \\ h_h &:= \frac{1}{h} \int_{(k-1)h}^{kh} h(s,t)dt \text{ for } (k-1)h \leqslant t \leqslant kh, \\ f_h &:= \frac{1}{h} \int_{(k-1)h}^{kh} f(x,t)dt \text{ for } (k-1)h \leqslant t \leqslant kh, \end{aligned}\right\} \tag{3.2}$$

where for simplicity it is assumed that T/h is integer. From (3.2) it follows that u_0^h is bounded in Q_T while g_h, h_h and f_h are time independent in each interval $((k-1)h, kh)$. Moreover (3.2) implies that we can determine $u_{hl}(t)$ inductively for $t \in ((k-1)h, kh)$ as a solution of an elliptic problem.

For the proof of this, we observe that monotonicity of function b implies that

$$\Psi(b(z)) - \Psi(b(z_0)) \geqslant (b(z) - b(z_0))z_0 \text{ for all } z, z_0 \in \mathbb{R}. \tag{3.3}$$

Later, if we choose $z = \frac{w}{\sigma|w|}$ with some $w \in \mathbb{R}$, $w \neq 0$ and $\sigma > 0$ we derive from (1.3) that

$$\Psi(w) \geqslant \frac{1}{\sigma}\left(|w| - \sup_{|z|\leqslant\frac{1}{\sigma}} |b(z)|\right)$$

so that

$$|w| \leqslant \delta\Psi(w) + c_\delta, \quad c_\delta = \sup_{|\zeta|\leqslant\frac{1}{\delta}} |b(\zeta)|. \tag{3.4}$$

In particular, from (3.2) it follows that since $\Psi(b(z)) \leqslant zb(z)$ then for any $g \in \mathbb{R}$, $z \in \mathbb{R}$ and $\delta > 0$

$$b(g+z) \leqslant \delta(g+z)b(g+z) + c_\delta.$$

Hence using boundedness of g we can derive from here that

$$b(g_h + z)g_h \leqslant \delta(g_h + z)b(g_h + z) + C(\delta) \quad \text{for any} \quad \delta > 0. \tag{3.5}$$

If $u_{hl}(t-h)$ is known, then left side of (3.1) defines a continuous mapping $\Phi_{hl} : \mathbb{R}^l \to \mathbb{R}^l$. Namely, if $v = \sum_{i=1}^l \alpha_i e_i$ and $\zeta = \sum_{i=1}^l \alpha_i e_i$ we have

$$\left.\begin{aligned} \Phi_{hl}(\alpha)\alpha = \mathcal{L}_{hl}\left\{u_{hl}, \sum_{i=1}^{l} \alpha_i e_i\right\} \geqslant \nu_0 \int_\Omega |\nabla v|^m dx + \\ +\frac{1}{h}[b(g_h + v) - b(u_{hl}(t-h))]v\,dx - c\int_\Omega (1 + \Psi(g_h + v) + |\nabla g_h|^m)dx + \\ +|\int_{\gamma_s} h_h v\,ds|, \end{aligned}\right\} \tag{3.6}$$

where c depends, in particular, on $\|f_h\|_{m',\Omega}$ and $\|g_h\|_{m',\gamma_s}$. Proving (3.6) we have used inequalities

$$\left.\begin{aligned} \left|\int\limits_{\Omega} f_h v\,dx\right| &\leqslant \varepsilon \int\limits_{\Omega} |v|^m dx + c(\varepsilon)\int\limits_{\Omega} |f_h|^{m'} dx,\\ \left|\int\limits_{\gamma_s} h_\varepsilon v ds\right| &\leqslant \varepsilon \int\limits_{\gamma_\varepsilon} |v|^m ds + c(\varepsilon)\int\limits_{\gamma_\varepsilon} |h_h|^{m'} ds,\\ \int\limits_{\gamma_\varepsilon} |v|^m ds &\leqslant c\int\limits_{\Omega} |\nabla v|^m dx. \end{aligned}\right\} \tag{3.7}$$

Obviously from (3.2) it follows that

$$\left.\begin{aligned} \int\limits_{\Omega} |f_h|^{m'} dx &\leqslant c(h^{-1}) \iint\limits_{Q_T} |f|^{m'} dx\, dt,\\ \int\limits_{\gamma_S} |h_\varepsilon|^{m'} ds &\leqslant c(h^{-1}) \iint\limits_{\Gamma_S} |h|^{m'} dx\, dt. \end{aligned}\right\}$$

Using (1.2), (3.7), and (3.5) we derive from (2.34) that

$$\left.\begin{aligned} \Phi_{hl}(\alpha)\alpha \geqslant \frac{\nu_0}{2}\int\limits_{\Omega} |\nabla v|^m\, dx + \left(\frac{1}{2h} - c\right)\int\limits_{\Omega} (g_h + v) b(g_h + v) dx - \\ -c(h^{-1})\left(1 + \int\limits_{\Omega} |b(u_{hl}(t-h)|^{m'}\, dx\right). \end{aligned}\right\} \tag{3.8}$$

If h is small enough independently of l, the second term in (3.8) is non-negative. Hence $\Phi_{hl}(\alpha)\alpha \geqslant 0$ if the norm of v in V_l is sufficient large, i.e., Φ_{hl} has a zero, that is u_{hl} exists for all $h \in (0, h_0)$ with some $h_0 > 0$ and any $l \in \mathbb{N}$.

From (3.1) it follows that for any $\tau \in (0,T)$

$$\left.\begin{aligned}\int_0^\tau\int_\Omega [\partial_t^{-h} b(u_{hl})\zeta + a(b(u_{hl},\nabla u_{hl})\nabla\zeta]dx\,dt = \\ = \int_0^\tau\int_\Omega f_h\zeta dx\,dt + \int_0^\tau\int_{\gamma_s} h_h\zeta ds\,dt, \\ \text{for all}\quad \zeta \in L_m(0,T;V_m)\quad \text{wich are} \\ \text{time independent in each interval}\quad ((k-1)h,kh).\end{aligned}\right\} \tag{3.9}$$

Testing in (3.7) with $\zeta = u_{hl} - g_h$ we obtain the a priori estimate in the energy norm. Really, using (3.3) we have for a.a. $t > 0$ pointwise in Q_T

$$\Psi(b(u_{hl}(t))) - \Psi(b(u_{hl}(t-h)) \leqslant (b(u_{hl}(t)) - b(u_{hl}(t-h))u_{hl}(t). \tag{3.10}$$

Recall that $u(t) := u_0^h$ for $-h < t < 0$ and $b(u(t-h)) = b_0$ if $t < h$.

From (3.9) and (3.7) it follows that $\frac{1}{h}[b(u_{hl}(t)) - b(u_{hl}(t-h))]$ belongs to $L_{m'}(0,T;V_*)$ and for any $\zeta \in L_m(0,T;V) \cap L_\infty(Q_T)$ we have

$$\int_0^\tau\int_\Omega \partial_t^{-h} b(u_{hl})\zeta dx\,dt = \int_0^\tau \langle\partial_t^{-h} b(u_{hl}),\zeta\rangle dt. \tag{3.11}$$

Then using boundedness of function $\zeta = u_{hl} - g_h$ we derive from (3.10) that

$$\left.\begin{aligned}\int_0^\tau\int_\Omega [\Psi(b(u(t)) - \Psi(b(u(t-h))]dx\,dt \leqslant \\ \leqslant \int_0^\tau\int_\Omega [b(u(t)) - b(u(t-h))]u_{hl}(t)dx\,dt = \\ = h\int_0^\tau \langle\partial_t^{-h} b(u_{hl}), u_{hl} - g_h\rangle dt + \\ + \int_0^\tau\int_\Omega [b(u(t)) - (b(u(t-h))]g_h dx\,dt.\end{aligned}\right\} \tag{3.12}$$

From (3.12) it follows that we can estimate the parabolic parm of (3.9) as follows:

$$\left.\begin{aligned} &\int_0^\tau\int_\Omega \partial_t^{-h} b(u_{hl})(u_{hl}-g_h)dx\,dt \geqslant \\ &\geqslant \frac{1}{h}\int_{\tau-h}^{\tau}\int_\Omega \Psi(b(u_{hl}))dx\,dt - \int_\Omega \Psi(b_0)dx + \\ &\quad + \int_0^\tau\int_\Omega (b(u_{hl})-b(u_h^0))\partial_t^h g_h\,dx\,dt - \\ &-\frac{1}{h}\int_{\tau-h}^{\tau}\int_\Omega (b(u_{hl})-b(u_h^0))g_h(t+h)dx\,dt. \end{aligned}\right\} \tag{3.13}$$

For the elliptic part we get the usual estimates and estimates wich can be derived from (3.7). Hence using the assumption on g, inequality (3.4) and Gronwall argument, we obtain the apriori estimate

$$\sup_{\tau\in[0,T]}\int_\Omega \Psi(b(u_{hl}))dx + \iint_{Q_T} |\nabla u_{hl}|^m dx\,dt \leqslant c \tag{3.14}$$

from these it follows that $u_{hl} \to u$ weakly in $L_m(0,T;V)$ for a subsequence $(h,l)\to(0,\infty)$.

Now let $s\in\mathbb{N}$ and use as test function

$$\zeta(t) := \partial_t^{sh}(u_{hl}-g_h)(\tau) \quad \text{for} \quad jh \leqslant t \leqslant (j+s)h$$

with $(j-1)h \leqslant \tau \leqslant jh$ and $0 \leqslant j \leqslant \frac{T}{h}-s$. Then we obtain using the above energy estimate

$$\int_0^{T-sh}\int_\Omega (b(u_{hl}(\tau+sh))-b(u_{hl}(\tau)))(u_{hl}(\tau+sh)-u_{hl}(\tau))dx\,dt \leqslant csh.$$

But since u_{hl} is a step function in time we see that this estimate is also satisfied if we replace sh by any positive number. This implies that $b(u_{hl})$ converge to $b(u)$ in $L_1(Q_T)$ for a subsequence, which is guaranteed by the proof of Lemma 2.2. This lemma is also shows that $\Psi(b(u)) \in L_\infty(0,T;L_1(\Omega))$.

The a priori estimates furthermore implies that there are functionals λ_{hl} bounded in $L_{m'}(0,T;V^*)$ such that

$$\left.\begin{aligned}&\int_0^T \langle\lambda_{hl},\zeta\rangle dt = \iint_{Q_T} \partial_t^{-h} b(u_{hl})\zeta\, dx\, dt =\\ &= -\int_0^{T-h}\int_\Omega (b(u_{hl}) - b(u_h^0))\partial_t^h \zeta\, dx\, dt\\ &\text{for all}\quad \zeta\in L_m(0,T;V_l) \text{ with } \zeta(t)=0 \text{ for } t\geqslant T-h.\end{aligned}\right\} \tag{3.15}$$

Hence for a subsequence

$$\lambda_{hl}\to\lambda \quad \text{weakly in} \quad L_{m'}(0,T;V^*). \tag{3.16}$$

Then from (3.15), (3.16) it follows

$$\left.\begin{aligned}&\int_0^T \langle\lambda,\zeta\rangle dt = -\iint_{Q_T} (b(u)-b(u_0))\partial_t\zeta\, dx\, dt\\ &\text{for all}\quad \zeta\in L_m(0,T;V) \text{ with } \zeta(t)=0,\end{aligned}\right\}$$

that is all conditions of Proposition 2.1 are fulfilled and hence $\lambda=\partial_t b(u)$ and therefore this proposition can be applied to function u.

It is easy to see that function u satisfies for any $\tau\in(0,T]$ the identity

$$\left.\begin{aligned}&\int_0^\tau \langle\partial_t b(u),\zeta\rangle dt + \iint_{Q_\tau} A\nabla\zeta\, dx\, dt = \iint_{Q_\tau} f\zeta\, dx\, dt + \int_0^\tau\int_{\gamma_S} h\zeta\, ds\, dt\\ &\text{for all } \zeta\in L_m(0,T;V) \text{ and some } A\in L_{m'}(Q_T;\mathbb{R}^n).\end{aligned}\right\} \tag{3.17}$$

Obviously,

$$a(b(u_{hl}),\nabla u_{hl})\to A \quad \text{weakly in} \quad L_{m'}(Q_T;\mathbb{R}^n). \tag{3.18}$$

§4. The proof of Theorem 2.1

In the previous section we demonstrated the method of Alt and Luckhaus who dealt with the case $f=0$ in Q_T, $h=0$ on Γ_S. Obviously, the existence of weak solution of problem (1.7) will be proved if we show that

$$A = a(b(u),\nabla u) \quad \text{a.e. in} \quad Q_T. \tag{4.1}$$

Instead of (4.1) Alt and Luckhaus proposed in [1] the proof of strong convergence of ∇u_{hl} to ∇u in $L_m(Q_T)$. We give here a more easy proof of (4.1) assuming that we consider operator $\mathcal{L}u$ instead of $\mathcal{M}u$. Recall that both of these operators satisfy condition (2.18). Moreover it is easy to prove (see, also, [12], p. 79) that from (2.13), (2.18) it follows that (1.2) is true. So, for the problem (2.13) all results of section 3 are valid.

For the proof of (4.1) it suffices to show that

$$\nabla u_{h,l} \to \nabla u \quad \text{a.e. in} \quad Q_T. \tag{4.2}$$

Indeed, if (4.2) is true, then $a(b(u_{hl}), \nabla u_{hl}) \to a(b(u), \nabla u)$ a.e. in Q_T (beginning since this moment function $a(w,p)$ is defined by formula (2.14)). Then inequality $\iint\limits_{Q_T} |a(b(u_{hl}), \nabla u_{hl})|^{m'} dx\, dt \leqslant c$ allows us to apply the Vitali theorem, showing that

$$a(b(u_{hl}, \nabla u_{hl}) \to a(b(u), \nabla u) \text{ weakly in } L_{m'}(Q_T). \tag{4.3}$$

Clearly (3.18) and (4.3) imply (4.1).

Let $\tau \in (0,T)$ is fixed. For any $h > 0$ there is a number $\tau_h = k_h h$ with $(k_h - 1)h < \tau \leqslant k_h h$. Obviously that $\lim_{h\to 0} \tau_h = \tau$. To prove (4.2), first we establish the relation

$$\lim_{(h,l)\to(0,\infty)} \iint\limits_{Q_{k_h h}} a(b(u_{hl}), \nabla u_{hl}) \nabla u_{hl} dx\, dt = \iint\limits_{Q_\tau} A \nabla u dx\, dt \tag{4.4}$$

for a.e. $\tau \in (0,T)$.

Obviously (4.4) will follow from (3.9), (3.17) if we prove

$$\lim_{(h,l)\to(0,\infty)} \int\limits_0^{k_h h} \langle \partial_t^{-h} b(u_{hl}), u_{hl} \rangle dt = \int\limits_0^{\tau} \langle \partial_t b(u), u \rangle dt \tag{4.5}$$

for a.e. $\tau \in (0,T)$. Let $v_{hl} \in L_m(0,T;V_m)$ are approximations of $u - g$ in $L^m(0,T;V)$ and time independent in each interval $((k-1)h, kh)$. In order to prove (4.5) take

$$\zeta = u_{hl} - (g_h + v_{hl}), \quad v_{hl} \to u - g \text{ strongly in } L_m(0,T;V), \tag{4.6}$$

as test function in (3.9) with τ_h instead of τ, and put

$$\left.\begin{aligned} \hat{p}_{hl} := p_{hl} + G(b(u_{hl})), \quad p_{hl} = \nabla u_{hl}, \quad \hat{q}_{hl} := q_{hl} + G(b(v_{hl})), \\ q_{hl} = \nabla v_{hl}, \quad G(w) := k(w)\vec{e}. \end{aligned}\right\} \tag{4.7}$$

For the sake of brevity we shall assume that $g = 0$ in Q_T (and, hence, $g_h = 0$ in Q_T). Then, using the notation (2.15), we can write

$$\left.\begin{aligned} a(b(u_{hl}), \nabla u_{hl})(\nabla u_{hl} - \nabla v_{hl}) = \Phi_m(\hat{p}_{hl})(p_{hl} - q_{hl}) = \\ = [\Phi_m(\hat{p}_{hl}) - \Phi_m(\hat{q}_{hl})](\hat{p}_{hl} - \hat{q}_{hl}) - \\ -[\Phi_m(\hat{p}_{hl}) - \Phi_m(\hat{q}_{hl})][G(b(u_{hl}) - G(b(v_{hl}))] + \\ +\Phi_m(\hat{q}_{hl})(p_{hl} - q_{hl}) =: J_1 + J_2 + J_3. \end{aligned}\right\} \tag{4.8}$$

We put, also,

$$y_{hl} := \Phi_m(\hat{p}_{hl}), \quad z_{hl} := \Phi_m(\hat{q}_{hl}). \tag{4.9}$$

Obviously

$$\Phi_{m'}(y_{hl}) = \hat{p}_{hl}, \quad \Phi_{m'}(z_{hl}) = \hat{q}_{hl}. \tag{4.10}$$

From (2.16) it follows that

$$J_1 \geqslant 0 \quad \text{in} \quad Q_T, \tag{4.11}$$

while (1.2), (3.14), and (4.6) imply

$$\lim_{(h,l)\to(0,\infty)} \iint\limits_{Q_{\tau_h}} J_2 dx\, dt = 0. \tag{4.12}$$

We shall prove now that

$$\lim_{(h,l)\to(0,\infty)} \iint\limits_{Q_{\tau_h}} J_3 dx\, dt = 0. \tag{4.13}$$

We denote $p = \nabla u$, $\hat{p} = p + G(b(u))$. Then

$$\left.\begin{aligned} \iint\limits_{Q_{\tau_h}} J_3 dx\, dt = \iint\limits_{Q_{\tau_h}} \Phi_m(\hat{p})(p_{hl} - q_{hl}) dx\, dt + \\ + \iint\limits_{Q_{\tau_h}} [\Phi_m(\hat{q}_{hl}) - \Phi_m(\hat{p})](p_{hl} - q_{hl}) dx\, dt = \\ =: I_1 + I_2. \end{aligned}\right\} \tag{4.14}$$

Because $\lim_{(h,l)\to(0,\infty)} I_1 = 0$ for the proof of (4.13) it suffices to show that

$$\lim_{(h,l)\to(0,\infty)} I_2 = 0. \tag{4.15}$$

In the case $m \geqslant 2$ we have $m' \in (1,2)$ and then from (2.16), (4.9), (4.10) we derive

$$|\Phi_m(\hat{q}_{hl}) - \Phi_m(\hat{p})| \leqslant c|\hat{q}_{hl} - \hat{p}|(|\Phi_m(\hat{q}_{hl})| + |\Phi_m(\hat{p})|)^{2-m'}. \tag{4.16}$$

Hence, using (1.2), (3.14), and (4.16) we can estimate

$$\left.\begin{aligned} |I_2| \leqslant c \left(\iint\limits_{Q_{\tau_h}} |\Phi_m(\hat{q}_{hl}) - \Phi_m(\hat{p})|^{m'} dx\, dt \right)^{1/m'} \leqslant \\ \leqslant c \left(\iint\limits_{Q_{\tau_h}} |\hat{q}_{hl} - \hat{p}|^m dx\, dt \right)^{1/m} \left(\iint\limits_{Q_{\tau_h}} (|\hat{q}_{hl}| + |p|)^m dx\, dt \right)^{\frac{m-2}{m}} \leqslant \\ \leqslant c_1 \left(\iint\limits_{Q_{\tau_h}} |\hat{q}_{hl} - \hat{p}|^m dx\, dt \right)^{1/m} \to 0 \text{ as } (h,l) \to (0,\infty), \end{aligned}\right\} \tag{4.17}$$

i.e., (4.15) is proved.

In the case $m \in (1.2)$ we have $m' \geqslant 2$ and from (2.16), (4.9), (4.10) we derive

$$|\Phi_m(\hat{q}_{hl}) - \Phi_m(\hat{p})|^{m'-1} \leqslant c|\hat{q}_{hl} - \hat{p}|. \tag{4.18}$$

Hence, using (1.2), (3.14), and (4.18) and taking into account that $\frac{1}{m'-1} =$

$m-1$, we obtain

$$\left.\begin{aligned} |I_2| \leqslant c \iint\limits_{Q_{\tau_h}} |\hat{q}_{hl} - \hat{p}|^{m-1} |p_{hl} - q_{hl}| dx\, dt \leqslant \\ \leqslant c \left(\iint\limits_{Q_{\tau_h}} |\hat{p}_{hl} - \hat{p}|^m \right)^{1/m'} \\ \left(\iint\limits_{Q_{\tau_h}} (|p_{hl}|^m + |q_{hl}|^m) dx\, dt \right)^{1/m} \leqslant \\ \leqslant c_1 \left(\iint\limits_{Q_{\tau_h}} |\hat{p}_{hl} - \hat{p}|^m dx\, dt \right)^{1/m'} \to 0 \text{ as } (h,l) \to (0,\infty), \end{aligned}\right\} \tag{4.19}$$

i.e., (4.15) is true.

Using (4.11)–(4.13) we derive from (3.9), (4.6)

$$\limsup_{(h,l)\to(0,\infty)} \int\limits_0^{\tau_h} \int\limits_\Omega \partial_t^{-h} b(u_{hl})(u_{hl} - v_{hl}) dx\, dt \leqslant 0. \tag{4.20}$$

Then from (4.20), (3.15), and (3.10) it follows that for any $\tau \in (0,T)$

$$\limsup_{(|h,l|\to(0,\infty)} \int\limits_0^{\tau_h} \langle \partial_t^{-h} b(u_{hl}, u_{hl} \rangle dt \leqslant \int\limits_0^{\tau} \langle \partial_t b(u), u \rangle dt. \tag{4.21}$$

For the sake of completeness we prove non inequality (3.3) which will be used later.

Let $\eta(\zeta)$ be a strictly convex smooth function on $\mathbb{R}$ such that $\eta(0) = 0$ and, for some $M > 1$

$$\left.\begin{aligned} \eta'(\zeta) \text{ is monotone increasing from } -(M+1) \text{ to } -M \text{ if } \zeta \leqslant -M, \\ \eta'(\zeta) = \zeta \text{ if } \zeta \in [-M, M], \\ \eta'(\zeta) \text{ is monotone increasing from } M \text{ to } M+1 \text{ if } \zeta \geqslant M. \end{aligned}\right\}$$

Let $\eta^* : \mathbb{R} \to (-\infty, +\infty]$ be the function

$$\eta^*(w) := \sup_{z\in\mathbb{R}} \left\{ \eta'(z)(w - b(z)) + \int\limits_0^z \eta'(\zeta) b(d\zeta) \right\};$$

this function is convex and semicontinuous. From the monotonicity of b and η' it follows that

$$\eta^*(w) - \eta^*(b(z)) \geqslant \eta'(z)(w - b(z)), \quad w \in \mathbb{R}, \quad z \in \mathbb{R}. \tag{4.22}$$

Obviously that, as $M \to +\infty$, we have

$$\eta'(u) \to u, \quad \eta^*(b(u)) \to \Psi(b(u)) \tag{4.23}$$

and hence, (3.3) is true.

From Proposition 2.1 it follows that for a.a. $\tau \in (0, T)$

$$\int_0^\tau \langle \partial_t b(u), u \rangle dt = \int_\Omega \Psi(b(u(\tau)))dx - \int_\Omega \Psi(b_0)dx. \tag{4.24}$$

Fix some $\tau \in (0, T)$ for which (4.21) is true. Recall that for any $h > 0$ there exists number $t_h = k_h h$ with $(k_h - 1)h < \tau \leqslant k_h \tau$. From the structury of u_{hl} and (3.3) it follows that

$$\left.\begin{aligned} &\int_0^{k_h h} \partial_t^{-h} b(u_{hl}) u_{hl} dx\, dt \geqslant \\ &\geqslant \frac{1}{h} \int_0^{k_h h} \int_\Omega [\Psi(b(u_{hl}(t))) - \Psi(b(u_{hl}(t-h)))] dx\, dt = \\ &= \int_\Omega \Psi(b(u_{hl}(\tau)))dx - \int_\Omega \Psi(b(u_0^h))dx. \end{aligned}\right\} \tag{4.25}$$

On the other hand from (4.21) and (4.24) it follows that

$$\limsup_{(h,l)\to(0,\infty)} \int_0^{k_h h} \langle \partial_h^{-t} b(u_{hl}), u_{hl} \rangle dt \leqslant \int_\Omega \Psi(b(u(\tau)))dx - \int_\Omega \Psi(b_0)dx. \tag{4.26}$$

Then from (4.25), (4.26) we derive than for a.a. $\tau \in (0, T)$

$$\limsup_{(h,l)\to(0,\infty)} \int_\Omega \Psi(b(u_{hl}(\tau)))dx \leqslant \int_\Omega \Psi(b(u(\tau)))dx.$$

Recalling that nonnegative functions $\Psi(b(u_{hl}))$ converge to $\Psi(b(u))$ a.e. in Q_T as $(h, l) \to (0, \infty)$ we can apply the Faton lemma and write for a.a. $\tau \in (0,T)$ the inequality

$$\int_\Omega \Psi(b(u(\tau)))dx \leqslant \liminf_{(h,l)\to(0,\infty)} \int_\Omega \Psi(b(u_{h,l}(\tau)))dx.$$

From the last two inequalities we derive

$$\lim_{(h,l)\to(0,\infty)} \int_\Omega \Psi(b(u_{hl}(\tau)))dx = \int_\Omega \Psi(b(u))dx. \tag{4.27}$$

From (4.25), (4.26) it follows that for some subsequence $\{h,l\}$

$$\left.\begin{aligned} &\int_\Omega \Psi(b(u_{hl}(\tau)))dx - \int_\Omega \Psi(b(u_0^h))dx \leqslant \int_0^{k_h h} \langle \partial_t^{-h} b(u_{hl}), u_{hl}\rangle dt \leqslant \\ &\leqslant \int_\Omega \Psi(b(u(\tau)))dx - \int_\Omega \Psi(b_0)dx. \end{aligned}\right\} \tag{4.28}$$

From (4.28), (4.27), and (4.24) we derive that for this subsequence $\{h,l\}$ we have

$$\lim_{(h,l)\to(0,\infty)} \int^{k_h h} \langle \partial_t^{-h} b(u_{hl}), u_{hl}\rangle dt = \int_0^\tau \langle \partial_t b(u), u\rangle dt. \tag{4.29}$$

So, (4.5) is proved and, hence, (4.4) is true.

Using (4.4) we can prove (4.2) exactly in the same way as in [12], p. 90–92, where (4.2) was established in the case $m \in (1,2)$. However in the case $m \geqslant 2$ the same arguments as in [12], p. 90–92, can be applied even with some simplicity. It completes the proof of Theorem 2.1 in the case $g = 0$ in Q_T. The proof of Theorem 2.1 in the case where the last condition is violated we gives to the reader. Theorem 2.1 is proved.

Now we present a version of Theorem 2.1 which is useful for the study of the obstable problems for equation (1.10).

Suppose that the following assumptions holds:

$$\left.\begin{aligned} g = g(x,t,u) \text{ is a bounded function on } Q_T \times \mathbb{R} \\ \text{satisfying the Caratheodory condition;} \\ b(u) = b_1(u) + b_2(u), \text{ where the } b_2(u) \text{ are continuous} \\ \text{monotone function on } \mathbb{R};\ b(u) \text{ is strictly monotone on } \mathbb{R}; \\ \zeta = \zeta(x,t) \text{ is a function on } Q_T,\ \zeta \in L_{m'}(Q_T) \end{aligned}\right\} \tag{4.30}$$

As before, let $a(w,p)$ be the function defined by (2.14) and $f = f(x,t)$, $g = g(x,t)$, $h = h(x,t)$, and $b_0 = b_0(x)$ are as in condition 3) in §2.

We consider the initial boundary value problem

$$\left.\begin{aligned} \partial_t b(u) - \operatorname{div} a(b_2(u), \nabla u) = f + \xi g \text{ in } Q_T, \\ u = g \text{ on } \Gamma_D,\ Nu = h \text{ on } \Gamma_S,\ b(u) = b_0 \text{ on } \Omega\times\{t=0\}. \end{aligned}\right\} \tag{4.31}$$

Proposition 4.1. *If conditions* (2.1), (2.19), *and* (4.31) *are fulfilled, then problem* (4.31) *has at least one weak solution.*

The proof of Proposition 4.1 can be given exactly in the same way as in the case of Theorem 2.1.

§5. The proof of Theorem 2.2

The proof of Theorem 2.2 is based on the following analogy of Proposition 2.3.

Proposition 5.2. *Suppose that conditions* (2.13) *and* (2.20) *are fulfilled. Let u_1 be a subsolution and u_2 be a supersolution such that*

$$\left.\begin{aligned} \mathcal{L}u_1 \leqslant f_1 \text{ in } Q_\Gamma, \quad u_1 \leqslant g \text{ on } \Gamma_D, \\ Nu_1 \leqslant h_1 \text{ on } \Gamma_S,\ b(u_1) \leqslant b_0^{(1)} \text{ on } \Omega\times\{t=0\} \end{aligned}\right\} \tag{5.1}$$

and

$$\left.\begin{aligned} \mathcal{L}u_2 \geqslant f_2 \text{ in } Q_\Gamma, \quad u_2 \geqslant g \text{ on } \Gamma_D, \\ Nu_2 \geqslant h_2 \text{ on } \Gamma_S,\ b(u_2) \geqslant b_0^{(2)} \text{ on } \Omega\times\{t=0\}, \end{aligned}\right\} \tag{5.2}$$

where $b_0^{(i)}$, f_i, h_i and g satisfy conditions 2)–4) *in* §2. *Then for any nonnegative function $\gamma \in C_0^\infty(\mathbb{R}^n\times(-\infty,T))$ we have inequality*

$$\left.\begin{aligned} &\iint\limits_{Q_T} \{[(b_0^{(2)} - b_0^{(1)})^+ - (b(u_1) - b(u_2))^+]\partial_t\gamma + \\ &+\operatorname{sign}^+(u_1-u_2)[a_0(b(u_1),\nabla u_1) - a_0(b(u_2),\nabla u_2)]\nabla\gamma\}\,dx\,dt \leqslant \\ &\leqslant \iint\limits_{Q_T} (f_1 - f_2)\operatorname{sign}^+(u_1-u_2)\gamma\,dx\,dt + \\ &+ \iint\limits_{\Gamma_S} (h_1-h_2)\operatorname{sign}^+(u_1-u_2)\gamma\,ds\,dt. \end{aligned}\right\}. \tag{5.3}$$

In particular, if $f_1 = f_2$ *a.e. in* Q_T, $h_1 = h_2$ *a.e. in* Γ_S *we have for a.a.* $t \in (0, T)$

$$\int\limits_{\Omega} (b(u_1(t) - b(u_2(t))^+ \, dx \leqslant \int\limits_{\Omega} (b_0^{(1)} - b_0^{(2)})^+ \, dx. \tag{5.4}$$

Proof. The proof of Theorem 2.2 is the same as one of Theorem 2.2 of paper [12] (see [12], p. 94–98). In addition we have to use here inequality (3.7).

Remark 5.1. The proof of Theorem 2.2 of [12] is a some supplement to the results of paper [2] by Otto. The techniques employed in [2] was inspired by the method of doubling the variables, which was used by S. N. Kruzhkov in the proof of the comparison theorem for the entropy solutions of scalar conservation laws (see [14]).

Addid in proof. We knew after this paper was submited to the present volume that in the case when $S_T = \varnothing$ the independent proof of Theorems 1 and 2 can be derived from the results of paper [15] and [16].

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St.-Petersburg Department
of Steklov Mathematical Institute
of the Russian Academy of Sciences

Поступило 8 апреля 1999 г.

CMAF University of Lisbon
Av. Prof. Gama Pinto,2, 164-003,
Lisbon, Portugal