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NOTE ON BI-IDEALS IN ORDERED SEMIGROUPS AND IN ORDERED GROUPS

ABSTRACT. It is shown that an ordered semigroup is right and left simple if and only if it does not contain proper bi-ideals. An example is constructed showing that an ordered semigroups without proper bi-ideals need not be an ordered group.

This paper is inspired by the paper of S. Lajos in [7].

The theory of semigroups based on ideals is extended to the theory of *poe*-semigroups based in ideal elements and to the theory of ordered semigroups based on ideals. The technical details to do that, are explained in detail in the published work of the first author (cf., for example [6]). Here, when we speak about an ordered semigroup we assume that it does not contain a greatest elements, in general.

A semigroup S is called left (respectively, right) simple if S does not contain proper left (respectively, right) ideals. In other words, if S is the only left (respectively, right) ideal of S. We notice that a semigroup is a group if and only if it is left and right simple. How can we extend the results of semigroups which are groups? We have two choices: one is to consider an ordered semigroup which is left and right simple. The other is to consider an ordered group.

As an example, we examine here, the following result of S. Lajos announced in [7]: "A semigroup is a group if and only if it has no proper bi-ideals."

From the published work of the first author we guess that an ordered semigroup should be left and right simple if and only if has no proper bi-ideals. In this note we prove that this is really true. And what happens if the ordered semigroup is a group instead of being left and right simple? That is, is an ordered semigroup an (ordered) group if and only if it has no proper bi-ideals? We prove here that if an ordered semigroup is an (ordered) group, then it has no proper bi-ideals, but the converse statement is not true, in general.

Let $(S, ., \leqslant)$ be an ordered semigroup. A non-empty subset A of S is called a left (respectively, right) ideal of S if 1) $SA \subseteq A$ (respectively,

$AS \subseteq A$); 2) $a \in A$, $S \ni b \leqslant a \Rightarrow b \in A$. A is called an ideal of S if it is both a left and right ideal of S [1]. A non-empty subset A of S is called a bi-ideal of S if

1) $ASA \subseteq A$;

2) $a \in A$, $S \ni b \leqslant a \Rightarrow b \in A$ [4].

A bi-ideal A of S is called subidempotent if $A^2 \subseteq A$.

A left, right, or bi-ideal A of S is called proper if $A \neq S$.

S is called left (respectively, right) simple if S does not contain proper left (respectively, right) ideals. (Cf. [2; Definition 3], [3; Definitions 3, 4, Corollary 1]). Equivalently, if for every left (respectively, right) ideal A of S, we have $A = S$.

For $H \subseteq S$, we denote $(H\,] := \{t \in S \mid t \leqslant h \text{ for some } h \in H\}$.

S is called regular if, for every $a \in S$, there exists $x \in S$ such that $a \leqslant axa$. Equivalent Definitions: 1) $A \subseteq (ASA]\ \forall A \subseteq S$; or 2) $a \in (aSa\,]\ \forall a \in S$.

We notice that an ordered semigroup S is left (respectively, right) simple if and only if $(Sa\,] = S$ (respectively, $(aS\,] = S$) for every $a \in S$ (cf. [3; Corollary 2]).

If an ordered semigroup S is left simple and right simple, then it is regular.

Indeed: let $a \in S$. Since $(Sa\,] = S$, $(aS\,] = S$, we have $a \in (aS\,] = (a(Sa\,]\,] = (aSa\,]$. Furthermore, if S is regular then the bi-ideals and the subidempotent bi-ideals of S are the same.

In fact: if B is a bi-ideal of S, then

$$BSB \subseteq B, \quad (BSB\,] \subseteq (B\,] = B.$$

Since S is regular, $B \subseteq (BSB\,]$. Thus, $B = (BSB\,]$, and

$$B^2 = (BSB\,](BSB\,] \subseteq (BSB^2SB\,] \subseteq (BSB\,] = B.$$

Proposition 1. *An ordered semigroup $(S, \,.\,, \leqslant)$ is left and right simple if and only if S does not contain proper bi-ideals.*

Proof. $\Rightarrow$. Let A be a bi-ideal of S. Let $a \in S$. Let $b \in A$ $(A \neq \varnothing)$.

We consider the left ideal of S generated by b, that is the set $L(b) := (b \cup Sb\,]$. Since S is left simple, we have $S = L(b)$. Since $a \in L(b)$, we have $a \leqslant b$ or $a \leqslant xb$ for some $x \in S$. Let $a \leqslant b$. Then, since $S \ni a \leqslant b \in A$, A is a bi-ideal of S, we have $a \in A$. Let $a \leqslant xb$ for some $x \in S$. We consider the right ideal of S generated by b, that is the set $R(b) := (b \cup bS\,]$. Since S is right simple, we have $S = R(b)$. Since $x \in R(b)$, we have $x \leqslant b$ or $x \leqslant by$ for some $y \in S$. Let $x \leqslant b$. Then, we have $a \leqslant xb \leqslant b^2 \in A$, and

$a \in A$. Let $x \leqslant by$ for some $y \in S$. Then $a \leqslant xb \leqslant byb \in ASA \subseteq A$, and $a \in A$.

$\Leftarrow$. Let L be a left ideal of S. Then L is a bi-ideal of S. By hypothesis, $L = S$. Similarly, S is right simple.

Proposition 2. *Let $(S, ., \leqslant)$ be an ordered group. Then S does not contain proper bi-ideals. An ordered semigroup which does not contain proper bi-ideals is not an ordered group, in general.*

Proof. An ordered group is a group. Since groups do not contain proper bi-ideals (announced in [7]), the ordered groups do not contain proper bi-ideals, as well. An independent proof is the following: Let $(S, ., \leqslant)$ be an ordered group, A a bi-ideal of S, $a \in S$. Let $b \in A$ $(A \neq \varnothing)$. Since $b \in S$, there exists $b^{-1} \in S$ such that $bb^{-1} = b^{-1}b = e$ (e is the identity of S). Then we have

$$a = eae = b\big(b^{-1}ab^{-1}\big)b \in ASA \subseteq A, \quad \text{and} \quad a \in A.$$

For the converse statement we give the following

Example. We consider the set $S := (0, \infty)$ of the real numbers with the usual operation "+" of addition and the usual order "$\leqslant$".

$(S, +, \leqslant)$ is an ordered semigroup.

S does not contain proper bi-ideals.

In fact:

let A be a bi-ideal of S, let $a \in S$, and let $b \in A$.

Since "$\leqslant$" is a linear order on S, we have either $a \leqslant b$ or $b < a$.

A) Let $a \leqslant b$. Then, since $b \in A$, we have $a \in A$.

B) Let $b < a$. We have

$$a = b + a - b = \frac{b}{2} + \frac{b}{2} + a - b = \frac{b}{2} + (a - b) + \frac{b}{2};$$

here $a - b \in S$.

Since $S \ni \frac{b}{2} \leqslant b \in A$, we have $\frac{b}{2} \in A$.

Hence, we have

$$a = \frac{b}{2} + (a - b) + \frac{b}{2} \in A + S + A \subseteq A.$$

S is not an (ordered) group. Indeed: There is no $t \in S$ such that $t + x = x$ for all $x \in S$.

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