Записки научных
семинаров ПОМИ
Том 266, 2000 г.

Alexander Fel'shtyn

NIELSEN ZETA FUNCTION, 3–MANIFOLDS, AND ASYMPTOTIC EXPANSIONS IN NIELSEN THEORY

0. Introduction

Before moving on the results of the paper, we briefly describe the few basic notions of Nielsen fixed point theory which will be used. We assume X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Let $p : \tilde{X} \to X$ be the universal cover of X and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lifting of f, i.e., $p \circ \tilde{f} = f \circ p$. Two liftings $\tilde{f}$ and $\tilde{f}'$ are called *conjugate* if there is a $\gamma \in \Gamma \cong \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. The subset $p(\mathrm{Fix}(\tilde{f})) \subset \mathrm{Fix}(f)$ is called *the fixed point class of f determined by the lifting class* $[\tilde{f}]$. Two fixed points x_0 and x_1 of f belong to the same fixed point class iff there is a path c from x_0 to x_1 such that $c \cong f \circ c$ (homotopy relative endpoints). This fact can be considered as an equivalent definition of a non-empty fixed point class. Every map f has only finitely many non-empty fixed point classes, each a compact subset of X.

A fixed point class is called *essential* if its index is nonzero. The number of lifting classes of f (and hence the number of fixed point classes, empty or not) is called the *Reidemeister number* of f, denoted $R(f)$. This is a positive integer or infinity. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$.

The Nielsen number is always finite. $R(f)$ and $N(f)$ are homotopy invariants. In the category of compact, connected polyhedra the Nielsen number of a map is, apart from in certain exceptional cases, equal to the least number of fixed points of maps with the same homotopy type as f.

Taking a dynamical point on view, we consider the iterates of f, and we may define several zeta functions connected with Nielsen fixed point theory (see [4, 5, 7]). The Nielsen zeta function of f is defined as power

---

Part of this work was conducted during the author's stay in the Max-Planck-Institut für Mathematik in Bonn.

series:

$$N_f(z) := \exp\left(\sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n\right).$$

The Nielsen zeta function $N_f(z)$ is homotopy invariant. The function $N_f(z)$ has a positive radius of convergence which has a sharp estimate in terms of the topological entropy of the map f [7].

We begin the article by proving in Sect. 1 that the Nielsen zeta function is a rational function or a radical of a rational function for orientation preserving homeomorphisms of special Haken or special Seifert 3-manifolds.

In Sect. 2, we obtain an asymptotic expansion for the number of twisted conjugacy classes or for the number of Nielsen fixed point classes whose norm is at most x in the case of pseudo-Anosov homeomorphism of surface.

The author would like to thank M. Gromov, Ch. Epstein, R. Hill, L. Potyagailo, R. Sharp, V. G. Turaev for stimulating discussions. In particular, Turaev proposed the author in 1987 the conjecture that the Nielsen zeta function is a rational function or a radical of a rational function for homeomorphisms of Haken or Seifert manifolds.

1. Nielsen zeta function and homeomorphisms of 3-manifolds

1.1. Periodic maps and homeomorphisms of hyperbolic manifolds.

We prove in Corollary 1 of this subsection that the Nielsen zeta function is a radical of rational function for any homeomorphism of a compact hyperbolic 3-manifold. Lemma 1 and Corollary 1 play also important role in the proof of the main theorem of this section.

We denote $N(f^n)$ by N_n. We shall say that $f: X \to X$ is a periodic map of period m, if f^m is the identity map $\mathrm{id}_X : X \to X$. Let $\mu(d)$, $d \in \mathbb{N}$, be the Möbius function of number theory. As is known, it is given by the following equations: $\mu(d) = 0$ if d is divisible by a square different from one; $\mu(d) = (-1)^k$ if d is not divisible by a square different from one, where k denotes the number of prime divisors of d; $\mu(1) = 1$.

We give the proof of the following key lemma for the completeness.

Lemma 1 [7]. *Let f be a periodic map of the least period m of the connected compact polyhedron X. Then the Nielsen zeta function is equal to*

$$N_f(z) = \prod_{d|m} \sqrt[d]{(1 - z^d)^{-P(d)}},$$

where the product is taken over all divisors d of the period m, and $P(d)$ is the integer

$$P(d) = \sum_{d_1|d} \mu(d_1) N_{d|d_1}.$$

Proof. Since $f^m = \mathrm{id}$, for each j, $N_j = N_{m+j}$. If $(k,m) = 1$, then there exist positive integers t and q such that $kt = mq + 1$. So $(f^k)^t = f^{kt} = f^{mq+1} = f^{mq}f = (f^m)^q f = f$. Consequently, $N((f^k)^t) = N(f)$. Let two fixed point x_0 and x_1 belong to the same fixed point class. Then there exists a path α from x_0 to x_1 such that $\alpha * (f \circ \alpha)^{-1} \simeq 0$. We have $f(\alpha * (f \circ \alpha)^{-1}) = (f \circ \alpha) * (f^2 \circ \alpha)^{-1} \simeq 0$ and a product $\alpha * (f \circ \alpha)^{-1} * (f \circ \alpha) * (f^2 \circ \alpha)^{-1} = \alpha * (f^2 \circ \alpha)^{-1} \simeq 0$. It follows that $\alpha * (f^k \circ \alpha)^{-1} \simeq 0$ is derived by the iteration of this process. So x_0 and x_1 belong to the same fixed point class of f^k. If two points belong to different fixed point classes f, then they belong to different fixed point classes of f^k. So, each essential class (class with nonzero index) for f is an essential class for f^k; in addition, different essential classes for f are different essential classes for f^k. So $N(f^k) \geq N(f)$. Analogously, $N(f) = N((f^k)^t) \geq N(f^k)$. Consequently, $N(f) = N(f^k)$. One can prove completely analogously that $N_d = N_{di}$, if $(i, m/d) = 1$, where d is a divisor of m. Using these series of equal Nielsen numbers, one can regroup the terms of the series in the exponential of the Nielsen zeta function so as to get logarithmic functions by adding and subtracting missing terms with necessary coefficients. We show how to do this first for period $m = p^l$, where p is a prime number. We have the following series of equal Nielsen numbers:

$$\begin{aligned}
&N_1 = N_k, \quad (k, p^l) = 1, \ (\text{ i.e, no } N_{ip}, N_{ip^2}, \ldots, N_{ip^l},\ i = 1, 2, 3, \ldots), \\
&N_p = N_{2p} = N_{3p} = \cdots = N_{(p-1)p} = N_{(p+1)p} = \cdots = \\
&\quad = \ldots \ (\text{ no } \ N_{ip^2}, N_{ip^3}, \ldots, N_{ip^l})
\end{aligned}$$

etc.; finally,

$$N_{p^{l-1}} = N_{2p^{l-1}} = \ldots \ (\text{ no } \quad N_{ip^l}),$$

and separately the number N_{p^l}.

Further,

$$\sum_{i=1}^{\infty}\frac{N_i}{i}z^i=\sum_{i=1}^{\infty}\frac{N_1}{i}z^i+\sum_{i=1}^{\infty}\frac{(N_p-N_1)}{p}\frac{z^{pi}}{i}+$$
$$+\sum_{i=1}^{\infty}\frac{(N_{p^2}-(N_p-N_1)-N_1)}{p^2}\frac{z^{p^2 i}}{i}+\cdots+$$
$$+\sum_{i=1}^{\infty}\frac{(N_{p^l}-\cdots-(N_p-N_1)-N_1)}{p^l}\frac{z^{p^l i}}{i}=$$
$$=-N_1\cdot\log(1-z)+\frac{N_1-N_p}{p}\cdot\log(1-z^p)+$$
$$+\frac{N_p-N_{p^2}}{p^2}\cdot\log(1-z^{p^2})+\cdots+$$
$$+\frac{N_{p^{l-1}}-N_{p^l}}{p^l}\cdot\log(1-z^{p^l}).$$

For an arbitrary period m, we get completely analogously,

$$N_f(z)=\exp\left(\sum_{i=1}^{\infty}\frac{N(f^i)}{i}z^i\right)=$$
$$=\exp\left(\sum_{d|m}\sum_{i=1}^{\infty}\frac{P(d)}{d}\cdot\frac{z^{di}}{i}\right)=$$
$$=\exp\left(\sum_{d|m}\frac{P(d)}{d}\cdot\log(1-z^d)\right)=$$
$$=\prod_{d|m}\sqrt[d]{(1-z^d)^{-P(d)}},$$

where the integers $P(d)$ are calculated recursively by the formula

$$P(d)=N_d-\sum_{d_1|d;\,d_1\neq d}P(d_1).$$

Moreover, if the last formula is rewritten in the form

$$N_d=\sum_{d_1|d}\mu(d_1)\cdot P(d_1)$$

and one uses the Möbius Inversion law for real function in number theory, then

$$P(d) = \sum_{d_1 | d} \mu(d_1) \cdot N_{d/d_1},$$

where $\mu(d_1)$ is the Möbius function in number theory. The lemma is proved. $\square$

Corollary 1. *Let $f : M^n \to M^n$, $n \geq 3$ be a homeomorphism of a compact hyperbolic manifold. Then by Mostow rigidity theorem f is homotopic to a periodic homeomorphism g. So Lemma 1 applies and the Nielsen zeta function $N_f(z)$ is equal to*

$$N_f(z) = N_g(z) = \prod_{d|m} \sqrt[d]{(1 - z^d)^{-P(d)}},$$

where the product is taken over all divisors d of the least period m of g, and $P(d)$ is the integer $P(d) = \sum_{d_1|d} \mu(d_1) N(g^{d|d_1})$.

The proof of the following lemma is based on Thurston's theory of homeomorphisms of surfaces [18].

Lemma 2 [7]**.** *The Nielsen zeta function of a homeomorphism f of a compact surface F is either a rational function or the radical of a rational function.*

Proof. The case of a surface with $\chi(F) > 0$ and the case of torus were considered in [7]. If surface has $\chi(F) = 0$ and F is not a torus, then any homeomorphism is isotopic to periodic one (see [11]) and the Nielsen zeta function is a radical of a rational function by Lemma 1. In the case of a hyperbolic ($\chi(F) < 0$) surface, according to Thurston's classification theorem, the homeomorphism f is isotopic either to a periodic or a pseudo-Anosov, or a reducible homeomorphism. In the first case the assertion of the lemma follows from Lemma 1. If f is a pseudo-Anosov homeomorphism of a compact surface, then for each $n > 0$, $N(f^n) = F(f^n)$ [20]. Consequently, in this case the Nielsen zeta function coincides with the Artin–Mazur zeta function: $N_f(z) = F_f(z)$. Since in [3] Markov partitions are constructed for a pseudo-Anosov homeomorphism, Manning's proof [13] of the rationality of the Artin–Mazur zeta function for diffeomorphisms satisfying Smale's axiom A carries over to the case of pseudo-Anosov homeomorphisms. Thus, the Nielsen zeta function $N_f(z)$ is also rational. Now if f is isotopic to a reduced homeomorphism ϕ, then there exists a reducing system S of disjoint circles $S_1, S_2, \ldots, S_m$ on $\operatorname{int} F$ such that

1) each circle S_i does not bound a disk in F;
2) S_i is not isotopic to S_j, $i \neq j$;
3) the system of circles S is invariant with respect to ϕ;
4) the system S has an open ϕ-invariant tubular neighborhood $\eta(S)$ such that each ϕ-component Γ_j of the set $F - \eta(S)$ is mapped into itself by some iterate ϕ^{n_j}, $n_j > 0$ of the map ϕ; here ϕ^{n_j} on Γ_j is either a pseudo-Anosov or a periodic homeomorphism;
5) each band $\eta(S_i)$ is mapped into itself by some iterate ϕ^{m_i}, $m_i > 0$; here ϕ^{m_i} on $\eta(S_i)$ is a generalized twist (possibly trivial).

Since the band $\eta(S_i)$ is homotopically equivalent to the circle S^1 the Nielsen zeta function $N_{\phi_i^m}(z)$ is rational (see [7]). The zeta functions $N_\phi(z)$ and $N_{\phi_i^m}(z)$ are connected on the ϕ-component Γ_j by the formula $N_\phi(z) = \sqrt[n_j]{N_{\phi_j^n}(z^{n_j})}$; analogously, on the band $\eta(S_i)$, $N_\phi(z) = \sqrt[m_j]{N_{\phi_j^m}(z^{m_j})}$. The fixed points of ϕ^n, belonging to different components Γ_j and bands $\eta(S_i)$ are nonequivalent [9], so the Nielsen number $N(\phi^n)$ is equal to the sum of the Nielsen numbers $N(\phi^n/\Gamma_j)$ and $N(\phi^n/\eta(S_i))$ of ϕ-components and bands. Consequently, by the properties of the exponential, the Nielsen zeta function $N_\phi(z) = N_f(z)$ is equal to the product of the Nielsen zeta functions of the ϕ-components Γ_j and the bands $\eta(S_i)$, i.e., is the radical of a rational function. □

1.2. Homeomorphisms of Seifert fibre spaces. Let M be a compact 3-dimensional Seifert fibre space. That is a space which is foliated by simple closed curves, called fibres, such that a fibre L has a neighborhood which is either a solid Klein bottle or a fibred solid torus T_r, where r denotes the number of times a fibre near L wraps around L. A fibre is regular if it has neighborhoods fibre equivalent to solid torus $S^1 \times D^2$, otherwise it is called a critical fibre. There is a natural quotient map $p : M \to F$, where F is a 2-dimensional orbifold, and hence, topologically a compact surface. The projection of the critical fibres, which we denote by S, consists of a finite set of points in the interior of F together with a finite subcollection of the boundary components. The reader is referred to [17] for details, for definitions about 2-dimensional orbifolds, their Euler characteristics and other properties of Seifert fibre spaces. An orbifold is hyperbolic if it has negative Euler characteristic. Hyperbolic orbifold admits hyperbolic structure with totally geodesic boundary.

Any fibre-preserving homeomorphism $f : M \to M$ naturally induces a relative surface homeomorphism of the pair (F, S), which we will denote by $\hat{f}$. Recall from [17] that there is a unique orientable Seifert fibre space with orbifold $P(2,2)$ — projective plane with cone singular points of order 2, 2. We denote this manifold by $M_{P(2,2)}$.

Lemma 3 [17, 11]**.** *Suppose M is a compact orientable Seifert fiber space which is not T^3, $S^1 \times D^2$, $T^2 \times I$ or $M_{P(2,2)}$. Then there is a Seifert fibration $p : M \to F$, so that any orientation preserving homeomorphism on M is isotopic to a fiber preserving homeomorphism with respect to this fibration.*

Observe that if M and F are both orientable, then M admits a coherent orientation of all of its fibres and the homeomorphism f either preserves fibre orientation of all the fibres or it reverses fibre orientation.

Lemma 4 [12]**.** *Let M be a compact, orientable, aspherical, 3-dimensional Seifert fibre space such that the quotient orbifold F is orientable and all fibres have neighborhoods of type T_1. Let $f : M \to M$ be a fibre-preserving homeomorphism inducing $\hat{f} : F \to F$. If f preserves fibre orientation, then the Nielsen number $N(f) = 0$. If f reverses fibre orientation, then $N(f) = 2N(\hat{f})$.*

Proof. First, by small isotopy, arrange that f has a finite number of fibres which are mapped to themselves. If f preserves fibre orientation, a further isotopy, which leaves $\hat{f}$ unchanged, ensures that none of these fibres contains a fixed point. Thus, $\mathrm{Fix}(f) = \emptyset$ and so, $N(f) = 0$. If f reverses fibre orientation, there are exactly 2 fixed points on each invariant fibre. Since M is aspherical Lemma 3.2 in [17] ensures that for any invariant fibre the 2 fixed points on that fibre are in distinct fixed point classes. On the other hand, the restriction on the fibre types in the hypothesis implies that a Nielsen path in F can always be lifted to a Nielsen path in M. In fact, there will be two distinct lifts of each path in F. As a result, f has two fixed point classes covering each fixed point class of $\hat{f}$. Since the index of a fixed point class of f is the same as that of its projection under p the result follows. $\square$

Remark 1. As one can see the Lemma deals with a restricted class of Seifert fibre spaces. M. Kelly gave some examples in [12] which indicate how critical fibres of Seifert fibre space effect Nielsen classes in the case when $r > 1$ in T_r and which difficulties arise in this case.

Definition 1. *A special Seifert fibre space is a Seifert fiber space such that the quotient orbifold F is orientable and all fibres have neighborhoods of type T_1.*

Theorem 1. *Suppose M is a closed orientable aspherical manifold which is a special Seifert fibre space and $f : M \to M$ is an orientation preserving homeomorphism. Then the Nielsen zeta function $N_f(z)$ is a rational function or a radical of a rational function.*

Proof. If f preserves fibre orientation, then f^n also preserves fibre orientation and so, by Lemma 4, the Nielsen numbers $N(f^n) = 0$ for all n and the Nielsen zeta function $N_f(z) = 1$. If f reverses fibre orientation, then f^2 preserves fibre orientation, f^3 reverses fibre orientation and so on. Thus, we have $N(f^{2k+1}) = 2N(\hat{f}^{2k+1})$ and $N(f^{2k}) = 0$ for $k = 0, 1, 2, \ldots$. As result, the Nielsen zeta function $N_f(z)$ equals

$$\begin{aligned}
N_f(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n\right) = \\
&= \exp\left(\sum_{k=0}^{\infty} \frac{N(f^{2k+1})}{2k+1} z^{2k+1}\right) = \\
&= \exp\left(\sum_{k=0}^{\infty} \frac{2N(\hat{f}^{2k+1})}{2k+1} z^{2k+1}\right) = \\
&= \exp\left(2 \cdot \sum_{n=1}^{\infty} \frac{N(\hat{f}^n)}{n} z^n - 2 \cdot \sum_{k=1}^{\infty} \frac{N(\hat{f}^{2k})}{2k} z^{2k}\right) = \\
&= (N_{\hat{f}}(z))^2 \cdot \exp\left(-\sum_{k=1}^{\infty} \frac{N((\hat{f}^2)^k)}{k} (z^2)^k\right) = \\
&= (N_{\hat{f}}(z))^2 / N_{\hat{f}^2}(z^2).
\end{aligned}$$

From Lemma 2 it follows that the Nielsen zeta functions $N_{\hat{f}}(z)$ and $N_{\hat{f}^2}(z^2)$ are either rational functions or the radicals of rational functions. Consequently, the Nielsen zeta function $N_f(z)$ is a rational function or a radical of a rational function. □

1.3. Main theorem. Basic concepts about 3-manifold topology can be found in [10], in particular, the Jaco–Shalen–Johannson decomposition of Haken manifolds and Seifert fiber spaces. See [18] for discussions on hyperbolic 3-manifolds. We recall some basic facts about compact connected orientable 3-manifolds. A 3-manifold M is irreducible if every embedded 2-sphere bounds an embedded 3-disk. By the sphere theorem [8], an irreducible 3-manifolds is a $K(\pi, 1)$ Eilenberg–MacLane space if and only if it is a 3-disk or has infinite fundamental group.

A properly-embedded orientable connected surface in a 3-manifold is incompressible if it is not a 2-sphere and the inclusion induces a injection on the fundamental groups. An irreducible 3-manifold is Haken if it contains an embedded orientable incompressible surface. We use notation $N(X)$ to denote a regular neighborhood of set X. A Haken 3-manifold M can be decomposed along a canonical set $\mathcal{T}$ of incompressible tori into pieces such that each component of $M - N(\mathcal{T})$ is either a hyperbolic manifold, or a twisted I-bundle over Klein bottle, or a Seifert fiber space with hyperbolic orbifold. This decomposition is called Jaco–Shalen–Johannson decomposition.

Definition 2. *A special Haken manifold is a Haken manifold M such that each component of $M - N(\mathcal{T})$ in JSJ decomposition of M is either a hyperbolic manifold, or a twisted I-bundle over Klein bottle, or an aspherical special Seifert fiber space with hyperbolic orbifold.*

We need some definitions from the paper [11].

Definition 3. *Suppose $f : M \to M$ is a map, and A, B are f-invariant sets of M. If there is a path γ from A to B such that $\gamma \sim f \circ \gamma \ \mathrm{rel}(A, B)$, then we say that A, B are f-related.*

Definition 4. *Suppose M is a compact 3-manifold with torus boundaries. A map $f : M \to M$ is standard on boundary if for any component T of ∂M, the map f/T is one of the following types:*

(1) *a fixed point free map;*
(2) *a periodic map with isolated fixed points;*
(3) *a fiber preserving, fiber orientation reversing map with respect to some S^1 fibration of T.*

Definition 5. *A map f on a compact 3-manifold M is said to have FR-property (fixed-point relating property) if the following is true: if $A \in \mathrm{Fix}(f)$ and B is either a fixed point of f or an f-invariant component of ∂M, and A, B are f-related by a path γ, then γ is A, B homotopic to a path in $\mathrm{Fix}(f)$.*

Definition 6. *A map f is a type I standard map if*

(1) *f has FR-property,*

(2) f *is standard on boundary,*
(3) $\mathrm{Fix}(f)$ *consists of isolated points, and*
(4) f *is of flipped pseudo-Anosov type at each fixed point.*

A map f is type II standard map if it satisfies (1) *and* (2) *above, as well as*

(3) $\mathrm{Fix}(f)$ *is a properly embedded* 1*-dimensional submanifold, and*
(4) f *preserves a normal structure on* $\mathrm{Fix}(f)$*.*

The main result of this section is the following theorem.

Theorem 2. *Suppose M is a closed orientable manifold which is a special Haken manifold, and $f : M \to M$ is an orientation preserving homeomorphism. Then the Nielsen zeta function $N_f(z)$ is a rational function or a radical of a rational function.*

Proof. Let $\mathcal{T}$ be the (possibly empty) set of invariant tori of the JSJ decomposition of M. Then each component of $M - \mathrm{int}\, N(\mathcal{T})$ is either a hyperbolic manifold, or a twisted I-bundle over Klein bottle, or a aspherical special Seifert fiber space with hyperbolic orbifold. Isotop f so that it maps $N(\mathcal{T})$ homeomorphically to itself. Suppose that P is a Seifert fibered component of $M - \mathrm{int}\, N(\mathcal{T})$ such that $f(P) = P$. By [17], f/P is isotopic to a fiber preserving map. Recall that a torus T in M is a vertical torus if it is union of fibers in M. By [11, Lemma 1.10], we can find a set of vertical tori $\mathcal{T}^*$ in P, cutting P into pieces which are either a twisted I-bundle over Klein bottle, or have hyperbolic orbifold, and a fiber preserving isotopy of f, so that after isotopy, the restriction of f on each invariant piece has periodic or pseudo-Anosov orbifold map. Adding all such $\mathcal{T}^*$ to $\mathcal{T}$, we get a collection of tori $\mathcal{T}'$, such that:

(1) $f(N(\mathcal{T}') = N(\mathcal{T}')$;
(2) each component M_i of $M - \mathrm{int}\, N(\mathcal{T}')$ either is a twisted I-bundle over Klein bottle or has hyperbolic orbifold;
(3) if f maps a Seifert fibered component M_i to itself, and if M_i has hyperbolic orbifold, then f is fiber preserving, and the orbifold map $\hat{f}$ on F_i is either periodic or pseudo-Anosov.

By [11] we can isotop f so that restriction f/M_i is a standard map for all M_i and after that we can further isotop f, rel ∂N_j, on each component N_j of $N(\mathcal{T}')$, so that it is a standard map on N_j. By the definition of standard maps, $\mathrm{Fix}(f)$ intersects each of M_i and N_j in points and 1-manifolds, so $\mathrm{Fix}(f)$ is a disjoint union of points, arcs, and circles. Jiang, Wang and Wu proved [11, Theorem 9.1], that different components of

$\mathrm{Fix}(f)$ are not equivalent in Nielsen sense. We repeat their proof here for the completeness. If exist two different components C_0, C_1 of $\mathrm{Fix}(f)$ which are equivalent, then there is a path α connecting C_0, C_1 such that $f\circ\alpha \sim \alpha$ rel ∂. Denote by T'' the set of tori $\partial N(\mathcal{T}')$. Among all such α, choose one such that the number of components $\sharp(\alpha - \mathcal{T}'')$ of the set $\alpha - \mathcal{T}''$ is minimal. In below we will find another such curve α'' with $\sharp(\alpha'' - \mathcal{T}'') < \sharp(\alpha - \mathcal{T}'')$, which would contradict the choice of α. Let D be a disk, and let $h : D \to M$ be a homotopy $f\circ\alpha \sim \alpha$ rel ∂. We may assume that h is transverse to $\mathcal{T}''$, and $\sharp h^{-1}(\mathcal{T}'')$ is minimal among all such h. Then $h^{-1}(\mathcal{T}'')$ consists of a properly embedded 1-manifold on D, together with possibly one or two isolated points mapped to the ends of α. $\mathcal{T}''$ is π_1-injective in M, so one can modify h to remove all circles in $h^{-1}(\mathcal{T}'')$. Note that $h^{-1}(\mathcal{T}'')$ must contain some arcs, otherwise α would lie in some M_i or $N(T_j)$, which is impossible because the restriction of f in each piece has FR-property. Now consider an outermost arc b in $h^{-1}(\mathcal{T}'')$. Let $\beta = h(b)$. The ends of β can not both be on α, otherwise we can use the outermost disk to homotope α and reduce $\sharp(\alpha - \mathcal{T}'')$, contradicting the choice of α. Since f is a homeomorphism, the same thing is true for $f\circ\alpha$. Therefore, β has one end on each of α and $f\cdot\alpha$. The arc b cuts off a disk Δ on D whose interior is disjoint from $h^{-1}(\mathcal{T}'')$. The boundary of Δ gives rise to a loop $h(\partial\Delta) = \alpha_1 \cup \beta \cup (f\circ\alpha_1)^{-1}$, where α_1 is a subpath of α starting from an end point x of α. Let T be the torus in $\mathcal{T}''$ which contains β. Then the restriction of h on Δ gives a homotopy $\alpha_1 \sim f\cdot\alpha_1$ rel(x, T). Since f has FR-property on each component of $M - \mathrm{int}\, N(\mathcal{T})$ and $N(\mathcal{T})$, by definition there is a path γ in $\mathrm{Fix}(f)$ such that $\gamma \sim \alpha_1$ rel(x,T). Since γ is in $\mathrm{Fix}(f)$, the path $\alpha' = \gamma^{-1}\cdot\alpha$ has the property that $f\circ\alpha' = (f\circ\gamma^{-1})\cdot(f\circ\alpha) = \gamma^{-1}\cdot(f\circ\alpha) \sim \gamma^{-1}\cdot\alpha = \alpha'$ rel ∂. Since $\alpha_1 \sim \gamma$ rel(x,T), the path $\gamma^{-1}\cdot\alpha_1$ is rel ∂ homotopic to a path δ on T. Write $\alpha = \alpha_1\cdot\alpha_2$. Then $\alpha' = \gamma^{-1}\cdot\alpha = (\gamma^{-1}\cdot\alpha_1)\cdot\alpha_2 \sim \delta\cdot\alpha_2$ rel ∂. By a small perturbation on δ, we get a path $\alpha'' \sim \alpha'$ rel ∂ such that $\sharp(\alpha'' - \mathcal{T}'') < \sharp(\alpha - \mathcal{T}'')$. Since $f\circ\alpha'' \sim f\circ\alpha' \sim \alpha' \sim \alpha''$ rel ∂, this contradicts the minimality of $\sharp(\alpha - \mathcal{T}'')$.

We can prove in the same way that the fixed points of each iteration f^n belonging to different components M_i and N_j are nonequivalent, so the Nielsen number $N(f^n)$ equals to the sum of the Nielsen numbers $N(f^n/M_i)$ and $N(f^n/N_j)$ of components. Consequently, by the properties of the exponential, the Nielsen zeta function $N_f(z)$ is equal to the product of the Nielsen zeta functions for the induced homeomorphisms of the components M_i and N_j. By Corollary 1 the Nielsen zeta function of the hyperbolic component M_i is a radical of a rational function. The Nielsen zeta function is a radical of a rational function for the component M_i which is an aspherical special Seifert fibre space by Theorem 1. The Nielsen zeta function is a rational function or a radical of a rational function for a homeomorphism of torus or Klein bottle by Lemma 2. This implies that the Nielsen zeta function is a rational function or a radical of a rational function for the induced homeomorphism of component N_j which is an I-bundle over torus and for the induced homeomorphism of component M_i which is a twisted I-bundle over Klein bottle. So, the Nielsen zeta function $N_f(z)$ of the homeomorphism f of the whole manifold M is a rational function or a radical of a rational function as a product of the Nielsen zeta functions for the induced homeomorphisms of the components M_i and N_j. $\square$

2. Asymptotic expansions for fixed point classes and twisted conjugacy classes of pseudo-Anosov homeomorphism

2.1. Twisted conjugacy and Reidemeister Numbers. Let Γ be a group and $\phi : \Gamma \to \Gamma$ an endomorphism. Two elements $\alpha, \alpha' \in \Gamma$ are said to be *ϕ-conjugate* iff there exists $\gamma \in \Gamma$ with

$$\alpha' = \gamma\alpha\phi(\gamma)^{-1}.$$

We shall write $\{x\}_\phi$ for the ϕ-conjugacy class of the element $x \in \Gamma$. The number of ϕ-conjugacy classes is called the *Reidemeister number* of ϕ, denoted by $R(\phi)$. If ϕ is the identity map then the ϕ-conjugacy classes are the usual conjugacy classes in the group Γ.

In [6] we have conjectured that the Reidemeister number is infinite as long as the endomorphism is injective and the group has exponential growth. Below we prove this conjecture for surface groups and pseudo-Anosov maps and, in fact, we obtain an asymptotic expansion for the number of twisted conjugacy classes whose norm is at most x.

Let $f : X \to X$ be given, and let a specific lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ be chosen as reference. Let Γ be the group of covering translations of $\tilde{X}$ over X. Then every lifting of f can be written uniquely as $\gamma \circ \tilde{f}$, with $\gamma \in \Gamma$. So elements of Γ serve as coordinates of liftings with respect to the reference $\tilde{f}$. Now for every $\gamma \in \Gamma$ the composition $\tilde{f} \circ \gamma$ is a lifting of f so there is a unique $\gamma' \in \Gamma$ such that $\gamma' \circ \tilde{f} = \tilde{f} \circ \gamma$. This correspondence $\gamma \to \gamma'$ is determined by the reference $\tilde{f}$, and is obviously a homomorphism.

Definition 7. *The endomorphism $\tilde{f}_* : \Gamma \to \Gamma$ determined by the lifting*

$\tilde f$ of f is defined by

$$\tilde f_*(\gamma) \circ \tilde f = \tilde f \circ \gamma.$$

It is well known that $\Gamma \cong \pi_1(X)$. We shall identify $\pi = \pi_1(X, x_0)$ and Γ in the following way. Pick base points $x_0 \in X$ and $\tilde x_0 \in p^{-1}(x_0) \subset \tilde X$ once and for all. Now points of $\tilde X$ are in 1-1 correspondence with homotopy classes of paths in X which start at x_0: for $\tilde x \in \tilde X$ take any path in $\tilde X$ from $\tilde x_0$ to $\tilde x$ and project it onto X; conversely, for a path c starting at x_0, lift it to a path in $\tilde X$ which starts at $\tilde x_0$, and then take its endpoint. In this way, we identify a point of $\tilde X$ with a path class $\langle c\rangle$ in X starting from x_0. Under this identification, $\tilde x_0 = \langle e\rangle$ is the unit element in $\pi_1(X, x_0)$. The action of the loop class $\alpha = \langle a\rangle \in \pi_1(X, x_0)$ on $\tilde X$ is then given by

$$\alpha = \langle a\rangle : \langle c\rangle \to \alpha \cdot c = \langle a \cdot c\rangle.$$

Now we have the following relationship between $\tilde f_* : \pi \to \pi$ and

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, f(x_0)).$$

Lemma 5. *Suppose $\tilde f(\tilde x_0) = \langle w\rangle$. Then the following diagram commutes:*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(X, f(x_0)) \\ & \tilde f_* \searrow & \downarrow w_* \\ & & \pi_1(X, x_0), \end{array}$$

where w_* is the isomorphism induced by the path w.

In other words, for every $\alpha = \langle a\rangle \in \pi_1(X, x_0)$, we have

$$\tilde f_*(\langle a\rangle) = \langle w(f \circ a)w^{-1}\rangle.$$

Remark 2. In particular, if $x_0 \in p(\mathrm{Fix}(\tilde f))$ and $\tilde x_0 \in \mathrm{Fix}(\tilde f)$, then $\tilde f_* = f_*$.

Lemma 6. *Lifting classes of f (and hence fixed point classes, empty or not) are in $1-1$ correspondence with $\tilde f_*$-conjugacy classes in π, the lifting class $[\gamma \circ \tilde f]$ corresponding to the $\tilde f_*$-conjugacy class of γ. We therefore have $R(f) = R(\tilde f_*)$.*

We shall say that the fixed point class $p(\mathrm{Fix}(\gamma \circ \tilde f))$, which is labeled with the lifting class $[\gamma \circ \tilde f]$, *corresponds* to the $\tilde f_*$-conjugacy class of γ. Thus $\tilde f_*$-conjugacy classes in π serve as coordinates for fixed point classes of f, once a reference lifting $\tilde f$ is chosen.

2.2. Asymptotic expansions. We assume X to be a compact surface of negative Euler characteristic and $f : X \to X$ to be a pseudo-Anosov homeomorphism, i.e., there is a number $\lambda > 1$ and a pair of transverse measured foliations (F^s, μ^s) and (F^u, μ^u) such that $f(F^s, \mu^s) = (F^s, \frac{1}{\lambda}\mu^s)$ and $f(F^u, \mu^u) = (F^u, \lambda\mu^u)$. The mapping torus T_f of $f : X \to X$ is the space obtained from $X \times [0,1]$ by identifying $(x, 1)$ with $(f(x), 0)$ for all $x \in X$. It is often more convenient to regard T_f as the space obtained from $X \times [0, \infty)$ by identifying $(x, s+1)$ with $(f(x), s)$ for all $x \in X$, $s \in [0, \infty)$. On T_f there is a natural semi-flow $\phi : T_f \times [0, \infty) \to T_f$, $\phi_t(x, s) = (x, s+t)$ for all $t \geq 0$. Then the map $f : X \to X$ is the return map of the semi-flow ϕ. A point $x \in X$ and a positive number $\tau > 0$ determine the orbit curve $\phi_{(x,\tau)} := \phi_t(x)_{0 \leq t \leq \tau}$ in T_f. The fixed points and periodic points of f then correspond to closed orbits of various periods. Take the base point x_0 of X as the base point of T_f. According to van Kampen Theorem the fundamental group $G := \pi_1(T_f, x_0)$ is obtained from π by adding a new generator z and adding the relations $z^{-1}gz = \tilde{f}_*(g)$ for all $g \in \pi = \pi_1(X, x_0)$, where z is the generator of $\pi_1(S^1, x_o)$ This means that G is a semi-direct product $G = \pi \bowtie \mathbb{Z}$ of π with $\mathbb{Z}$.

We now describe some known results.

Lemma 7. *If Γ is a group and ϕ is an endomorphism of Γ then an element $x \in \Gamma$ is always ϕ-conjugate to its image $\phi(x)$.*

Proof. If $\gamma = x^{-1}$ then one has immediately $\gamma x = \phi(x)\phi(\gamma)$. The existence of a γ satisfying this equation implies that x and $\phi(x)$ are ϕ-conjugate. ☐

Lemma 8. *Two elements x, y of π are $\tilde{f}_*$-conjugate iff xz and yz are conjugate in the usual sense in G. Therefore $R(f) = R(\tilde{f}_*)$ is the number of usual conjugacy classes in the coset $\pi \cdot z$ of π in G.*

Proof. If x and y are $\tilde{f}_*$-conjugate then there is a $\gamma \in \pi$ such that $\gamma x = y\tilde{f}_*(\gamma)$. This implies $\gamma x = yz\gamma z^{-1}$ and therefore $\gamma(xz) = (yz)\gamma$ so xz and yz are conjugate in the usual sense in G. Conversely suppose xz and yz are conjugate in G. Then there is a $\gamma z^n \in G$ with $\gamma z^n xz = yz\gamma z^n$. From the relation $zxz^{-1} = \tilde{f}_*(x)$ we obtain $\gamma \tilde{f}_*^n(x) z^{n+1} = y\tilde{f}_*(\gamma)z^{n+1}$ and therefore $\gamma \tilde{f}_*^n(x) = y\tilde{f}_*(\gamma)$. This shows that $\tilde{f}_*^n(x)$ and y are $\tilde{f}_*$-conjugate. However by Lemma 7 x and $\tilde{f}_*^n(x)$ are $\tilde{f}_*$-conjugate, so x and y must be $\tilde{f}_*$-conjugate. ☐

There is a canonical projection $\tau : T_f \to \mathbb{R}/\mathbb{Z}$ given by $(x, s) \mapsto s$. This induces a map $\pi_1(\tau) : G = \pi_1(T_f, x_0) \to \mathbb{Z}$.

We see that the Reidemeister number $R(f)$ is equal to the number of homotopy classes of closed paths γ in T_f whose projections onto $\mathbb{R}/\mathbb{Z}$ are homotopic to the path

$$\begin{array}{rccc} \sigma : & [0,1] & \to & \mathbb{R}/\mathbb{Z} \\ & s & \mapsto & s. \end{array}$$

Corresponding to this there is a group theoretical interpretation of $R(f)$ as the number of usual conjugacy classes of elements $\gamma \in \pi_1(T_f)$ satisfying $\pi_1(\tau)(\gamma) = z$.

Lemma 9 [19, 14]**.** *The interior of the mapping torus* $\mathrm{int}(T_f)$ *admits a hyperbolic structure of finite volume if and only if f is isotopic to a pseudo-Anosov homeomorphism.*

So, if the surface X is closed and f is isotopic to a pseudo-Anosov homeomorphism, the mapping torus T_f can be realized as a hyperbolic 3-manifold, $\mathbb{H}^3/G$, where $\mathbb{H}^3$ is the Poincare upper half space $\{(x, y, z) : z > 0,\ (x, y) \in \mathbb{R}^2\}$ with the metric $ds^2 = (dx^2 + dy^2 + dz^2)/z^2$. The closed geodesics on a hyperbolic manifold are in one-to-one correspondence with the free homotopy classes of loops. These classes of loops are in one-to-one correspondence with the conjugacy classes of loxodromic elements in the fundamental group of the hyperbolic manifold. This correspondences allow Ch. Epstein (see [2], p. 127) to study the asymptotics of such functions as $p_n(x) = \#\{$primitive closed geodesics of length less than x represented by an element of the form $gz^n\}$ using the Selberg trace formula. A primitive closed geodesic is one which is not an iterate of another closed geodesic. Later, Phillips and Sarnak [15] generalized results of Epstein and obtained for n-dimensional hyperbolic manifold the asymptotics of the number of primitive closed geodesics of length at most x lying in fixed homology class. The proof of this result makes routine use of the Selberg trace formula. In the more general case of variable negative curvature such asymptotics was obtained by Pollicott and Sharp [16]. They used dynamical approach based on the geodesic flow. We will only need an asymptotics for $p_1(x)$. Note, that closed geodesics represented by an element of the form gz are automatically primitive, because they wrap exactly once around the mapping torus (once around generator z).

We have the following asymptotic expansion [2, 15, 16]

$$p_1(x) = \frac{e^{hx}}{x^{b/2+1}} \left(\sum_{n=0}^{N} \frac{C_n}{x^{n/2}} + O(\frac{1}{x^{N/2}}) \right), \tag{1}$$

for any $N > 0$, where $h = \dim T_f - 1 = 2$ is the topological entropy of the geodesic flow on the unit-tangent bundle ST_f, $b = \dim H_1(T_f; Z) > 0$ is the first Betti number of the mapping torus T_f, the constant $C_0 > 0$ depends on the volume of hyperbolic 3-manifold T_f. N. Anantharaman [an] has shown that constants C_n vanish if n is odd. So, we have the following asymptotics

$$p_1(x) \sim C_0 \frac{e^{hx}}{x^{b/2+1}}, \quad x \to \infty. \tag{2}$$

Notation. We write $f(x) = g(x) + O(h(x))$ if there exists $C > 0$ such that $|f(x) - g(x)| \leq Ch(x)$. We write $f(x) \sim g(x)$ if $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$.

Now, using one-to-one correspondences in Lemmas 6 and 8, we define a norm of fixed point class, or corresponding to him lifting class, or corresponding to them twisted conjugacy class $\{g\}_{\tilde{f}_*}$ in the fundamental group of surface $\pi = \pi_1(X, x_0)$ as the length of the primitive closed geodesic γ on T_f, which is represented by an element of the form gz. So, for example, the norm function l^* on the set of twisted conjugacy classes equals $l^* = l \circ B$, where l is length function on geodesics ($l(\gamma)$ is the length of the primitive closed geodesic γ) and B is bijection between the set of twisted conjugacy classes $\{g\}_{\tilde{f}_*}$ in the fundamental group of surface $\pi = \pi_1(X, x_0)$ and the set of closed geodesics represented by an elements of the form gz in the fundamental group $G := \pi_1(T_f, x_0)$. We introduce the following counting functions:

$FPC(x) = \#\{$fixed point classes of f of norm less than $x\}$,

$L(x) = \#\{$lifting classes of f of norm less than $x\}$,

$Tw(x) = \#\{$twisted conjugacy classes for $\tilde{f}_*$ in the fundamental group of surface of norm less than $x\}$.

Theorem 3. *Let X be a closed surface of negative Euler characteristic and $f : X \to X$ be a pseudo-Anosov homeomorphism. Then*

$$FPC(x) = L(x) = Tw(x) = \frac{e^{2x}}{x^{b/2+1}} \left(\sum_{n=0}^{N} \frac{C_n}{x^{n/2}} + O(\frac{1}{x^{N/2}}) \right),$$

where $b = \dim H_1(T_f; Z) > 0$ is the first Betti number of the mapping torus T_f, the constant $C_0 > 0$ depends on the volume of hyperbolic 3-manifold T_f, constants C_n vanish if n is odd.

Proof. The proof follows from Lemmas 8 and 9 and asymptotic expansion (1). $\square$

Corollary 2. *For a pseudo-Anosov homeomorphism of surface the Reidemeister number is infinite.*

We can generalize Theorem 3 in the following way.

Theorem 4. *Let M be a compact manifold and $f : M \to M$ be a homeomorphism. Suppose that the mapping torus T_f admits a Riemannian metric of negative sectional curvatures and the first Betti number $b = \dim H_1(T_f; Z) > 0$. Then there exist constants C_0, C_1, $C_2, \ldots$ with $C_0 > 0$ such that*

$$FPC(x) = L(x) = Tw(x) = \frac{e^{hx}}{x^{b/2+1}} \left(\sum_{n=0}^{N} \frac{C_n}{x^{n/2}} + O(\frac{1}{x^{N/2}}) \right)$$

for any $N > 0$, where $h > 0$ is the topological entropy of the geodesic flow on the unit-tangent bundle ST_f and constants C_n vanish if n is odd.

Question 1. For which compact manifolds M and homeomorphisms $f : M \to M$ the mapping torus T_f admits a Riemannian metric of negative curvature or negative sectional curvatures?

Question 2. How to define the norm of the fixed point class or twisted conjugacy class in general case? Does exist a twisted Selberg trace formula for the discrete group? Such formula can give asymptotic expansion of counting function for twisted conjugacy classes.

References

1. N. Anantharaman, *Precise counting results for closed orbits of Anosov flows.* preprint.
2. C. Epstein, *The spectral theory of geometrically periodic hyperbolic 3-manifolds.* — Memoirs of the AMS **58** no. 335 (1985).
3. A. Fathi, M. Shub, *Some dynamics of pseudo-Anosov diffeomorphisms.* — Asterisque **66-67** (1979), 181–207.
4. A. L. Fel'shtyn, *New zeta functions for dynamical systems and Nielsen fixed point theory.* — Lecture Notes in Math. **1346** (1988), 33–55.
5. A. L. Fel'shtyn, *Dynamical zeta functions, Nielsen theory and Reidemeister torsion.* — Memoirs of the AMS. (to appear).
6. A. L. Fel'shtyn, R. Hill, *Trace formulae, zeta functions, congruences and Reidemeister torsion in Nielsen theory.* — Forum Mathematicum **10** no. 6 (1998), 641–663.
7. A. L. Fel'shtyn, V. B. Pilyugina, *The Nielsen zeta function.* — Funct. Anal. Appl. **19** no. 4 (1985), 61–67.

8. J. Hempel, *3-manifolds,* Annals of Math. Studies, vol. 86. Princeton, 1976.
9. N. V. Ivanov, *Nielsen numbers of maps of surfaces.* — Journal Sov. Math. **26** (1984), 1636–1641.
10. W. Jaco, *Lectures on three-manifold topology,* Regional Conference Series in Mathematics, AMS, vol. 43. 1977.
11. B. Jiang, S. Wang, Y. Wu, *Homeomorphisms of 3-manifolds and realisation of Nielsen number.* — Preprint **18** (1997), Institute of Mathematics, Peking University.
12. M. Kelly, *Nielsen numbers and homeomorphisms of geometric 3-manifolds.* — Topology Proceedings **19** (1994).
13. A. Manning, *Axiom A diffeomorphisms have rational zeta function.* — Bull. London Math. Soc. **3** (1971), 215–220.
14. J.-P. Otal, *Le theoreme d'hyperbolisation pour les varietes fibrees de dimension 3.* — Asterisque **235** (1996).
15. R. Phillips, P. Sarnak, *Geodesics in homology classes.* — Duke Math. Journal **55** no. 2 (1987), 287–297.
16. M. Pollicott, R. Sharp, *Asymptotic expansions for closed orbits in homology classes.* preprint ESI 594, Vienna.
17. P. Scott, *The geometry of 3-manifolds.* — Bull. London Math. Soc. **15** no. 56, 401–487.
18. W. Thurston, *The geometry and topology of 3-manifolds.* Princeton University.
19. W. Thurston, *Hyperbolic structures on 3-manifolds, II: surface groups and 3-manifolds which fibers over the circle.* preprint.
20. W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces.* — Bull. AMS **19** (1988), 417–431.

Institut für Mathematik,
Ernst-Moritz-Arndt-Universität Greifswald
Jahn-strasse 15a, D-17489 Greifswald, Germany
Email felshtyn@mail.uni-greifswald.de

10 октября 1999 .