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**S. I. Repin**

# A POSTERIORI ESTIMATES FOR THE STOKES PROBLEM

Abstract. In this paper, a posteriori estimates for the velocity, stress and pressure fields are obtained for the stationary Stokes problem. The general estimate is a sum of three physically meaningful parts which correspond to the errors in the constitutive relations, in the incompressibility condition and in the equilibrium equation for the associated stress field. It is proved that the right-hand side of the estimate tends to zero for any sequence converging to the exact solution in the energy space.

## 1. Introduction

The mathematical analysis of boundary value problems is often focused on getting various estimates of their solutions via the external data properties. In general, these estimates have the form

$$\| u \| \leq \mathrm{M}(\mathcal{D}), \tag{1.1}$$

where $u$ is a solution, $\| \, . \, \|$ is an appropriate norm and $\mathcal{D}$ is the set of external data. This type estimates give *a priori* information about a solution $u$ and usually play important role in its subsequent analysis. Numerous a priori estimates for elliptic and parabolic problems (see e.g. [1]), for problems in hydrodynamics like the Stokes or Navier–Stokes equations (see [2]) and for many other problems are known in the literature.

Another line of the mathematical analysis that has exceptional importance for the approximation theory is related to the so-called *a posteriori* estimates. Their destination is to give an upper bound of the difference $v - u$ for any function $v$ from the basic functional space associated with the problem. The function $v$ (obtained by some analytical or numerical procedure) is included in the right-hand side of such estimate, so that the latter has the form

$$\| u - v \| \leq \mathrm{M}(v, \mathcal{D}). \tag{1.2}$$

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It is evident, that the estimate (1.2) is practically interesting only if it possesses the following *consistency* property:

$$\mathrm{M}(v_k, \mathcal{D}) \to 0 \tag{1.3}$$

for any sequence $v_k$ converging to $u$ in the norm $\|\!| \,.\, \|\!|$.

Recent years have seen rapidly increasing interest to a posteriori error estimates in the numerical analysis. Being confined to the limits of a short note we are not able to present here a consequent overview of the results obtained in this field. Readers can find a more detailed exposition supplied with additional references in e.g. [3, 4].

A posteriori estimates for the Stokes problem and for some other viscous flow problems were studied by a number of authors (see, e.g. [5–9]). Typically, they have been obtained in the frame of the so-called "residual method" originally proposed in [10, 11] for the finite element approximations. This type estimates are crucially based on the Galerkin orthogonality condition. Therefore, they are only valid for exact solutions of the corresponding finite dimensional problems which form a very special subset in the natural set of admissible functions.

The problem considered in this paper appears as that of establishing the *general* form of a posteriori estimate (1.2) for the Stokes problem. Our approach is based on duality theory of the calculus of variations. It have been earlier applied to deriving a posteriori estimates in various nonlinear boundary value problems (see [12–18]). In the present paper, we obtain practically convenient and physically meaningful estimates of the difference $v - u$ valid for any $v$ from a wide set of functions that are not necessarily satisfy the incompressibility condition or the Galerkin orthogonality condition in some finite dimensional subproblem.

## 2. Preliminaries

**2.1. Notations.** Let $\Omega$ be a bounded, connected and open set in $\mathbb{R}^n$ ($n$=2 or 3) with a Lipschitz continuous boundary $\partial\Omega$. By $\mathbb{R}^n$ and $\mathbb{M}_s^{n\times n}$ we denote the spaces of real vectors and real symmetric $n \times n$ matrixes. As usual $\mathbb{R}^n$ and $\mathbb{M}_s^{n\times n}$ are equipped with the norms $|\,.\,|$ and $\boldsymbol{|}\,.\,\boldsymbol{|}$ related to the scalar products in $\mathbb{R}^n$ and $\mathbb{M}_s^{n\times n}$ denoted by the one and two dots, i.e. for $a \in \mathbb{R}^n$ and $\tau \in \mathbb{M}_s^{n\times n}$ we have

$$|\,a\,| := \sqrt{a \cdot a}, \qquad \boldsymbol{|}\,\tau\,\boldsymbol{|} := \sqrt{\tau : \tau}.$$

Henceforth, the symbol := means "equals by definition" and the abbreviation "iff" is used instead of the words "if and only if".

Since no confusion can arise, we use one common symbol $\|.\|$ for the norms in the Lebesgue spaces of square summable scalar, vector and tensor valued functions $L_2(\Omega)$, $L_2(\Omega, \mathbb{R}^n)$ and $L_2(\Omega, \mathbb{M}_s^{n\times n})$. For the sake of simplicity we also introduce special notations for some spaces: $W(\Omega) := W_2^1(\Omega)$, $V(\Omega) := W_2^1(\Omega, \mathbb{R}^n)$, $\Sigma(\Omega) := L_2(\Omega, \mathbb{M}_s^{n\times n})$. In addition, $W_0(\Omega)$ and $V_0(\Omega)$ denote subspaces of functions from $W(\Omega)$ and $V(\Omega)$ vanishing on the boundary $\partial\Omega$ in the sense of traces.

In order to study the Stokes problem we, as usual, require some extra spaces related to the divergence free functions. Let

$$J^\infty(\Omega) := \big\{ v \in C_0^\infty(\Omega, \mathbb{R}^n) \,\|\, \operatorname{div} v = 0, \\ \operatorname{supp} v \;\text{ is a compact subset of }\; \Omega \big\}.$$

The closure of this set in the norm $\|.\|$ gives a space $\mathcal{H}^0(\Omega)$. Another Hilbert space $\mathcal{H}^1(\Omega)$ is the closure of $J^\infty(\Omega)$ in the norm

$$\|v\|_{(1)}^2 := \int_\Omega |\nabla v|^2 \; dx.$$

In the sequel, we also will use the norm

$$\|g\|_{(-1)} := \sup_{w \in V_0(\Omega)} \frac{\int_\Omega g w \, dx}{\|w\|_{(1)}}$$

which is a norm in the space topologically dual to $V_0(\Omega)$. Lastly, we use the notation $L_2^0(\Omega)$ for the space of square summable scalar functions such that

$$\int_\Omega p \, dx = 0. \tag{2.1}$$

**2.2. The Dirichlet problem in the velocity-pressure formulation.** The classical Stokes problem consists of determining a vector valued function $u$ (velocity) and a scalar valued function $p$ (pressure) satisfying the following system of equations

$$-\nu \Delta u + \nabla p = f \quad \text{in} \quad \Omega, \tag{2.2}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega, \tag{2.3}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.4}$$

where $\nu$ is a positive constant. Given

$$f \in \mathrm{L}_2(\Omega, \mathbb{R}^n) \tag{2.5}$$

we define a generalized solution $u \in \mathcal{H}^1(\Omega)$ as a function that meets the integral identity

$$\nu \int\limits_\Omega \nabla u : \nabla v \, dx = \int\limits_\Omega f \cdot v \, dx \quad \forall\, v \in \mathcal{H}^1(\Omega). \tag{2.6}$$

Under the made above assumptions this solution exists and unique. Moreover, one can prove that $u \in W^2_{2,\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $\nabla p \in W^1_{2,\mathrm{loc}}(\Omega)$ (see e.g. [2]). The function $u$ can be equivalently defined as a minimizer of the following variational problem (see e.g. [19, 20]).

**Problem $\mathcal{P}$.** Find $u \in \mathcal{H}^1(\Omega)$ such that

$$I(u) \leqslant I(v) \quad \forall\, v \in \mathcal{H}^1(\Omega), \tag{2.7}$$

where

$$I(v) = \int\limits_\Omega \left( \frac{\nu}{2} \, |\nabla v|^{\,2} - f \cdot v \right) \, dx.$$

The exact lower bound of this problem we denote $\inf \mathcal{P}$.

Let $L : \mathcal{H}^1(\Omega) \times \Sigma(\Omega) \to R$ be the Lagrangian

$$L(v, \tau) = \int\limits_\Omega \left( \tau : \nabla v - \frac{1}{2\nu} \, |\tau|^{\,2} \right) \, dx - \int\limits_\Omega f v \, dx.$$

It is easy to see that Problem $\mathcal{P}$ admits a minimax representation:

$$\inf_{v \in \mathcal{H}^1} I(v) = \inf_{v \in \mathcal{H}^1} \sup_{\tau \in \Sigma} L(v, \tau).$$

The Lagrangian $L$ is linear with respect to $v$ and concave with respect to $\tau$. It also possesses the required continuity and coercivity properties, so that known results of convex analysis imply the basic equality

$$\inf \sup L = \sup \inf L = \inf \mathcal{P}.$$

The complementary problem

$$\sup_{\tau \in \Sigma} \inf_{v \in \mathcal{H}^1} L(v, \tau)$$

that naturally arises from the above minimax setting is equivalent to a new variational setting.

**Problem $\mathcal{P}^*$.** Find $\sigma \in \Sigma_f(\Omega)$ such that

$$I^*(\sigma) = \sup_{\tau \in \Sigma_f} I^*(\tau),$$

where

$$I^*(\tau) = -\frac{1}{2\nu} \int_\Omega |\tau|^2 \, dx$$

and

$$\Sigma_f(\Omega) := \left\{ \tau \in \Sigma(\Omega) \;\middle\|\; \int_\Omega \tau : \nabla w \, dx = \int_\Omega f w \, dx \quad \forall w \in \mathcal{H}^1(\Omega) \right\}.$$

In the framework of the duality theory Problem $\mathcal{P}^*$ is called *dual* to Problem $\mathcal{P}$.

The assertion stated below readily follows from the general theory of the calculus of variations (see e.g. [21]).

**Theorem 1.** *There exists a unique minimizer $u$ of Problem $\mathcal{P}$ and unique maximizer $\sigma$ of Problem $\mathcal{P}^*$. These two functions meet the equalities*

$$I^*(\sigma) = \sup \mathcal{P}^* = \inf \mathcal{P} = I(u), \tag{2.8}$$

$$\sigma = \nu \nabla u. \tag{2.9}$$

## 3. Main results

To clarify the main principles of our approach we rewrite the classical Stokes system (2.2)–(2.4) in a somewhat different form

$$\operatorname{div} \sigma = \nabla p - f \quad \text{in} \quad \Omega, \tag{3.1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega, \tag{3.2}$$

$$\sigma = \nu \nabla u \quad \text{in} \quad \Omega, \tag{3.3}$$

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.4}$$

This system includes additional variable $\sigma$ which physically corresponds to the field of stresses. Now we can consider the Stokes problem as that of finding a triplet of functions $(u, \sigma, p)$. The natural "energy" set for this velocity-stress-pressure formulation is

$$\mathcal{E} := \mathcal{H}^1(\Omega) \times \Sigma_f(\Omega) \times L_2^0(\Omega).$$

The basic error equality for functions from this set is obtained in Sec. 4.

**Theorem 2.** *Under the assumptions of Theorem 1 for any $v \in \mathcal{H}^1(\Omega)$ and any $\tau_f \in \Sigma_f(\Omega)$ the following equality holds*

$$\nu \,\|\nabla(v-u)\|^2 + \nu^{-1}\,\|\tau_f - \sigma\|^2 = 2\,(\,I(v) - I^*(\tau_f)\,)\,. \tag{3.5}$$

Regrettably, elements of $\mathcal{H}^1(\Omega)$ and $\Sigma_f(\Omega)$ are subjected to differential relations, so that (3.5) can be used only for rather special classes of functions. For this reason, our subsequent efforts will be concentrated on getting an estimate applicable for functions from a wider and more simple set

$$\widetilde{\mathcal{E}} := V_0(\Omega) \times \Sigma(\Omega) \times L_2^0(\Omega).$$

The general form of a posteriori estimate for functions $(v, \tau, q) \in \widetilde{\mathcal{E}}$ is derived in Sec. 4. Let us state this result precisely.

**Theorem 3.** *For any $v \in V_0(\Omega)$ and any $\tau \in \Sigma(\Omega)$ the estimates:*

$$\|\, v - u \,\|_{(1)} \leq 2\mathbf{R}_1(v) + \nu^{-1}\mathbf{R}_2(v,\tau) + \nu^{-1}\mathbf{R}_3(\tau),$$
$$\|\, \sigma - \tau \,\| \leq \nu\mathbf{R}_1(v) + \mathbf{R}_2(v,\tau) + 2\mathbf{R}_3(\tau)$$

*hold, where*

$$\mathbf{R}_1(v) = \inf_{v_0 \in \mathcal{H}^1} \|\nabla(v - v_0)\|, \tag{3.6}$$

$$\mathbf{R}_2(v,\tau) = \|\, \nu\nabla v - \tau \,\|\,, \tag{3.7}$$

$$\mathbf{R}_3(\tau) = \inf_{\tau_f \in \Sigma_f} \|\tau_f - \tau\|. \tag{3.8}$$

It is easy to see that $\mathbf{R}_1(v)$ measures the distance between the function $v \in V_0(\Omega)$ and the set $\mathcal{H}^1(\Omega)$ which contains the desired velocity field $u$.

Analogously, $\mathbf{R}_3(\tau)$ measures the distance between $\tau \in \Sigma(\Omega)$ and the set $\Sigma_f(\Omega)$ which contains the stress field $\sigma$. The quantity $\mathbf{R}_2(v,\tau)$ gives the error in the duality relation (3.3). Hence, the foregoing theorem shows that upper bounds of the norms $\|v - u\|_{(1)}$ and $\|\tau - \sigma\|$ are linear combinations of three terms which represent errors in the *incompressibility condition*, *constitutive law* and *necessary condition for dual problem*, respectively.

As is shown in Sec. 5 upper bounds of $\mathbf{R}_1$ and $\mathbf{R}_3$ can be expressed via the quantities

$$R_1(v) := \|\,\operatorname{div} v\,\|\,, \quad R_3(\tau, q) := \|\operatorname{div}\tau + f - \nabla q\|_{(-1)} \tag{3.9}$$

and three positive constants that depend on the geometrical properties of the domain $\Omega$ only. They are none other than the constants $c_\Omega$ and $C_\Omega$

in the Friederichs–Poincare inequality for the spaces $W_0(\Omega)$ and $V_0(\Omega)$, respectively and the constant $\mathbb{C}_\Omega$ in the Ladyzhenskaya–Babuska–Brezzi condition (5.10). Thanks to this fact we obtain a posteriori estimates in a more transparent form.

**Theorem 4.** *Let all assumptions of Theorem 1 hold. Then for any* $(v, \tau, q) \in \widetilde{\mathcal{E}}$

$$\begin{aligned} \|v-u\|_{(1)} &\le \gamma_{11}\, R_1(v) + \gamma_{12}\, R_2(v,\tau) + \gamma_{13}\, R_3(\tau, q) \\ \|\tau - \sigma\| &\le \gamma_{21}\, R_1(v) + \gamma_{22}\, R_2(v,\tau) + \gamma_{23}\, R_3(\tau, q) \\ \|p - q\| &\le \gamma_{31}\, R_1(v) + \gamma_{32}\, R_2(v,\tau) + \gamma_{33}\, R_3(\tau, q), \end{aligned} \tag{3.10}$$

*where*

$$\begin{aligned} &\gamma_{11} = 2c_\Omega C_\Omega, && \gamma_{12} = \nu^{-1}, && \gamma_{13} = \nu^{-1}, \\ &\gamma_{21} = 2\nu c_\Omega, && \gamma_{22} = 1, && \gamma_{23} = 2, \\ &\gamma_{31} = 2\nu c_\Omega C_\Omega {\mathbb{C}_\Omega}^{-1}, && \gamma_{32} = 2{\mathbb{C}_\Omega}^{-1}, && \gamma_{33} = 2{\mathbb{C}_\Omega}^{-1}, \end{aligned}$$

*and* $R_2(v, \tau) := \|\tau - \nu\nabla v\|$.

These estimates have clear physical sense. Really, the first of them $R_1(v)$ equals zero iff $v \in \mathcal{H}^1(\Omega)$, i.e., if

$$\operatorname{div} v = 0.$$

Therefore, $R_1(v)$ is a measure of the *error in the incompressibility condition* (3.2).

The quantity $R_2(v, \tau)$ equals zero iff

$$\tau = \nu \nabla v,$$

so that it measures the *error in the duality relations* (3.3) for the pair $(v, \tau)$.

Lastly, the quantity $R_3(\tau, q)$ equals zero iff $\tau$ and $q$ meets the equation

$$\operatorname{div} \tau = \nabla q - f$$

in the sense of distributions. Thus, $R_3(\tau, q)$ is a *residual error* associated with Eq. (3.1).

Now we see the principal structure of the estimates (3.10):

> *A posteriori error majorants are sums of three penalty terms which penalize violations of Eqs.*(3.1)–(3.3). *The penalty coefficients $\gamma_{ij}$ are completely defined by the physical constant $\nu$ and three geometrical constants $c_\Omega$, $C_\Omega$ and $\mathbb{C}_\Omega$.*

The case when $\tau$ and $q$ are more regular is analyzed in Sec. 5. More precisely, we assume that

$$q \in W(\Omega), \qquad \tau \in Q(\Omega) := \{\tau \in \Sigma(\Omega) \;\|\; \operatorname{div} \tau \in L_2(\Omega)\}\,. \tag{3.11}$$

In this case, the negative norm in $R_3$ can be replaced by easily computable Lebesgue norm

$$R_3(\tau, q) = C_\Omega \,\|\operatorname{div} \tau - f + \nabla q\|$$

what gives practically more attractive estimates (5.17). It is easy to see that the right-hand sides of (5.17) tend to zero for any sequences $(\tau_k, v_k, q_k)$ such that $\tau_k \to \sigma \in Q(\Omega)$, $v_k \to u$ in $V_0(\Omega)$ and $q_k \to p$ in $W(\Omega)$. A simple method of getting such sequences directly follows from the above a posteriori estimates. We present it at the end of Sec. 5 together with some practical comments.

## 4. Proofs of Theorems 2 and 3

**4.1. Proof of Theorem 2.** The minimizer $u$ of Problem $\mathcal{P}$ meets the relation (2.6). Therefore, for any $v \in \mathcal{H}^1(\Omega)$ we obtain

$$\begin{aligned} I(v) - I(u) &= \int\limits_\Omega \Big(\frac{\nu}{2}\,|\nabla v|^{\,2} - \frac{\nu}{2}\,|\nabla u|^{\,2} - f\cdot(v-u)\Big)\,dx = \\ &= \int\limits_\Omega \Big(\frac{\nu}{2}\,|\nabla (v-u)|^{\,2} + \nu\nabla u : \nabla(v-u) - f\cdot(v-u)\Big)\,dx = \\ &= \frac{\nu}{2}\int\limits_\Omega |\nabla(v-u)|^{\,2}\,dx. \end{aligned} \tag{4.1}$$

It means that

$$\frac{\nu}{2}\,\|\nabla(v-u)\|^2 = I(v) - \inf \mathcal{P} \qquad\qquad \forall v \in \mathcal{H}^1(\Omega). \tag{4.2}$$

The maximizer $\sigma$ of Problem $\mathcal{P}^*$ meets the equation

$$\int\limits_\Omega \sigma : (\tau_f - \sigma)\,dx = 0 \qquad \forall \tau_f \in \Sigma_f(\Omega).$$

By virtue of this equation we deduce

$$\sup \mathcal{P}^* - I^*(\tau_f) = I^*(\sigma) - I^*(\tau_f) = \frac{1}{2\nu}\,\|\tau_f - \sigma_f\|^2 \quad \forall\, \tau_f \in \Sigma_f(\Omega). \tag{4.3}$$

Now (4.2) and (4.3) imply the basic variational equality for the Stokes problem

$$\nu \left\| \nabla(v-u) \right\|^2 + \nu^{-1} \left\| \tau_f - \sigma \right\|^2 = 2 \left( I(v) - I^*(\tau_f) \right) \tag{4.4}$$

which is valid for any $v \in \mathcal{H}^1(\Omega)$ and any $\tau_f \in \Sigma_f(\Omega)$.

**4.2. Proof of Theorem 3.** Let $v_0 \in \mathcal{H}^1(\Omega)$ and $\tau_f \in \Sigma_f(\Omega)$. Then for any $\theta > 0$ the following inequalities holds

$$\left\| \nabla(v-u) \right\|^2 \leq \left(1 + \frac{1}{\theta}\right) \left\| \nabla(v - v_0) \right\|^2 + (1+\theta) \left\| \nabla(u - v_0) \right\|^2, \tag{4.5}$$

$$\left\| \sigma - \tau \right\|^2 \leq \left(1 + \frac{1}{\theta}\right) \left\| \tau - \tau_f \right\|^2 + (1+\theta) \left\| \sigma - \tau_f \right\|^2, \tag{4.6}$$

From these inequalities and (4.4) we have:

$$\left\| \nabla(v-u) \right\|^2 \leq \left(1 + \frac{1}{\theta}\right) \left\| \nabla(v - v_0) \right\|^2 + \frac{2(1+\theta)}{\nu} \Big[ I(v_0) - I^*(\tau_f) \Big], \tag{4.7}$$

$$\left\| \sigma - \tau \right\|^2 \leq \left(1 + \frac{1}{\theta}\right) \left\| \tau - \tau_f \right\|^2 + 2\nu(1+\theta) \Big[ I(v_0) - I^*(\tau_f) \Big] \tag{4.8}$$

for any $v \in V_0(\Omega)$ and any $\tau \in \Sigma(\Omega)$.

The difference in square brackets we represent as follows

$$I(v_0) - I^*(\tau_f) = i_1 + i_2 + i_3,$$

where

$$\begin{aligned} i_1 &:= I(v_0) - I(v) = \\ &= \int\limits_\Omega \Big( \frac{\nu}{2} \left| \nabla(v_0 - v) \right|^2 + \nu \nabla v : \nabla(v_0 - v) - f(v_0 - v) \Big)\, dx, \end{aligned} \tag{4.9}$$

$$i_2 := I(v) - I^*(\tau) = \frac{1}{2\nu} \int\limits_\Omega \left| \nu \nabla v - \tau \right|^2 \, dx + \int\limits_\Omega (\nabla v : \tau - f v)\, dx, \tag{4.10}$$

$$i_3 := I^*(\tau) - I^*(\tau_f) = \frac{1}{2\nu} \int\limits_\Omega \left| \tau_f - \tau \right|^2 dx + \frac{1}{\nu} \int\limits_\Omega \tau : (\tau_f - \tau)\, dx. \tag{4.11}$$

Since

$$\int\limits_\Omega \tau_f : \nabla v_0 \, dx = \int\limits_\Omega f \cdot v_0 \, dx$$

we obtain

$$I(v_0) - I^*(\tau_f) =$$
$$= \int_\Omega \left\{ \frac{\nu}{2} |\nabla(v_0 - v)|^2 + \frac{1}{2\nu} \left( |\nu\nabla v - \tau|^2 + |\tau_f - \tau|^2 \right) \right\} dx +$$
$$+ \int_\Omega \left\{ (\nu\nabla v - \tau) : \nabla(v_0 - v) + \frac{1}{\nu}(\nu\nabla v - \tau) : (\tau - \tau_f) + (\tau - \tau_f) : \nabla(v_0 - v) \right\}.$$

Estimating the scalar products in the above integral via the Young's inequality we get

$$I(v_0) - I^*(\tau_f) \leq \left(1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) \frac{\nu}{2} \|\nabla(v_0 - v)\|^2 +$$
$$+ \left(1 + \alpha_1 + \frac{1}{\alpha_3}\right) \frac{1}{2\nu} \|\nu\nabla v - \tau\|^2 + (1 + \alpha_2 + \alpha_3) \frac{1}{2\nu} \|\tau_f - \tau\|^2,$$

where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are arbitrary positive numbers. Now (4.7) and (4.8) come in the forms

$$\|\nabla(v - u)\|^2 \leq$$
$$\leq \left(1 + \frac{1}{\theta}\right) \|\nabla(v - v_0)\|^2 + (1 + \theta) \Big[ \left(1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) \|\nabla(v_0 - v)\|^2 +$$
$$+ \left(1 + \alpha_1 + \frac{1}{\alpha_3}\right) \frac{1}{\nu^2} \|\nu\nabla v - \tau\|^2 + (1 + \alpha_2 + \alpha_3) \frac{1}{\nu^2} \|\tau_f - \tau\|^2 \Big],$$

$$\|\sigma - \tau\|^2 \leq$$
$$\leq \left(1 + \frac{1}{\theta}\right) \|\tau - \tau_f\|^2 + (1 + \theta) \Big[ \left(1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) \nu^2 \|\nabla(v_0 - v)\|^2 +$$
$$+ \left(1 + \alpha_1 + \frac{1}{\alpha_3}\right) \|\nu\nabla v - \tau\|^2 + (1 + \alpha_2 + \alpha_3 +) \|\tau_f - \tau\|^2 \Big].$$

We recall that $v_0$ and $\tau_f$ are arbitrary functions from the sets $\mathcal{H}^1(\Omega)$ and $\Sigma_f(\Omega)$, respectively. Therefore, passing to the infimum over these sets we obtain another pair of estimates

$$\|\nabla(v - u)\|^2 \leq (1 + \theta) \Big[ \Big(1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\theta}\Big) \mathbf{R}_1^2(v) +$$

$$+\left(1+\alpha_1+\frac{1}{\alpha_3}\right)\frac{1}{\nu^2}\mathbf{R}_2^2(v,\tau)+(1+\alpha_2+\alpha_3)\,\frac{1}{\nu^2}\mathbf{R}_3^2(\tau)\Big], \qquad (4.12)$$

$$\|\sigma-\tau\|^2 \leq (1+\theta)\Big[\Big(1+\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\Big)\nu^2\mathbf{R}_1^2(v)+$$

$$+\left(1+\alpha_1+\frac{1}{\alpha_3}\right)\mathbf{R}_2^2(v,\tau)+\left(1+\alpha_2+\alpha_3+\frac{1}{\theta}\right)\mathbf{R}_3^2(\tau)\Big], \qquad (4.13)$$

where $\mathbf{R}_1(v)$, $\mathbf{R}_2(v,\tau)$ and $\mathbf{R}_3(\tau)$ are defined by the relations (3.6)–3.8). The parameters $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\theta$ should be choosed to minimize the right-hand sides of (4.12) and (4.13). This procedure results in the following estimates

$$\|\, v-u\,\|_{(1)} \leq 2\mathbf{R}_1(v)+\nu^{-1}\left(\mathbf{R}_2(v,\tau)+\mathbf{R}_3(\tau)\right), \qquad (4.14)$$

$$\|\,\sigma-\tau\,\| \leq \nu\mathbf{R}_1(v)+\mathbf{R}_2(v,\tau)+2\mathbf{R}_3(\tau). \qquad (4.15)$$

**4.3. Consistency of a posteriori estimates.** In the remainder of this section, we show that the estimates (4.14)–(4.15) are consistent (cf. (1.3)).

**Proposition 1.** *For any sequences* $\{v_m\}\in V_0(\Omega)$ *and* $\{\tau_m\}\in\Sigma(\Omega)$ *such that*

$$v_m \to u \ \text{in}\ V\,, \qquad \tau_m \to \sigma \ \text{in}\ \Sigma \qquad (4.16)$$

*the sequences of real numbers* $\mathbf{R}_1(v_m,)$, $\mathbf{R}_2(v_m,\tau_m)$ *and* $\mathbf{R}_3(\tau_m)$ *tend to zero.*

**Proof.** This assertion is a simple consequence of (4.16) and the inequalities

$$\begin{aligned}\mathbf{R}_1(v) &\leq \|v_m-u\|_{(1)}\,,\\ \mathbf{R}_2(v,\tau) &\leq \|v_m-u\|_{(1)}+\|\tau_m-\sigma\|\,,\\ \mathbf{R}_3(\tau) &\leq \|\tau_m-\sigma\|_{(1)}\,. \qquad \square\end{aligned}$$

The task that arise next is to find a practical way of evaluating $\mathbf{R}_1(v)$, $\mathbf{R}_2(v,\tau)$ and $\mathbf{R}_3(\tau)$. We are focused on this subject in the subsequent section.

## 5. Proof of Theorem 4

In this section we justify the estimates (3.10). The proof consists of three steps which are presented in three subsections below.

**5.1. An estimate for $R_1(v)$.** It is not difficult to see that

$$\frac{1}{2}\mathbf{R}_1^2(v) = \inf_{v\in\mathcal{H}^1} \frac{1}{2}\left\|\nabla(v_0 - v)\right\|^2 =$$

$$= \inf_{w\in V_0} \sup_{\phi\in W_0} \phi \in W_0 \int\limits_{\Omega} \left(\frac{1}{2}\left|\nabla(w-v)\right|^2 + \nabla\phi\cdot w\right) dx. \quad (5.1)$$

The minimax problem (5.1) has another equivalent form, that is

$$\frac{1}{2}\mathbf{R}_1^2(v) = \sup_{\phi\in W_0}\Big[\inf_{w\in V_0}\int\limits_{\Omega}\left(\frac{1}{2}\left|\nabla w\right|^2 + \nabla\phi\cdot(w+v)\right) dx\Big]. \quad (5.2)$$

At first we obtain an upper bound of the term in square brackets. Let $w_0$ be a nonzero element of the space $V_0(\Omega)$. Since $tw_0 \in V_0(\Omega)$ for any $t \in \mathbb{R}$ we get

$$\inf_{w\in V_0}\int\limits_{\Omega}\left(\frac{1}{2}\left|\nabla w\right|^2 + \nabla\phi\cdot(w+v)\right) dx \leq$$

$$\leq \frac{1}{2}t^2\left\|\nabla w_0\right\|^2 + t\left\|\nabla\phi\right\|\left\|w_0\right\| + \int\limits_{\Omega}\nabla\phi\cdot v\, dx. \quad (5.3)$$

Setting $t = t_* := -\left\|\nabla\phi\right\|\left\|w_0\right\|\left\|\nabla w_0\right\|^{-2}$ we maximize the right-hand side of (5.3). Thus, we obtain

$$\inf_{w\in V_0}\int\limits_{\Omega}\left(\frac{1}{2}\left|\nabla w\right|^2 + \nabla\phi\cdot(w+v)\right) dx \leq$$

$$\leq -\frac{1}{2}\frac{\left\|w_0\right\|^2}{\left\|\nabla w_0\right\|^2}\left\|\nabla\phi\right\|^2 + \int\limits_{\Omega}\nabla\phi\cdot v\, dx =$$

$$= -\frac{1}{2}\frac{\left\|w_0\right\|^2}{\left\|\nabla w_0\right\|^2}\left\|\nabla\phi\right\|^2 - \int\limits_{\Omega}\phi\,\mathrm{div}\, v\, dx = \quad (5.4)$$

$$= -\frac{1}{2}\frac{\left\|w_0\right\|^2}{\left\|\nabla w_0\right\|^2}\left\|\nabla\phi\right\|^2 + c_\Omega\left\|\mathrm{div}\, v\right\|\left\|\nabla\phi\right\|,$$

where $c_\Omega$ denotes a constant in the Friederichs–Poincare inequality for the space $W_0(\Omega)$.

Now we proceed to estimating the value of supremum in (5.2). Let $C_\Omega$ denote a constant in the Friederichs–Poincare inequality for the space $V_0(\Omega)$. Then

$$\sup_{\phi\in V_0}\left[-\frac{1}{2}\frac{\|w_0\|^2}{\|\nabla w_0\|^2}\|\nabla\phi\|^2+c_\Omega\,\|\mathrm{div}\,v\|\,\|\nabla\phi\|\right]\leq$$

$$\leq\sup_{t>0}\left[-\frac{1}{2}\frac{\|w_0\|^2}{\|\nabla w_0\|^2}t^2+c_\Omega\,\|\mathrm{div}\,v\|\;t\right]= \tag{5.5}$$

$$=\frac{1}{2}c_\Omega^2\,\|\mathrm{div}\,v\|^2\,\frac{\|\nabla w_0\|^2}{\|w_0\|^2}\leq\frac{1}{2}c_\Omega^2 C_\Omega^2\;\|\mathrm{div}\,v\|^2$$

and we end up with the estimate

$$\mathbf{R}_1(v)\leq c_\Omega\,C_\Omega\;\|\mathrm{div}\,v\|\,. \tag{5.6}$$

**5.2. An estimate for $R_3(\tau)$.** To get an upper bound for $\mathbf{R}_3(\tau)$ we note that for any $q\in L_2^0(\Omega)$ the set

$$\Sigma_{fq}(\Omega):=\left\{\tau\in\Sigma\left\|\int\limits_\Omega\tau:\nabla w\,dx=\int\limits_\Omega(f\cdot w+q\,\mathrm{div}\,w)\,dx\;\forall w\in V_0(\Omega)\right.\right\}$$

is a subset of $\Sigma_f(\Omega)$. Therefore,

$$\frac{1}{2}\mathbf{R}_3^2(\tau)=\inf_{\tau_f\in\Sigma_f}\frac{1}{2}\|\tau_f-\tau\|^2\leq$$

$$\leq\inf_{\tau_f\in\Sigma_{fq}}\frac{1}{2}\|\tau_f-\tau\|^2\leq\inf_{\eta^*\in\Sigma}\sup_{w\in V_0}\times \tag{5.7}$$

$$\times\left\{\frac{1}{2}\|\eta^*-\tau\|^2+\int\limits_\Omega\Big(\eta^*:\nabla w-f\cdot w+q\,\mathrm{div}\,w\Big)\,dx\right\}.$$

The value of inf sup in (5.7) can be estimated by applying duality arguments again (see [12, 13] for more details). This procedure results in the following inequality

$$\frac{1}{2}\mathbf{R}_3^2(\tau)\leq$$

$$\leq-\inf_{w\in V_0}\int\limits_\Omega\left(\frac{1}{2}\,|\nabla w|^2-\tau:\nabla w+f\cdot w+q\mathrm{div}\,w\right)\,dx. \tag{5.8}$$

Since

$$\int_\Omega (-\tau : \nabla w + f \cdot w + q \operatorname{div} w\} \, dx \ \geq - \|\nabla w\| \, \|\operatorname{div} \tau + f - \nabla q\|_{(-1)} \,,$$

we estimate the infimum in (5.8) via the negative norm and finally obtain

$$\mathbf{R}_3(\tau) \leq \|\operatorname{div} \tau + f - \nabla q\|_{(-1)} \,. \tag{5.9}$$

**5.3. An estimate for the pressure field.** Now we proceed to getting upper bounds for $\|p - q\|$. Our analysis is based on the "inf-sup" condition (see e.g. [19, 22, 23]) which is required for the well–posedness of the problem for pressure. This condition reads: there exists a positive constant $\mathbb{C}_\Omega$ such that

$$\inf_{\substack{q \in L_2^0 \\ q \neq 0}} \ \sup_{\substack{w \in V_0 \\ w \neq 0}} \frac{|\int_\Omega \ q \operatorname{div} w \, dx\,|}{\|q\| \, \|w\|_{(1)}} \geq \mathbb{C}_\Omega \,. \tag{5.10}$$

From (5.10) it follows that for any functions $p$, $q \in L_2^0(\Omega)$ and any small positive number $\varepsilon$ there exists a function $w_{pq}^\varepsilon \in V_0(\Omega)$ satisfying the inequality

$$\int_\Omega (p - q) \operatorname{div} w_{pq}^\varepsilon \, dx \geq (\mathbb{C}_\Omega - \varepsilon) \, \|p - q\| \, \left\|\nabla w_{pq}^\varepsilon\right\| . \tag{5.11}$$

By virtue of the equality

$$\nu \int_\Omega \nabla u : \nabla w_{pq}^\varepsilon \, dx = \int_\Omega \left(f w_{pq}^\varepsilon + p \operatorname{div} w_{pq}^\varepsilon\right) \, dx$$

we rewrite the left-hand side of (5.11) as follows

$$\int_\Omega (p - q) \operatorname{div} \ w_{pq}^\varepsilon \, dx =$$

$$= \int_\Omega \left\{\nu \nabla(u - v) : \nabla w_{pq}^\varepsilon + \left(\nu \nabla v : \nabla w_{pq}^\varepsilon + \nabla q \cdot w_{pq}^\varepsilon - f w_{pq}^\varepsilon\right)\right\} \, dx =$$

$$= \int_\Omega \nu \nabla(u - v) : \nabla w_{pq}^\varepsilon \, dx + \int_\Omega (\nu \nabla v - \tau) : \nabla w_{pq}^\varepsilon \, dx +$$

$$+\int\limits_{\Omega} \left(\tau : \nabla w^{\varepsilon}_{pq} + \nabla q \cdot w^{\varepsilon}_{pq} - f w^{\varepsilon}_{pq}\right)\, dx. \tag{5.12}$$

The relations (5.11) and (5.13) imply

$$\|p-q\| \leq \frac{1}{(\mathbb{C}_\Omega - \varepsilon)\left\|\nabla w^{\varepsilon}_{pq}\right\|}\left[\nu \int\limits_{\Omega} \nabla(u-v) : \nabla w^{\varepsilon}_{pq} + \right.$$

$$\left. + \int\limits_{\Omega} (\nu \nabla v - \tau) : \nabla w^{\varepsilon}_{pq}\, dx + \int\limits_{\Omega} \left(\tau : \nabla w^{\varepsilon}_{pq} - q \operatorname{div} w^{\varepsilon}_{pq} - f w^{\varepsilon}_{pq}\right)\right] \leq \tag{5.13}$$

$$\leq \frac{1}{(\mathbb{C}_\Omega - \varepsilon)}\left[\nu \|\nabla(u-v)\| + \|\nu \nabla v - \tau\| + \|\operatorname{div} \tau + f - \nabla q\|_{(-1)}\right].$$

By tending $\varepsilon$ to zero and making use of (4.14), (5.6) and (5.9) we get a posteriori estimate for the pressure field

$$\frac{1}{2}\|p-q\| \leq$$

$$\leq \frac{1}{\mathbb{C}_\Omega}\left[c_\Omega\, C_\Omega \|\operatorname{div} v\| + \|\nabla v - \tau\| + \|\operatorname{div} \tau + f - \nabla q\|_{(-1)}\right]. \tag{5.14}$$

The desired estimates (3.10) follow from (4.14), (4.15), (5.6), (5.9), and (5.14).

This completes the proof of Theorem 4.

**5.4. Computable estimates.** It is worthwhile to add some comments on practical applicability of earlier obtained a posteriori estimates. If $q \in W(\Omega) \cap L^0_2(\Omega)$ and

$$\tau \in Q(\Omega) := \{\tau \in \Sigma(\Omega) \ \ \|\ \operatorname{div} \tau \in L_2(\Omega)\} \tag{5.15}$$

then the negative norm in (5.19) can be replaced by the Lebesgue norm. In this case,

$$\mathbf{R}_3(\tau) \leq C_\Omega \ \|\operatorname{div} \tau + f - \nabla q\| \tag{5.16}$$

and the estimates (3.10) have the following matrix form

$$\left\{\begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix}\right\} =$$

$$= \left\{\begin{matrix} 2c_\Omega C_\Omega, & \nu^{-1}, & C_\Omega \nu^{-1}, \\ 2\nu c_\Omega, & 1, & 2C_\Omega, \\ 2\nu c_\Omega C_\Omega {\mathbb{C}_\Omega}^{-1}, & 2{\mathbb{C}_\Omega}^{-1}, & 2C_\Omega {\mathbb{C}_\Omega}^{-1} \end{matrix}\right\} \left\{\begin{matrix} R_1(v) \\ R_2(v,\tau) \\ R_3(\tau,q) \end{matrix}\right\}, \tag{5.17}$$

where $e_1 := \|v-u\|_{(1)}$, $e_2 := \|\tau-\sigma\|$ and $e_3 := \|p-q\|$.

The estimate (5.17) gives a simple and reliable method for getting approximate solutions of the Stokes problem with any a priori given tolerance. In the reminder of this section We briefly outline the corresponding numerical technology.

Let $\{V_k\}$, $\{Q_k\}$, and $\{W_k\}$ be sequences of finite dimensional subspaces in $V_0(\Omega)$, $Q(\Omega)$, and $W(\Omega)$, respectively. Assume that for any positive integer $k$

$$\mathcal{E}_k := V_k \times Q_k \times W_k \subset V_0(\Omega) \times Q(\Omega) \times L_2^0(\Omega)$$

and that $\bigcup_{k=1}^{+\infty} \mathcal{E}_k$ is dense in $V_0(\Omega) \times Q(\Omega) \times L_2^0(\Omega)$. One can easily find a sequence $(v_k, \tau_k, q_k)$ by solving the following auxiliary problems

$$J(v_k, \tau_k, q_k) = \inf_{(v,\tau,q)\in\mathcal{E}_k} J(v, \tau, q),$$

where

$$J(v,\tau,q) := \int\limits_{\Omega} \left(\lambda_1 \mid \operatorname{div} v \mid^2 + \lambda_2 \mid \tau - \nu \nabla v \mid^2 + \lambda_3 \mid \operatorname{div} \tau + f - \nabla q \mid^2\right) dx$$

and the numbers $\lambda_i$ $i = 1,2,3$ are assumed to be positive and such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$. It immediately follows from (5.17) that $(v_k, \tau_k, q_k) \to (u, \sigma, p)$ in the topology of the space $V_0(\Omega)\times\Sigma(\Omega)\times L_2^0(\Omega)$. Moreover, for any $k$ the errors $e_1$, $e_2$ and $e_3$ are explicitly controllable by the estimate (5.17).

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С.-Петербургский
Государственный
Технологический
Университет

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