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PREDICTION PROBLEMS AND HUNT–MUCKENHOUPT–WHEEDEN CONDITION

1. Introduction and Notations

Suppose we observe on interval I a vector valued process $x(t) = (x_1, \cdots, x_d)$ with stationary in wide sense increments and have to construct for random vector ξ (which coordinates belongs to the subspace $\mathbf{X}$ generated by all increments of coordinates of the process x) a linear predictor ξ_I, which depends only on our observations (namely on increments $x_i(t) - x_i(s)$, when $t, s \in I$ and $i = 1, \cdots, d$). Assume that there is a representation

$$\xi = \int \varphi(u)\, dx(u), \tag{$*$}$$

for some matrix function φ, and put

$$\xi_I = \int\limits_I \varphi(u)\, dx(u). \tag{$**$}$$

The first question is: under what condition there exists well defined representation $(*)$ for any element ξ of the space $\mathbf{X}$. The second one: for what case the predictor ξ_I in $(**)$ is a good predictor in the following sense:

$$\mathbf{E}\|\xi - \xi_I\|^2 \le C \inf_{\eta \in \mathbf{X}_I} \mathbf{E}\|\xi - \eta\|^2$$

for some constant C, which does not depend on ξ and I, where we define by $\mathbf{X}_I$ the linear subset generated by inctremens of coordinates of the process x on interval I.

For one-dimensional and descrete time case, when we observe stationary in wide sense sequence x_n, the problem may be reduced to a well known problem of analysis.

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Let x_n be a stationary in wide sense sequence with zero mean and spectral density f. We put for interval I

$$\mathbf{X}_I = \overline{sp}\{x_n, n \in I\}$$

and

$$\mathbf{X} = \overline{sp}\{x_n, n \in \mathbb{Z}\}.$$

Here, we denote by $L^2(d\mathbf{P})$ the L^2-space denerated by a probability measure $\mathbf{P}$ and by $\overline{sp}\{M\}$ the closure in $L^2(d\mathbf{P})$ of linear span of the set M.

Theorem 1.1. *The sequence x_n is a basis in $\mathbf{X}$ in the metric of the space $L^2(d\mathbf{P})$, that is for any element $\xi \in \mathbf{X}$ there exists a uniquely defined sequence a_n, such that*

$$\xi = \sum_{-\infty}^{\infty} a_n x_n, \quad \text{where series} \quad \sum a_n x_n \quad \text{is convergent in} \quad L^2(d\mathbf{P})$$

if and only if spectral density f of sequence x_n satisfies the Muckenhoupt condition:

$$\sup \frac{1}{|I|} \int_I f(u)\, du \, \frac{1}{|I|} \int_I \frac{1}{f(u)}\, du < +\infty, \tag{1.1}$$

where sup *is taken over all intervals I, and $|I|$ is the lenght of I. Under condition* (1.1) *for any interval I*

$$\mathbf{E}|\xi - \xi_I|^2 \le C \inf_{\eta \in \mathbf{X}_I} \mathbf{E}|\xi - \eta|^2,$$

for some constant C, which depends only on f. Here,

$$\xi_I = \sum_{n \in I} a_n x_n.$$

This theorem can be reduced to the well known theorem of analysis (see R. A. Hunt, B. Muckenhoupt, and R. L. Wheeden (1973)), which states (in the language of stationary processes) that operators P_n, which is defined by the relation

$$P_n \sum a_j x_j = \sum_{|j| \le n} a_j x_j, \tag{$***$}$$

is uniformly (in n) bounded if and only if the spectral density f of sequence x_n satisfies the Muckenhoupt condition. We asume in $(***)$ that only finite number of coefficients a_j are not zero.

Let $x(t) = (x_1(t), \dots, x_d(t))$, $t \in \mathbf{R}$, be a vector valued process with zero mean and stationary in the wide sense increments. In this case the map

$$x : \mathbf{R} \to \left(L^2(d\mathbf{P})\right)^d,$$

is such that the operators U_τ defined by the relation

$$U_\tau\{x_j(t) - x_j(s)\} = \{x_j(t+\tau) - x_j(s+\tau)\} \quad i = 1, \cdots, d; \quad t \in \mathbf{R}, \tag{1.2}$$

may be extended to the subspace $\mathbf{X}$,

$$\mathbf{X} = \overline{sp}\left\{x_j(t) - x_j(s), \quad i = 1, \cdots, d; \quad t \in \mathbf{R}\right\},$$

as isometric operators.

Let for random vectors $\xi = (\xi_1, \cdots, \xi_d)$ and $\eta = (\eta_1, \cdots, \eta_d)$ a matrix $\mathbf{R}(\xi, \eta)$ be the correlation matrix:

$$\mathbf{R}(\xi, \eta) = (r_{ij}), \qquad r_{ij} = \mathbf{E}\xi_i\overline{\eta_j}.$$

Here and further all the matrix A has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1d} \\ a_{21} & a_{22} & \dots & a_{2d} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ a_{d1} & a_{d2} & \dots & a_{dd} \end{pmatrix},$$

and we write $A = (a_{ij})$. We denote by $L_d^2 = L_d^2(d\mathbf{P}) = \left(L^2(d\mathbf{P})\right)^d$ the space which contains all random vectors $\xi = (\xi_1, \cdots, \xi_d)$ with coordinates $\xi_i \in L^2(d\mathbf{P})$. Further for simplicity of notation for matrix $A = (a_{ij})$ and vector $h = (h_1, \cdots, h_d)$ we denote by Ah such a vector $y = (y_1, \cdots, y_d)$ that

$$y_j = \sum_{i=1}^{d} a_{ji}h_i.$$

We will use the notation A^* for the conjugate matrix $A^* = \left(a_{ij}^*\right)$ where $a_{ij}^* = \overline{a_{ji}}$. It is clear that if

$$\xi = \sum_{j=1}^{d} A_j\xi^j, \quad \xi^j \in L_d^2(d\mathbf{P})$$

and

$$\eta = \sum_{j=1}^{d} B_j \eta^j, \quad \eta^j \in L_d^2(d\mathbf{P}),$$

then

$$\mathbf{R}(\xi, \eta) = \sum_{i,j=1}^{d} A_i \mathbf{R}(\xi^i, \eta^j) B_j^*.$$

Let $D_d = D_d(\mathbf{R})$ be the space of infinitely differentiable matrix-functions $\varphi = (\varphi_{ij})$ with compact support. The integral

$$x[\varphi] = \int \varphi(u)\, dx(u) \tag{1.3}$$

is well defined for $\varphi \in D_d$ and there exists the spectral representation

$$x[\varphi] = \int \widehat{\varphi}(v) dZ(v), \tag{1.4}$$

where

$\widehat{\varphi}$ is the Fourier transformation of matrix function φ.

$Z(v) = (Z_1(v), \cdots, Z_d(v))$ is a vector valued process with zero mean and orthogonal increments:

$$\mathbf{E}\,(Z_i(b) - Z_i(a))\,(Z_j(d) - Z_j(c)) = 0 \quad i, j = 1, \cdots, d;$$
$$\text{if} \quad (a, b] \cap (c, d] = \emptyset.$$

(integrals without limits are undestood to have limits $-\infty$ to ∞).

The process $x[\cdot]$ is stationary in wide sense (further we will write stationary) generalized process, i.e., an such operator

$$x[\cdot] : D_d \to \left(L^2(d\mathbf{P})\right)^d,$$

such that

$$\mathbf{R}(x[\varphi], x[\psi]) = \mathbf{R}(x[\tau_s\varphi], x[\tau_s\psi]), \quad \varphi, \ \psi \in D_d,$$

where shift operator τ_s is defined by

$$\tau_s\varphi(t) = \varphi(t + s).$$

The structure measure μ of process Z is a σ-additive function on the Borel σ-algebra $B(\mathbf{R})$ with values in the set of nonnegative matrix functions. It is defined for interval $I = (a, b]$ by the relation

$$\mu\{I\} = \mathbf{R}(Z(b) - Z(a), Z(b) - Z(a)).$$

The mesure μ is called the spectral measure of process $x[\cdot]$. If the measure μ is absolutely continuous with respect to the Lebesgue measure:

$$\mu\{A\} = \int\limits_A f(v)\,dv \quad \text{for some nonnegative matrix function} \quad f,$$

then f is called the spectral density of process $x[\cdot]$.

It is clear that it is convenient to investigate our problem in rerms of stationary generalized process $x[\cdot]$, which is defined by (1.3), and it is easily to see that only in the case, where the spectral density f satisfies the condition

$$\frac{1}{1+u^2} f(u) \in L^1,$$

the generalized process $x[\cdot]$ may be represented in the form (1.3) for some process $x(t)$ with stationary increments.

So now we consider a stationary in wide sense generalised process $x[\cdot]$ with spectral density f, which satisfies for some positive $n \in \mathbb{Z}$ to the condition

$$\frac{1}{1+u^{2n}} f(u) \in L^1,$$

and put

$$\mathbf{X} = \overline{sp}\{x^j[\varphi] : \varphi \in D_d, \quad j = 1.\cdots,d\},$$

where $x[\varphi] = \left(x^1[\varphi], \cdots, x^d[\varphi]\right)$

We denote by L_f^2 the space that contains all matrix function $h(u) = (h_{ij}(u))$, such that

$$\int \operatorname{tr}\{h(v)f(u)h^*(v)\}\,dv, \infty,$$

and introduce a matrix-valued inner product in L_f^2:

$$\mathbf{J}_f(h,g) = \int h(v)f(u)g^*(v)\,dv. \tag{1.5}$$

Since

$$\mathbf{R}(x[\varphi], x[\psi]) = \int \widehat{\varphi}(v)f(u)\widehat{\psi}^*(v)\,dv, \quad (\varphi, \psi \in D_d), \tag{1.6}$$

the operator $\mathbf{u}$, which is defined by the relation $\mathbf{u}x[\varphi] = \widehat{\varphi}$, satisfies the condition

$$\mathbf{J}_f(\widehat{\varphi}, \widehat{\psi}) = \mathbf{J}_f(\mathbf{u}x[\varphi], \mathbf{u}x[\psi]) = \mathbf{R}(x[\varphi], x[\psi]), \quad (\varphi, \psi \in D_d). \tag{1.7}$$

So operator $\mathbf{u}$ is an isometry operator with respect to the pair of norm:

$$\|h\|_f = \sqrt{\mathbf{J}_f(h,h)} \quad \text{and} \quad \|\xi\|_{L^2_d} = \sqrt{\mathbf{R}(\xi,\xi)},$$

and it can be extended to all $(\mathbf{X})^d$. Therefore, we conclude that the operator $x[\cdot]$, which was defined on D_d can be extended to the set $D(x)$ of all matrics function φ such that $\widehat{\varphi} \in L^2_f$.

In the case, where the stationary generalized process $x[\cdot]$ is represented in the form (1.3) we say that integral $\int \varphi(t)\, dx(t)$ is well defined, if $\widehat{\varphi} \in \mathrm{L}^2_f$.

2. The main result

Let $\mathbf{L}^2_{\mathrm{loc}}$ be the space which contains all matrix function $\varphi(t) = (\varphi_{ij}(t))$ such that functions φ_{ij} are locally square integrable. We say that a stationary generalized process $x[\cdot]$ is L^2_{loc}-basis in $\mathbf{X}^d$ if

1) process $x[\cdot]$ is defined on $\mathbf{1}_I(\cdot)\mathbf{L}^2_{\mathrm{loc}}$ for any finite interval I,

2) for any $\xi \in \mathbf{X}^d$ there exists uniquely defined matrix function $\varphi \in \mathbf{L}^2_{\mathrm{loc}}$ such that in the metric of the space $\{L^2(d\mathbf{P})\}^d$

$$x[\varphi_T] \to \xi, \quad \text{when} \quad T \to \infty,$$

$$\text{where} \quad \varphi_T(t) = \mathbf{1}_{[-T,T]}(t)\varphi(t).$$

In this case we write $\xi = x[\varphi]$. We denote here by $\mathbf{1}_M$ the indicator function of a set M.

We say that a process $x(t)$ with stationary increments is L^2_{loc}-basis in $\mathbf{X}^d$ if:

1) for any finite interval I the integral $\int_I \varphi(t)\, dx(t)$ is well defined, when $\varphi \in \mathbf{L}^2_{\mathrm{loc}}$,

2) for any $\xi \in \mathbf{X}^d$ there exists uniquely defined matrix function $\varphi \in \mathbf{L}^2_{\mathrm{loc}}$ such that

$$\xi = \int \varphi(t)\, dx(t),$$

where we define the integral above by the relation

$$\int \varphi(t)\, dx(t) = \lim_{T\to\infty} \int_{-T}^{T} \varphi(t)\, dx(t), \tag{2.1}$$

and the limit is taken in the space $\left(L^2(d\mathbf{P})\right)^d$.

It is clear that if a generalized process $x[\cdot]$ may be represented in the form (1.3), we have two difinitions of the same notion.

Sinse the mapping $\mathbf{u} : \mathbf{u}x[\varphi] \mapsto \widehat{\varphi}$ is an isometry

$$\mathbf{X}^d \to L_f^2,$$

we obtain that if a stationary generalized process $x[\cdot]$ is L^2_{loc}-basis, then operator $P_T, T > 0$, which is defined by the relation

$$P_T h\,(u) = \int\limits_{-T}^{T} a(u) \exp\{iuv\}\, du,$$

where

$$h(u) = \int a(u) \exp\{iuv\}\, du,$$

may be extended to the space L_f^2 as bounded operators and these extensions P_T are be uniformly (in T) bounded operators. Therefore, from the result of S. Treil and A. Volberg (1995) we obtain that a stationary generalized process $x[\cdot]$ is L^2_{loc}-basis only if the spectral density f satisfies the condition

$$\sup \left\| \left\{ \frac{1}{|I|} \int\limits_I f(u)\, du \right\}^{1/2} \left\{ \frac{1}{|I|} \int\limits_I f^{-1}(u)\, du \right\}^{1/2} \right\| < \infty, \tag{2.2}$$

where supremum is taken on all intervals I.

Theorem 2.1. *Suppose that the spectral density f of a stationary generalized process $x[\cdot]$ satisfies condition* (2.2) *and*

$$\sup_{|I|>\delta} \left\| \frac{1}{|I|} \int\limits_I f(u)\, du \right\| < \infty, \tag{2.3}$$

$$\sup_{|I|>\delta} \left\| \frac{1}{|I|} \int\limits_I f^{-1}(u)\, du \right\| < \infty. \tag{2.4}$$

Then stationary generalized process $x[\cdot]$ is L^2_{loc}-basis and for any $\xi \in \mathbf{X}^d$

$$\mathbf{E}||\xi - \xi_I||^2 \le C \inf_{\eta \in (\mathbf{X}_I)^d} \mathbf{E}||\xi - \eta||^2$$

for some constant C, which does not depend on ξ and I.

Proof. We denote by $\mathbf{L}^2(\mathbf{T})$ the space of square integrable functions on the unit circle $\mathbf{T} = \{z : |z| = 1\}$ with values in $\mathbf{C}^d$. Let $\mathbf{H}^2(\mathbf{T})$ be the corresponding Hardy space (see [5]) of analitic functions (we consider $\mathbf{H}^2(\mathbf{T})$ as a subspace of the space $\mathbf{L}^2(\mathbf{T})$) and $P_+(\mathbf{T})$ be the orthoprojector in $\mathbf{L}^2(\mathbf{T})$ onto $\mathbf{H}^2(\mathbf{T})$. For nonnegative matrix-function f_* on $\mathbf{T}$ we denote by $\mathbf{L}^2_{f_*}(\mathbf{T})$ the L^2-space of functions on the unit circle $\mathbf{T} = \{z : |z| = 1\}$ with values in $\mathbf{C}^d$ with inner product

$$(\varphi, \psi)_{f_*} = \int\limits_{|z|=1} \langle f_*(z)\varphi(z), \psi(z)\rangle \; dm,$$

where m is the normalized Lebesgue measure on T. If f is a nonnegative matrix-function on real line $\mathbf{R}$, then we denote by $\mathbf{L}^2(\mathbf{T}), \mathbf{L}^2_f(\mathbf{R})$ the corresponding L^2-spaces on $\mathbf{R}$. Let $\mathbf{H}^2(\mathbf{R})$ be the Hardy space of analitic functions in the upper half plane (we consider $\mathbf{H}^2(\mathbf{R})$ as a subspace of the space $\mathbf{L}^2(\mathbf{R})$). We denote by $P_+(\mathbf{R})$ the orthoprojector in $\mathbf{L}^2(\mathbf{R})$ onto $\mathbf{H}^2(\mathbf{P})$.

S. Treil and A. Volberg (1995) proved that under the condition

$$\sup \left\| \left\{ \frac{1}{|I|} \int\limits_I f_*(z)\, dm \right\}^{1/2} \left\{ \frac{1}{|I|} \int\limits_I f_*^{-1}(z)\, dm \right\}^{1/2} \right\| < \infty, \qquad (2.5)$$

(here supremum is taken over all intervals I), there exists a constant C, which depends only on f such that

$$\int\limits_{|z|=1} \langle f(z)P_+(\mathbf{T})\varphi(z), P_+(\mathbf{T})\varphi(z)\rangle \; dm \leq$$

$$\leqslant C \int\limits_{|z|=1} \langle f(z)\varphi(z), \varphi(z)\rangle \; dm. \qquad (2.6)$$

That is the operator $P_+(\mathbf{T})$, which is defined on $\mathbf{L}^2(\mathbf{T}) \cap \mathbf{L}^2_{f_*}(\mathbf{T})$ may be extended to $\mathbf{L}^2_{f_*}(\mathbf{T})$ as a bounded operator.

Let h be a function, defined on $\mathbf{T}$. We denote by Kh the function

$$Kh(\lambda) = h\left(\frac{\lambda - i}{\lambda + i}\right),$$

which is defined on real line $\mathbf{R}$. The mapping S, $Sh(\lambda) = \frac{1}{\lambda+i}$, is (to within some constant factor) an isometry $S : L^2(\mathbf{T}) \to \mathbf{L}^2(\mathbf{R})$, and $S\mathbf{H}^2(\mathbf{T}) = \mathbf{H}^2(\mathbf{R})$. Let f_* be a nonnegative function on $\mathbf{T}$ and f be a nonnegative function on $\mathbf{R}$, which is defined by $f = Kf_*$. It is clear that the operator S is (to within some constant factor) also an isometry $S : L^2_{f_*}(\mathbf{T}) \to \mathbf{L}^2_f(\mathbf{R})$. Therefore, we conclude that the operator $P_+(\mathbf{R})$ is bounded operator in $\mathbf{L}^2_f(\mathbf{R})$ under condition

$$\begin{aligned} \sup\Bigg\| \left\{ \frac{1}{\mu(I)} \int\limits_I f(u) \frac{1}{1+u^2}\, du \right\}^{1/2} \times \\ \times \left\{ \frac{1}{\mu(I)} \int\limits_I f^{-1}(u) \frac{1}{1+u^2}\, du \right\}^{1/2} \Bigg\| < \infty, \end{aligned} \tag{2.7}$$

where sup is taken over all intervals I and

$$\mu(I) = \int\limits_I \frac{1}{1+u^2}\, du.$$

It may be observed (by the same way as in the one dimentional case) that (2.7) follows from (2.2).

It is clear that operators P_T,

$$P_T \int a(t) \exp\{itu\}\, dt = \int\limits_{-T}^{T} a(t) \exp\{itu\}\, dt,$$

are uniformly (in T) bounded in $\mathbf{L}^2_f(\mathbf{R})$ if and only if the operators P_T^+,

$$P_T^+ \int a(t) \exp\{itu\}\, dt = \int\limits_{0}^{T} a(t) \exp\{itu\}\, dt,$$

are uniformly bounded in $\mathbf{L}^2_f(\mathbf{R})$. Since operators P_T are uniformly bounded as the operator P_+ is bounded operator in $\mathbf{L}^2_f(\mathbf{R})$, we conclude that under condition (2.2) operators P_T are uniformly bounded in $\mathbf{L}^2_f(\mathbf{R})$.

Let a subspace $\mathbf{H}_T$ be defined by $\mathbf{H}_T = P_T L^2_f$. Suppose a matrix-function $\psi \in L^2_f$ and $\psi = \psi_T + \psi^T$, where

$$\psi_T \in \mathbf{H}_T, \quad ||\psi^T||_f \to 0 \ \ (\text{as} \quad T \to \infty).$$

Then under condition (2.2) $||P_T\psi^T||_f \to 0$ we have. Therefore, for any matrix-function $\psi \in L_f^2$

$$\psi_T^* = P_T\psi \to \psi \quad \text{in the metric of the space} \quad L_f^2, \quad \text{as} \quad T \to \infty.$$

Under conditions (2.2), (2.3),(2.4) for any function $\psi \in L_f^2$ the functions ψ_T^* are square integrable. Since $P_T\psi_{T_1}^* = \psi_T^*$ as $T \le T_1$, we conclude that for any function $\psi \in L_f^2$ there exists function $\varphi \in L^2_{\mathrm{loc}}$ such that

$$\psi_T^*(u) = \int_{-T}^{T} \varphi(t)\exp\{itu\}\,dt, \quad \varphi \in L^2_{\mathrm{loc}}. \tag{2.8}$$

Now let a random vector ξ be an element of $(\mathbf{X})^d$, $\psi = \mathbf{u}\xi$ and function φ be defined as in (2.8). It is evident that in the metric of the space $\left(L^2(d\mathbf{P})\right)^d$

$$\xi = \lim_{T\to\infty} x[\mathbf{1}_{[-T,T]}(\cdot)\varphi(\cdot)],$$

since (see (1.7))

$$\mathbf{R}(x[\varphi] - x[\varphi_T], x[\varphi] - x[\varphi_T]) = \mathbf{J}_f(\psi - \psi_T^*, \psi - \psi_T^*),$$

where $\varphi_T(\cdot) = \mathbf{1}_{[-T,T]}(\cdot)\varphi(\cdot)$.

Now we define for interval I an operator $P(I)$ by the relation

$$P(I)\int a(t)\exp\{itu\}\,dt = \int_I a(t)\exp\{itu\}\,dt.$$

Under condition (2.2) the operator $P(I)$ is bounded operator in L_f^2. Let ψ be a matrix-function, $\psi = \psi_I + \psi^I$, where $\psi_I \in P(I)L_f^2$. Since under condition (2.2) the operators P_I are uniformly bounded on I, there exists a constant C, which depends only on f and d such that

$$\|\psi - P(I)\psi\|_f \le C\,\|\psi - \psi_I\|_f\,.$$

Therefore,

$$\|\psi - P(I)\psi\|_f \le C\inf_{\psi_I}\|\psi - \psi_I\|_f\,,$$

where inf is taken on $\psi_I \in P(I)L_f^2$. Thus, if random vector ξ is an element of $(\mathbf{X})^d$, $\psi = \mathbf{u}\xi$

$$\mathbf{E}\|\xi - \xi_I\|^2 \le C \inf_{\eta\in\{\mathbf{X}_I\}^d} \mathbf{E}\|\xi - \eta\|^2,$$

since
$$\mathbf{E}\|\xi - \xi_I\|^2 = \|\psi - P(I)\psi\|_f^2$$
and
$$\mathbf{E}\|\xi - \eta\|^2 = \|\psi - \psi_I\|_f^2,$$
where $\psi_I = \mathbf{u}\eta$.

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