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Marta Macho-Stadler

FOLIATIONS, GROUPOIDS AND BAUM-CONNES CONJECTURE

Abstract. Baum–Connes conjecture looks for the establishment of an analogue of the well-known isomorphism between the topological K-theory of a locally compact space M and the analytical K-theory of the C*-algebra of continuous functions on M vanishing at infinity, for the leaf spaces of foliated manifolds. In this work we describe the principal notions involved in the statement of the conjecture, and we point out the actual status of it.

1. Approach to Baum–Connes conjecture

In the "commutative world", the most immediate and powerful tools are Homology and Fundamental Group. But, they have not "non-commutative" obvious generalizations Nevertheless, topological K-theory [1] is the most successful tool, since it passes easily to non-commutative world.

It is well known that, for each locally compact space M, the C*-algebra of continuous functions vanishing at infinity, $C_0(M)$, allows us to "reconstruct" M, and there is an isomorphism between the *topological K-theory* of M and the *analytical K-theory* of $C_0(M)$.

Baum–Connes conjecture, independently of its meaning in the context of index theory, looks for the establishment of an analogue of this isomorphism for some "singular" spaces: the leaf spaces of foliated manifolds. Precisely, if $\mathcal{F}$ is a C^∞-foliation on a manifold M, $M/\mathcal{F}$ is a bad quotient in many cases and thus, to obtain information about the transverse structure of the foliation, it is necessary to use another type of objects:

(1) the dynamics of $\mathcal{F}$ is described by its holonomy groupoid G, which is a Lie groupoid. G can be considered as a desingularization of the leaf space $M/\mathcal{F}$. As for any Lie groupoid, we can associate to G a C*-algebra of functions, $C^*_{red}(G)$, which is interpreted as the "space of continuous functions vanishing at infinity" on $M/\mathcal{F}$. And the *analytical K-theory of the leaf space*,

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$K_{an}(M/\mathcal{F})$, is just defined as the K-theory of this C*-algebra, $K_*(C^*_{red}(G))$;

(2) moreover, we can construct a *classifying space* BG for G. G acts freely and properly on BG, which is not in general a manifold (and it has not even the homotopy type of a manifold!). BG can be considered as the leaf space, modulo homotopy. In [2], the authors introduce a generalized G-equivariant K-theory associated to this object, $K_{*,\tau}(BG)$, which is defined as the *topological K-theory* $K_{top}(M/\mathcal{F})$ *of the leaf space.*

Intuitively, G, $C^*_{red}(G)$ and BG are "items" completely determined by $\mathcal{F}$ and carrying the same information. Elliptic operators give a map between the K-theory groups previously described,

$$\mu : K_{top}(M/\mathcal{F}) \to K_{an}(M/\mathcal{F}),$$

and Baum–Connes conjecture asserts that μ is a group isomorphism, when the holonomy groups are torsion free.

The proof of the conjecture would furnish us a relation between the information given by the transverse structure of the foliation (through G) and the geometry granted by BG; in other words, it would "present" a geometrical interpretation of the analytical object $K_*(C^*_{red}(G))$. In this talk, we describe the principal notions involved in the statement of the conjecture, and we point out the actual status of it.

2. Lie groupoids

2.1 Algebraic groupoids.

Definition 1. *An algebraic groupoid is defined by:*

(1) *a couple of sets (M, G), where $M \subset G$ is the unit space and G is the total space,*

(2) *two surjective maps, $\alpha, \beta : G \to M$, the projections, the source and the range, respectively, where if $x \in M$, $\alpha(x) = \beta(x) = x$,*

(3) *a bijection $i : G \to G$, the inversion, such that $i = i^{-1}$ (we note it $i(\gamma) = \gamma^{-1}$),*

(4) *a partial composition law, $. : G^2 \to G$, the multiplication, where the set of composable pairs is $G^2 = \{(\gamma_2, \gamma_1) \in G \times G : \alpha(\gamma_2) = \beta(\gamma_1)\}$, and satisfying:*

- *Associativity: if $\gamma_1, \gamma_2, \gamma_3 \in G$ and we have $((\gamma_2, \gamma_1) \in G^2$ and $(\gamma_3, \gamma_2.\gamma_1) \in G^2)$ or $((\gamma_3, \gamma_2) \in G^2$ and $(\gamma_3.\gamma_2, \gamma_1) \in G^2)$, then $\gamma_3.(\gamma_2.\gamma_1) = (\gamma_3.\gamma_2).\gamma_1$,*

- *Units: for all $\gamma \in G$, we have $(\gamma, \alpha(\gamma)) \in G^2$ and $(\beta(\gamma), \gamma) \in G^2$, and thus $\gamma.\alpha(\gamma) = \gamma = \beta(\gamma).\gamma$,*
- *Inversion: for all $\gamma \in G$, we have $(\gamma, \gamma^{-1}) \in G^2$, $(\gamma^{-1}, \gamma) \in G^2$, and the identities $\gamma.\gamma^{-1} = \beta(\gamma)$, $\gamma^{-1}.\gamma = \alpha(\gamma)$.*

Examples 1. The first examples of algebraic groupoids are:

(1) if M is a point, the groupoid is reduced to a *group*;
(2) if $M = G$, $\alpha(\gamma) = \beta(\gamma) = \gamma$ and $\gamma^{-1} = \gamma$, thus G^2 is the diagonal of $G \times G$ and we obtain the *trivial groupoid*;
(3) given a set X, we consider $G = X \times X$, M is the diagonal of G (identified with X), and we define $\alpha(y, x) = x$, $\beta(y, x) = y$ and $(x, y)^{-1} = (y, x)$. The set of composable pairs is $G^2 = \{((y, x), (x, z)) : x, y, z \in X\}$ and the multiplication $(y, x).(x, z) = (y, z)$. This is the *coarse groupoid*;
(4) the *graph of an equivalence relation* R on a set X is a groupoid, where the unit space is the diagonal, and we define $\alpha(x, y) = x$, $\beta(x, y) = y$ and $(x, y)^{-1} = (y, x)$. $G^2 = \{((x, y)(y, z)) \in G \times G : x, y, z \in X\}$ is the set of composable pairs and $(x, y).(y, z) = (x, z)$.

Definition 2. *Given a groupoid G and $x, y \in M$, we define the α-fibre over x, $G_x = \{\gamma \in G : \alpha(\gamma) = x\}$, the β-fibre over y, $G^y = \{\gamma \in G : \beta(\gamma) = y\}$ and $G_x^y = G_x \cap G^y$. The set G_x^y can be empty. However, when $x \in M$, G_x^x is a group (with the point x as unit element), the isotropy group of G at x. The isotropy groupoid of G is $\mathrm{Is}(G) = \bigcup_{x \in M} G_x^x$ (with M as unit space and the obvious operations).*

Definition 3. *A homomorphism of groupoids from G_1 to G_2 is a function $f : G_1 \to G_2$, such that when $(\gamma_2, \gamma_1) \in G_1^2$, we have $(f(\gamma_2), f(\gamma_1)) \in G_2^2$, and then $f(\gamma_2.\gamma_1) = f(\gamma_2).f(\gamma_1)$.*

2.2 Lie groupoids.

Definition 4. *G is a Lie groupoid (where "differentiable" means C^∞) if:*

(1) *G and M are differentiable manifolds, and M is Hausdorff,*
(2) *the maps α, β, i and . are differentiable; α, β are submersions and i is a diffeomorphism.*

Examples 2. The first examples of Lie groupoids are:

(1) *action of a Lie group on a manifold*: we consider an action of a connected Lie group H on a manifold M, $\Phi : H \times M \to M$. We define in this way a Lie groupoid, with total space $G = H \times M$, unit space M and with the operations, $\alpha(g,x) = x$, $\beta(g,x) = \Phi(g,x)$, $(g,x)^{-1} = (g^{-1}, \Phi(g,x))$, and if $x_2 = \Phi(g_1, x_1)$, the multiplication is $(g_2, x_2).(g_1, x_1) = (g_2 g_1, x_1)$;

(2) *homotopy groupoid of a manifold*: given a differentiable manifold M, we consider $\mathcal{P}(M)$, the set of paths on M, provided with the compact-open topology. We consider the open equivalence relation on $\mathcal{P}(M)$:

$\gamma \sim \gamma'$ if γ is homotope to γ' with fixed extremities.

The quotient space, $\Pi_1(M) = \mathcal{P}(M)/\sim$, is a locally compact groupoid, whose unit space is the manifold M (identified with the class of constant paths) and where the structure of groupoid is defined through the following relations: $\alpha(\gamma) = \gamma(0)$, $\beta(\gamma) = \gamma(1)$, and the multiplication and the inversion of the groupoid are defined following the usual composition and inversion of paths, respectively. The continuous function $(\alpha, \beta) : \Pi_1(M) \to M \times M$ is a covering map, and thus we can lift a structure of differentiable manifold on $\Pi_1(M)$, which is compatible with the quotient topology, and then $\Pi_1(M)$ is a Lie groupoid. If $x \in M$, we have universal coverings of M, $\alpha : \Pi_1(M)^x \to M$ and $\beta : \Pi_1(M)_x \to M$. And the isotropy group at $x \in M$ is $\Pi_1(M)^x_x = \pi_1(M, x)$. The Lie groupoid $\Pi_1(M)$ is the *fundamental groupoid* or *homotopy groupoid* of M.

Definition 5. *Given two Lie groupoids G_1 and G_2, a homomorphism (of Lie groupoids) $f : G_1 \to G_2$ is a differentiable map, which is besides a groupoid homomorphism.*

3. G-fiber bundles

3.1 G-actions. Given a Lie groupoid G and Z a non Hausdorff, locally compact and differentiable manifold, provided with a differentiable map $\rho : Z \to M$, we consider the set $Z *_M G = \{(z,\gamma) \in Z \times G : \rho(z) = \beta(\gamma)\}$.

Definition 6. *A right differentiable action of G on Z is a differentiable map $\Phi : Z *_M G \to Z$, noted $\Phi(z,\gamma) = z.\gamma$, such that:*

(1) *$\rho(z.\gamma) = \alpha(\gamma)$, for each $(z,\gamma) \in Z *_M G$;*

(2) *if one of the expressions* $(z.\gamma).\gamma'$ *or* $z.(\gamma.\gamma')$ *is defined, the other one is also defined, and they coincide;*
(3) *for each* $z \in Z$, *we have* $z.\rho(z) = z$.

We say that Z *is a right* G*-differentiable space when* Z *is a non Hausdorff differentiable manifold, provided with a differentiable right* G*-action.*

Definition 7. *The action of* G *on* Z *is:*

(1) *proper when the map* $\Psi : Z *_M G \to Z \times Z$ *defined by* $\Psi(z,\gamma) = (z, z.\gamma)$ *is proper, and*
(2) *free when given* $(z,\gamma) \in Z *_M G$, *we have* $\gamma.z = z$ *if and only if* $\gamma \in M$.

Definition 8. *A* G*-space* Z *is principal when the action is proper and free. In this case, the canonical projection* $\pi : Z \to Z/G$ *is a submersion, and the quotient space* Z/G *is locally compact and Hausdorff.*

Example 1. If G is a Lie groupoid and we consider $Z = G$ and $\rho = \alpha$, then $Z *_M G = G^2$. The multiplication of the groupoid is a natural right G-action of G on itself. This is a free action, and G is a principal G-space. When $x \in M$, the orbit of this point by the action is $G(x) = G^x$.

Definition 9. *Given two right* G*-differentiable spaces* Z_1 *and* Z_2, *a* G*-map,* $f : Z_1 \to Z_2$, *is a differentiable and* G*-equivariant map, that is, if* $(z_1,\gamma) \in Z_1 *_M G$, *then* $(f(z_1),\gamma) \in Z_2 *_M G$ *and* $f(z_1.\gamma) = f(z_1).\gamma$.

3.2 Morita-equivalence of groupoids.

Definition 10. *If* Z *is a* G*-space, a* G*-bundle over* Z *is defined by a* G*-space* E *and a* G*-map, the projection,* $p : E \to Z$, *such that:*

(1) $p : E \to Z$ *is a complex bundle,*
(2) *for each* $(z,\gamma) \in Z *_M G$, *the map* $\phi : E_z \to E_{z.\gamma}$ *given by* $\phi(u) = u.\gamma$, *is a linear map.*

For our purposes, the good notion of equivalence of groupoids is the following:

Definition 11. *A Morita-equivalence between two Lie groupoids* G_1 *and* G_2 *is defined by:*

(1) *a non Hausdorff locally compact manifold,* Z_f, *provided with two submersions* $r : Z_f \to M_1$ *and* $s : Z_f \to M_2$;
(2) Z_f *is a left principal* G_1*-space and a right principal* G_2*-space, and the actions commute;*

(3) *r induces a diffeomorphism between Z_f/G_2 and M_1 and s induces a diffeomorphism from $G_1 \backslash Z_f$ onto M_2.*

4. From groupoids to foliations

4.1 Groupoids defining foliations.

Definition 12. *Given a Lie groupoid G, for each $\gamma \in G$, the differentiable map $L_\gamma : G^{\alpha(\gamma)} \longrightarrow G^{\beta(\gamma)}$, defined by $L_\gamma(\lambda) = \gamma.\lambda$, is a G-bundle homomorphism of β-fibres, with inverse map $L_{i(\gamma)}$. It is the left translation by γ. We define also the right translation $R_\gamma : G_{\beta(\gamma)} \longrightarrow G_{\alpha(\gamma)}$, by $R_\gamma(\lambda) = \lambda.\gamma$, which is a diffeomorphism of α-fibres. Evidently, the left and right translations commute.*

Thus, the projection by α of the β-fibres (respectively, the projection by β of the α-fibres) is a partition of M, and they are equal: for each $x \in M$, $\Phi(x) = \beta(G_x) = \alpha(G^x)$ is the element of the partition which contains this point. In fact, the connected components of the elements of the partition defined in this way are the leaves of a singular foliation $\mathcal{S}$ on M. It is a regular foliation when we make some restrictions on the groupoid action: if the intersection of the α-fibres and the β-fibres is of constant dimension (that is, if the isotropy groupoid $\mathrm{Is}(G)$ is a manifold), the dimension of the leaves is also constant, and thus $\mathcal{S}$ will be a regular foliation.

Definition 13. *A Lie groupoid G is regular when:*

(1) *the α-fibres are connected,*
(2) *the isotropy groups are discrete (thus, $\dim(G_x^x) = 0$, for each $x \in M$).*

And it is oriented when the induced foliation is orientable.

The inversion i changes the α-fibres and the β-fibres, thus in the conditions of Definition 13, the β-fibres are also connected and, following the previous discussion, M is provided with a regular foliation $\mathcal{S}$. We say that it is *defined by the action of G on M*. For example, the homotopy groupoid of a manifold (which induces a foliation by a unique leaf) is regular.

Definition 14. *A regular groupoid G defines a foliation $\mathcal{F}$ on M when the orbits of the groupoid action are the leaves of $\mathcal{F}$.*

Reciprocally, every foliation can be defined through an action of a regular Lie groupoid; the most important are the homotopy and the holonomy groupoids of the foliation, the others are "in the middle" (the holonomy groupoid is the biggest one, the homotopy groupoid is the smallest one, and the others are constructed as quotients of the holonomy groupoid).

4.2 Homotopy groupoid of a foliation. Let $(M, \mathcal{F})$ be a foliation of dimension p and codimension q, on a manifold M of dimension $n = p + q$. We consider $\mathcal{P}(\mathcal{F})$ the set of tangent paths of $\mathcal{F}$ (that is, the paths whose image is contained in a leaf), provided with the compact-open topology. We define the following open equivalence relation:

$\gamma \sim \gamma'$ if γ is homotope (with fixed extremities) to γ' in the leaf.

On the quotient by this action, $\Pi_1(\mathcal{F}) = \mathcal{P}(\mathcal{F})/\sim$, we define a structure of algebraic groupoid by: the unit space is M (identified with the class of constant paths), the projections are $\alpha(\gamma) = \gamma(0)$, $\beta(\gamma) = \gamma(1)$, the groupoid multiplication and the inversion are induced by the usual composition and inversion of paths, respectively. The topology of $\mathcal{P}(\mathcal{F})$ induces on $\Pi_1(\mathcal{F})$ a quotient topology, with respect to α, β, i and . are continuous maps, α and β are open and i is a homeomorphism. If $x \in M$, $\beta : \Pi_1(\mathcal{F})_x \to L_x$ and $\alpha : \Pi_1(\mathcal{F})^x \to L_x$ are universal coverings of L_x, the leaf containing x, and the isotropy group at x is just $\Pi_1(\mathcal{F})_x^x = \pi_1(L_x, x)$.

The differentiable structure is worked in detail in [11]. But, concisely, we can describe it in the following manner: if γ is a tangent path in $\mathcal{F}$, we consider two distinguished cubes $U_i = P_i \times T_i$, where U_i is a neighborhood of $\gamma(i)$ $(i \in \{0,1\})$, P_i is a plaque and T_i is a transversal of $\mathcal{F}$. Modulo possible restrictions of T_0 and T_1, the local triviality of the foliation allows us to define a holonomy diffeomorphism $h_\gamma : T_0 \to T_1$, represented by a "tube of tangent paths of $\mathcal{F}$", $\widehat{h_\gamma} : T_0 \times [0,1] \to M$. This tube is parametrized by T_0 and the map h_γ consists in going from the origin to the end of these paths. And $\widehat{h_\gamma}$ extends to a differentiable family of tangent paths $\mathcal{F}$, family parametrized by $P_0 \times T_0 \times P_1$, and which induces a diffeomorphism from U_0 to U_1. Passing to homotopy classes, we obtain a local chart on $\Pi_1(\mathcal{F})$. The atlas constructed in this way defines on $\Pi_1(\mathcal{F})$ a structure of differentiable manifold of dimension $2p + q$. Since all the objects defining the groupoid structure on $\Pi_1(\mathcal{F})$ are differentiable, we obtain a Lie groupoid, called *the homotopy or fundamental groupoid* of the foliation. It is "the foliated version" of the homotopy groupoid of a manifold.

4.3 Holonomy groupoid of a foliation. If we define on $\mathcal{P}(\mathcal{F})$ a finer equivalent relation "$\sim_h$":

$$\gamma \sim_h \gamma' \text{ if the holonomy germ of the path } (\gamma')^{-1}.\gamma \text{ is trivial,}$$

we obtain a Lie groupoid (of dimension $2p+q$), $\mathrm{Hol}(\mathcal{F}) = \mathcal{P}(\mathcal{F})/\sim_h$, called *the holonomy groupoid or graph of the foliation*. The isotropy group of $x \in M$ is $\mathrm{Hol}(\mathcal{F})_x^x = h(\pi_1(L_x, x))$, where the map $h : \pi_1(L_x, x) \to \mathrm{Diff}(T_0, x)$ (defined by $h(\gamma) = h_\gamma$) is the holonomy representation of the leaf L_x through x. The fibers of this groupoid are the holonomy coverings of the leaves of $\mathcal{F}$. Observe that the quotient $\mathrm{Hol}(\mathcal{F})^x / \mathrm{Hol}(\mathcal{F})_x^x$ is diffeomorphic to L_x, and thus $\mathrm{Hol}(\mathcal{F})$ is the natural desingularization of $M/\mathcal{F}$, and in fact it "unwraps" all the leaves at one time!

The lifting property of paths allows us to construct a Lie groupoid surjective homomorphism $\phi : \Pi_1(\mathcal{F}) \to \mathrm{Hol}(\mathcal{F})$. Unlike $\Pi_1(\mathcal{F})$, $\mathrm{Hol}(\mathcal{F})$ "forgets" the leaf structure and it carries uniquely information about the transverse structure of $\mathcal{F}$. In general, $\Pi_1(\mathcal{F})$ and $\mathrm{Hol}(\mathcal{F})$ are not Hausdorff.

If we consider the foliation restricted to a distinguished cube $U = P \times T$, $\mathcal{F}_U$, where P is a plaque and T is a transversal, the two groupoids previously defined coincide

$$\Pi_1(\mathcal{F}_U) = \mathrm{Hol}(\mathcal{F}_U) = P \times P \times T.$$

In fact, we have a parametrized family (by T) of coarse groupoids (see example 1 (3)): this is *the local triviality property* of the homotopy and holonomy groupoids of a foliation.

The research on the leaf space is a study of transverse properties, it is useful to introduce the following notion:

Definition 15. *Given G a regular groupoid defining a foliation $\mathcal{F}$ on M and T a transverse submanifold which intersects every leaf (a total transversal, eventually non connected), let $G_T^T = \{\gamma \in G : \alpha(\gamma), \beta(\gamma) \in T\}$. If we note α_T and β_T (which are local diffeomorphisms) the restrictions of α and β to G_T^T, we have a Lie groupoid G_T^T, the transverse groupoid associated with the transversal T.*

The transverse groupoid is simpler than the original one, and:

Proposition 1. *The natural immersion of the transversal on the manifold, $i_T : T \to M$, induces a Morita-equivalence between the groupoids G and G_T^T.*

5. C*-ALGEBRAS AND FOLIATIONS

5.1 What is a C*-algebra?

Definition 16. *A C*-algebra A is a complex Banach algebra, provided with an involution $*: A \to A$, satisfying the relation $\|a^*a\| = \|a\|^2$ with respect to the Banach norm.*

Examples 3. The first examples of C*-algebras are:

(1) if M is a locally compact Hausdorff space, the algebra of continuous functions on M, with complex values and vanishing at infinity, $C_0(M)$, with the involution $f^*(x) = \overline{f(x)}$ (for $x \in M$), is a commutative C*-algebra;
(2) if $\mathcal{H}$ is a Hilbert space and $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators, acting on $\mathcal{H}$, with the involution defined by the usual adjunction (T^* is the adjoint of $T \in \mathcal{B}(\mathcal{H})$, if $\langle Th, h\rangle = \langle h, T^*h\rangle$, for $h \in \mathcal{H}$), $\mathcal{B}(\mathcal{H})$ is a C*-algebra. Every auto-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, norm closed, is also a C*-algebra. For example, the subalgebra of compact operators $\mathcal{K}(\mathcal{H})$ is a C*-algebra.

We have a category whose objects are C*-algebras and whose morphisms are linear maps $f: A \to B$, which are multiplicative ($f(a_1a_2) = f(a_1)f(a_2)$) and preserve involution ($f(a^*) = f(a)^*$), they are the $*$-*homomorphisms*. In fact, every C*-algebra can be considered as an auto-adjoint and norm closed subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space.

The central theorem of this theory shows that the category of commutative C*-algebras and $*$-homomorphisms is dual of the category of locally compact spaces and proper continuous maps:

Theorem 1 (Gelfand–Naimark). *Every commutative C*-algebra is isomorphic to $C_0(M)$, for some Hausdorff locally compact space M.*

We intend to transpose these properties to "singular spaces".

Examples 4. We have more examples of C*-algebras:

(1) the universal C*-algebra generated by a unitary u, with relations $u^*u = uu^* = 1$, is $C^*(u) \simeq C(\mathbb{S}^1)$;
(2) the universal C*-algebra generated by $\{u_1, \dots, u_n\}$, with relations $u_i^*u_i = u_iu_i^* = 1$ and $u_iu_j = u_ju_i$, is $C^*(u_1, \dots, u_n) \simeq C(\mathbb{T}^n)$;
(3) the universal C*-algebra generated by $\{h_0, \dots, h_n\}$, with relations $h_i = h_i^*$, $h_ih_j = h_jh_i$ and $\sum_{i=0}^{n} h_i^2 = 1$, is $C^*(h_0, \dots, h_n) \simeq$

$C(\mathbb{S}^n)$;

(4) *C*-algebra of a locally compact group*: we complete the involutive algebra $C_c(G)$ of complex valued continuous functions with compact support on G with respect to the norms:
- the maximal norm of C*-algebra (which involves the unitary representations of $C_c(G)$), obtaining $C^*_{max}(G)$, or
- the norm induced by the left regular representation of the group, obtaining $C^*_{red}(G)$;

(5) *C*-algebra of a group action*: it is a generalization of (4), obtaining the C*-algebra *crossed product* $A \rtimes_\alpha G$ (which is a "twisted" group C*-algebra, with coefficients in A) and the C*-algebra *reduced crossed product*, $A \rtimes_{\alpha,red} G$. Some simple examples:
- if G acts by homeomorphism on a locally compact space M (thus by $*$-automorphisms on $C_0(M)$), $C_0(M) \rtimes G$ is the *transformation group* C*-algebra. If the action is free and proper, $C_0(M) \rtimes G$ is isomorphic to $C_0(M/G) \otimes \mathcal{K}(L^2(G))$;
- if $A = \mathbb{C}$ and α is trivial, the crossed product is $C^*(G)$ (and the reduced crossed product is $C^*_{red}(G)$);
- if A and G are arbitrary, and α is trivial, the crossed product is the tensor maximal product $A \otimes_{max} C^*(G)$.

5.2 Groupoid C*-algebras. The following construction is valid for many groupoids. But, to simplify, we suppose that G is the holonomy groupoid of a regular and orientable foliated space $(M, \mathcal{F})$.

Since G is not, in general, Hausdorff, it is necessary to adapt the definition of algebra of continuous functions with compact support on G to this situation:

Definition 17 [3]. *$C_c(G)$ is the vector space generated by finite sums of the form $f = \varphi \circ \chi$, where:*
- *$\chi : U \longrightarrow \mathbb{R}^k$ is a local chart for the manifold structure on G,*
- *φ is a continuous function with compact support on $\chi(U)$, that is, $f = \varphi \circ \chi$ on U and f is null on $G - U$.*

If G is Hausdorff, this definition coincides with the usual of $C_c(G)$.

The choice of a Riemannian metric on M defines an orthogonal decomposition of the tangent bundle $T(M) = \nu(\mathcal{F}) \oplus T(\mathcal{F})$. We consider a pure form ω, of type $(0, p)$ on M (p is the dimension of the foliation), which defines by restriction to the leaves the volume associated to the induced metric (the volume element ω changes differentiably on M). If $x \in M$ and L_x is the leaf through x, the restriction $\omega|_{L_x} = \omega_x$ is a volume form on L_x. Moreover, $\alpha : G^x \to L_x$ is a covering map (corresponding to the kernel of the holonomy representation), and thus it is possible to lift ω_x in λ^x on G^x. Globally, we lift ω by α, in a volume form $\lambda = \alpha^*(\omega)$ on G, which defines a volume by restriction on the β-fibres.

Definition 18. *We say that $\{\lambda^x\}_{x \in M}$ is a left Haar system on G. Intuitively, a Haar system is an object of metric nature, which plays the role of Haar measure for groups.*

This volume is invariant by left translations:

Lemma 1. *If $f \in C_c(G^x)$ and $\gamma_0 \in G_x$, we have*

$$\int_{G^x} f(\gamma) d\lambda^x(\gamma) = \int_{G^{\beta(\gamma_0)}} f(L_{\gamma_0}(\gamma)) d\lambda^{\beta(\gamma_0)}(\gamma).$$

And we have the continuity condition:

Lemma 2. *For $f \in C_c(G)$, the map $\lambda(f) : M \to \mathbb{C}$ defined by $\lambda(f)(x) = \int_{G^x} f(\gamma) d\lambda^x(\gamma)$ is continuous.*

From the previous lemmas, we deduce:

Proposition 2. *$C_c(G)$ is a $*$-algebra for the following operations. If $f, g \in C_c(G)$:*

(1) Involution: $f^*(\gamma) = \overline{f(\gamma^{-1})}$, *for* $\gamma \in G$,
(2) Convolution: $f * g(\gamma) = \int_{G^{\beta(\gamma)}} f(\gamma_1) g(\gamma_1^{-1}.\gamma) d\lambda^{\beta(\gamma)}(\gamma_1)$, *for* $\gamma \in G$.

We consider $R_x : C_c(G) \to \mathcal{L}(L^2(G^x))$, the non degenerated representation of $C_c(G)$ defined by

$$R_x(f)(\xi)(\gamma) = \int_{G^x} f(\gamma^{-1}.\gamma_1)\xi(\gamma_1) d\lambda^x(\gamma_1),$$

for $x \in M$, $f \in C_c(G)$, $\xi \in L^2(G^x)$ and $\gamma \in G^x$.

Definition 19. *The reduced C*-algebra of G, $C^*_{red}(G)$, is the completion of $C_c(G)$ with respect to the C*-algebra norm $\| f \| = \sup_{x \in M} \| R_x(f) \|$.*

The previous construction depends on the choice of the Riemannian metric on M. If λ and λ_1 are volumes on G, obtained as above through different Riemannian metrics, they are related by the identity $\lambda_1 = f.\lambda$, where the map $f : G \to \mathbb{R}$ is continuous, positive and constant on the fibers of the map $h : G \to M \times M$ defined by $h(\gamma) = (\beta(\gamma), \alpha(\gamma))$. The C*-algebras $C^*_{\lambda,red}(G)$ and $C^*_{\lambda_1,red}(G)$ coincide in the set theoretic viewpoint, that is, the same functions are bounded for the two norms. Moreover, for each $x \in M$, the measure λ^x is absolutely continuous with respect to λ_1^x, that is, there is a unique function (the Radon–Nikodym derivative) $f \geq 0$, λ-measurable, such that $\lambda_1(E) = \int_E f d\lambda$. Thus, the C*-algebra $C^*_{red}(G)$ does not depend on the choice of λ or λ_1. So, the previous construction is intrinsic.

5.3 Morita-equivalence of C*-algebras.

Definition 20. *Let A be a C*-algebra and $\mathcal{E}$ a right A-module. $\mathcal{E}$ is a Hilbert A-module if it is provided with an A-valued inner product: $\langle\, , \rangle_A : \mathcal{E} \times \mathcal{E} \to A$, linear with respect to the second variable ($\lambda\langle \xi, \eta\rangle_A = \langle \xi, \lambda\eta\rangle_A$) and anti-linear with respect to the first one ($\lambda\langle \xi, \eta\rangle_A = \langle \overline{\lambda}\xi, \eta\rangle_A$), such that, if $\xi, \eta \in \mathcal{E}$ and $a \in A$, then:*

1. $\langle \xi, \eta\rangle_A = \langle \eta, \xi\rangle_A^*$;
2. $\langle \xi, \eta a\rangle_A = \langle \xi, \eta\rangle_A a$ *and* $\langle \xi a, \eta\rangle_A = a^* \langle \xi, \eta\rangle_A$;
3. $\langle \xi, \xi\rangle_A \geq 0$ *(that is $\langle \xi, \xi\rangle_A$ is positive as an element of A) and* $\langle \xi, \xi\rangle_A = 0$ *if and only if* $\xi = 0$;
4. *the map $n : \mathcal{E} \to \mathbb{R}$ defined by $n(\xi) = ||\langle \xi, \xi\rangle_A||^{1/2}$ is a complete space norm on $\mathcal{E}$.*

Definition 21. *A Hilbert A-module $\mathcal{E}$ is full if the A-valued inner product on $\mathcal{E}$, $\langle\, , \rangle_A$, generates A as a closed bilateral ideal.*

Definition 22. *Given A and B two C*-algebras, they are Morita-equivalent when there is a Hilbert full B-module $\mathcal{E}$, such that A is isomorphic to $\mathcal{K}_B(\mathcal{E})$.*

Moreover, we can verify:

Proposition 3. *If G_1 and G_2 are Morita-equivalent groupoids, then $C^*_{red}(G_1)$ and $C^*_{red}(G_2)$ are Morita-equivalent C*-algebras.*

5.4 C*-algebra of a foliation.

Definition 23. *The C*-algebra of a foliated space $(M,\mathcal{F})$, $C^*(M/\mathcal{F})$, is defined as the reduced C*-algebra of its holonomy groupoid, $C^*_{red}(G)$.*

The construction of $C^*(M/\mathcal{F})$ is *local* in the following sense:

Proposition 4. *If $U \subset M$ is open and $\mathcal{F}_U$ is the restriction of $\mathcal{F}$ to U, then G_U (the graph of $(U,\mathcal{F}_U)$) is an open subgroupoid of G, and the inclusion $C_c(G_U) \subset C_c(G)$ extends to an $*$-isometric isomorphism from $C^*(U/\mathcal{F}_U)$ to $C^*(M/\mathcal{F})$.*

$C^*(M/\mathcal{F})$ is *stable*, that is:

Proposition 5. $C^*(M/\mathcal{F}) \otimes \mathcal{K} \cong C^*(M/\mathcal{F})$.

Using Propositions 1 and 3, we prove that the C*-algebra of a foliation *depends only on its transverse structure*:

Theorem 2. *If $(M,\mathcal{F})$ is a foliated manifold and T is a total transversal, then the C*-algebras $C^*(M/\mathcal{F})$ and $C^*_{red}(G^T_T)$ are Morita-equivalent.*

We prove easily:

Theorem 3. *If $(M_1,\mathcal{F}_1)$ and $(M_2,\mathcal{F}_2)$ are topologically equivalent foliations, then $C^*(M_1/\mathcal{F}_1) \otimes \mathcal{K} \cong C^*(M_2/\mathcal{F}_2) \otimes \mathcal{K}$.*

Some properties of a foliated manifold are reflected in the structure of the associated C*-algebra; for example:

Proposition 6. *If the graph is Hausdorff, we have:*

(1) *$C^*(M/\mathcal{F})$ is simple if and only if each leaf in $\mathcal{F}$ is dense,*
(2) *$C^*(M/\mathcal{F})$ has an injective irreducible representation if and only if there is a dense leaf,*
(3) *the foliation $\mathcal{F}$ is closed (that is, its leaves are closed), if and only if $C^*(M/\mathcal{F})$ has the compact operators as quotient.*

In the non Hausdorff case, for example, if the foliation is minimal, the C*-algebra is not necessarily simple.

Examples 5. Some examples of C*-algebras of foliations:

(1) if M is locally compact and it is foliated by points, then $G = M$ and $C^*(M/\mathcal{F}) = C_0(M)$;
(2) if M is locally compact and M is foliated by a unique leaf, then $G = M \times M$. A Haar system is simply a measure λ on M, with

support M. Evidently, the elements of the dense subalgebra of the definition can be chosen as integral operators with kernel of compact support on $L^2(M, \lambda)$. Its completion is obviously the family of compact operators $\mathcal{K}$ on $L^2(M, \lambda)$;

(3) if the foliation is defined by a fibration $F^p \to M \to B^q$, with F connected, then M is foliated by the inverse images of the points of B. All the leaves are closed and diffeomorphic to F. In this case, the holonomy groupoid is the graph of the equivalence relation corresponding to the partition of M in leaves, and the C*-algebra $C^*(M/\mathcal{F})$ is isomorphic to $C_0(B, \mathcal{K}(L^2(F))$;

(4) if the foliation comes from the action of a Lie group Γ, in such way that the graph is $M \times \Gamma$ (this fact is not, in general, true), then $C^*(M/\mathcal{F})$ is the reduced crossed product $C_0(M) \rtimes_{red} \Gamma$.

6. K-theory of a C*-algebra

6.1 K-theory of topological spaces. Let M be a compact space. If we note by $V(M)$ the set of all isomorphism classes of locally trivial complex bundles over M, it is well known that V is a contravariant functor from the category of compact spaces to the category of Abelian semigroups, homotopy invariant. $K^0(M)$ is defined as the Grothendieck group of $V(M)$, and it is also a contravariant functor from the category of compact spaces to those of Abelian groups. This definition generalizes to the case of locally compact M.

The *reduced suspension* of order n of M, $S^n(M)$, is defined as the non compact space $S^n(M) = M \times \mathbb{R}^n$. The *K-group* of order n of M is defined as $K^n(M) = K^0(S^n(M))$.

Let M be a locally compact space and E a complex vector bundle over M. Then

Proposition 7. *If* $\Gamma(M, E)$ *is the set of* continuous sections *on* E*, we have:*

(1) $\Gamma(M, E)$ *is a module on the ring of* $\mathbb{C}$*-valued continuous functions over* M, $C(M)$*;*

(2) *an isomorphism of complex vector bundles induces an isomorphism on the corresponding section modules;*

(3) *if* E *is the trivial bundle of dimension* n, *then* $\Gamma(M, E) \simeq C(M)^n$*;*

(4) *if* M *is compact, then* $\Gamma(M, E)$ *is a projective module of finite type.*

Thus, Γ *is a covariant functor from the category of complex vector bundles* *over a compact Hausdorff space M to the category of projective modules of finite type on $C(M)$, and we can prove that Γ is bijective: this is* Swan's theorem.

According to the previous arguments, if M is compact, $K^0(M)$ can be also described as the group of formal differences $[P]-[Q]$ of isomorphism classes of projective modules of finite type on $C(M)$. This result has an enormous importance, because there is a natural generalization in the non-commutative case.

6.2 K-theory of C*-algebras. If A is a C*-algebra, we define K_0 as a contravariant functor from the category of C*-algebras to those of Abelian groups, in such a way that the elements of $K_0(A)$ can be thought as formal differences $[p]-[q]$, where p and q are projections on $\mathbb{M}_k(\widetilde{A})$ ($\widetilde{A}$ is the unitary algebra associated to the C*-algebra A, in the usual manner), for a certain $k \in \mathbb{N}$ and $p-q \in \mathbb{M}_k(A)$.

When A is unitary, it is also possible (and very useful) to consider $K_0(A)$ as the formal differences $[\mathcal{E}]-[\mathcal{F}]$ of equivalence classes of A-projective modules of finite type.

Moreover, we can define the K-theory groups of higher dimension (introducing the notion of *suspension of a C*-algebra*), and Swan's theorem shows that for $A = C_0(M)$ (where M is a locally compact space), its K-analytical group, $K_j(A)$, is naturally isomorphic to the K-topological group $K^j(M)$.

The functor K_j is homotopy invariant, preserves inductive limits, is stable (that is, $K_j(A) \simeq K_j(A \otimes \mathcal{K})$, where $\mathcal{K}$ is the algebra of compact operators) and semi-exact (if J is an ideal in A and we have a short exact sequence $0 \longrightarrow J \xrightarrow{i} A \xrightarrow{\pi} A/J \longrightarrow 0$, then we have a short exact sequence of groups $K_j(J) \xrightarrow{i_*} K_j(A) \xrightarrow{\pi_*} K_j(A/J)$).

And the main result for our purposes is:

Proposition 8. *If A and B are Morita-equivalent C*-algebras, the corresponding K-theory groups $K_*(A)$ and $K_*(B)$ are isomorphic.*

6.3 K-orientation. Given a foliated manifold $(M, \mathcal{F})$, a theorem due to Buffet–Lor shows that there is a topological space BG and a continuous map $i : BG \longrightarrow M/\mathcal{F}$, with the universal property that, for each space X and map $f : X \longrightarrow M/\mathcal{F}$, there is $g : X \longrightarrow BG$ (defined modulo homotopy), such that $f = i \circ g$. BG is a CW-complex and there is a principal G-bundle EG (which is contractible) over BG, such that for each space X and principal G-bundle E over X, there is a map $f : X \to BG$, with $E = f^*(EG)$. BG is unique up to a homotopy.

We can regard BG (which is a transverse object) as a foliated space $(W, \mathcal{F}_W)$, where the foliation $\mathcal{F}_W$ has all leaves contractible, and whose holonomy transverse groupoids are locally isomorphic to the initial groupoid.

K-theory is a complex theory, with compact support. We work with real bundles (tangent bundles, tangent bundles of foliated spaces, ...), then we need an additional structure for "complexifying" them. K-orientation, defined through *spin*^{c}-structures, is the natural orientation which guarantees the existence of index, duality, Thom isomorphism theorems, ... , for this cohomology theory.

Definition 24. *Let* Ml_n^c *be the group* $\mathrm{Ml}_n(\mathbb{R}) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{U}(1)$, *where* $\mathrm{Ml}_n(\mathbb{R})$ *is the metalinear group, that is, the non trivial two-fold covering of the set of real matrices with positive determinant group* $\mathrm{Gl}_n^+(\mathbb{R})$. *The maximal compact of* Ml_n^c *is* $spin^c(n)$.

Definition 25. *The holonomy groupoid of* $\mathcal{F}$ *is K-oriented if the structural groupoid of the tangent bundle to the foliation* $T(\mathcal{F})$ *is reduced to* $spin^c$.

Definition 26. *If* X *is a* G*-manifold and* $\nu(\mathcal{F})$ *is the normal bundle of the foliation, a map* $f : X \to BG$ *is K-orientable if the bundle* $T(X) \oplus f^*(\nu(\mathcal{F}))$ *has a* $spin^c$*-structure. When we choose one of them, we say that* f *is K-oriented.*

7. Statement of Baum–Connes conjecture

Let $(M, \mathcal{F})$ be a foliated manifold and G its holonomy groupoid. A purely geometric manner of defining K-theory is the following:

Definition 27. *A K-cycle is a triple* (X, E, f), *where:*

(1) X *is a closed manifold and* E *is a complex* G*-bundle over* X,
(2) $f : X \to BG$ *is a K-oriented map (that is, a* $spin^c$*-structure over* $T(X) \oplus f^*(\nu(\mathcal{F}))$ *has been chosen).*

We do not suppose that X *is connected, and* E *can have different dimensions on different connected components of* X.

We define an equivalence relation "$\sim$" on the set of K-cycles, through disjoint sum, bordism and vector bundle modification conditions (relation connected with the multiplicativity of elliptic operators index on vector bundles), and we define:

Definition 28. *$K_{*,\tau}(BG)$ is the quotient of the K-cycles by the equivalence relation "$\sim$", where the subindex τ means "twist" by the transverse bundle of the foliation. This is the topological K-theory of the leaf space, $K_{top}(M/\mathcal{F})$.*

Examples 6. In the simplest cases, we have:

(1) if $\mathcal{F}$ is the foliation by a unique leaf, we have $\nu(\mathcal{F}) = 0$ and $K_{*,\tau}(BG) \simeq K^*(BG)$, and
(2) if M is foliated by points, we have $\nu(\mathcal{F}) \simeq T(M)$, BG is homotopy equivalent to M and $K_{*,\tau}(BG) \simeq K^*(M)$.

Definition 29. *The analytical K-theory of the leaf space, $K_{an}(M/\mathcal{F})$, is defined as the K-theory $K^*(C^*(M/\mathcal{F}))$.*

The longitudinal index theorem [5] defines the *K-index map*,

$$\mu : K_{top}(M/\mathcal{F}) \to K_{an}(M/\mathcal{F}),$$

and:

Conjecture 1 (Baum–Connes). *The K-index map is an isomorphism, when the holonomy groups are torsion free.*

8. Our work in this field

A particularly "favorable" situation is the case of a foliation $(M, \mathcal{F})$ for which the holonomy groupoid G has contractible fibres (α- or β-fibres): the case of a *classifying foliation*. In this case, BG is identified with the manifold M and the topological K-theory of the leaf space is reduced to the K-theory of M (or of $C_0(M)$). Besides, in this simple case, we cannot give an immediate formulation of Baum–Connes conjecture, since the "foreseen" isomorphism is defined in terms of a *K-oriented map* between leaf spaces (or groupoids), which can be interpreted as the projection from M to $M/\mathcal{F}$.

Our study is centered in classifying foliations, or more generally in foliations which can be transformed in a classifying one through simple topological manipulations (as topological equivalences, Morita-equivalences, ...). In these cases, we prove the "reduced formulation" of Baum–Connes conjecture:

Conjecture 2 (Reduced form of Baum–Connes conjecture). *If the holonomy groupoid of the foliation is classifying and K-oriented, the K-index map $\mu : K^*(M) \to K_{an}(M/\mathcal{F})$ is an isomorphism.*

Baum–Connes conjecture has already been verified in some cases of foliations:

(1) fibrations $F \to M \to B$: in this case, $C^*(M/\mathcal{F}) \simeq C_0(B) \otimes \mathcal{K}$ and $K_{an}(M/\mathcal{F}) \simeq K^*(B)$, where the leaf space is identified with the base space of the fibration B;
(2) following the Thom isomorphism of A. Connes [3], for foliations induced by free actions of $\mathbb{R}^n$, we have $C^*(M/\mathcal{F}) \simeq C_0(M) \rtimes \mathbb{R}^n$ and the K-theory is $K^*(M)$ when n is odd, and $K^{*+1}(M)$ in other case. For foliations induced by free actions of solvable simply connected Lie groups Γ, we obtain $K_{an}(M/\mathcal{F}) \simeq K^{*+dim(\Gamma)}(M)$;
(3) Reeb foliations on $\mathbb{T}^2$ and $\mathbb{S}^3$ [10];
(4) without holonomy foliations ([9] in the C^∞-case and [7] in the topological case): BG is a torus $\mathbb{T}^n$ (n is the order of the holonomy group of the foliation), provided with a linear foliation, and we use (2);
(5) almost without holonomy foliations [6]: by an argument of graphs of groups, we reduce this case to (4). The reasoning used in this case is also valid for other type of foliated spaces, in which a certain "scheme" in closed and open sets works;
(6) some non trivial examples: Sacksteder foliation, Hirsch foliation, $\mathbb{Z}$-periodic foliations, ...

In all these cases, BG has the homotopy type of a manifold, and we use "strongly" argumentations involving Morita-equivalences of groupoids and C*-algebras (thus Propositions 1, 3 and 8).

At present, we study (in collaboration with M. O'uchi, [8]) properties of *groupoid correspondences*, in order to solve the conjecture in other examples.

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Universidad del País Vasco-Euskal Herriko Unibertsitatea
Facultad de Ciencias.
Departamento de Matemáticas
Apartado 644. 48080 Bilbao (SPAIN)
E-mail: mtpmastm@lg.ehu.es

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