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A CLASSIFICATION APPROACH FOR OPEN MANIFOLDS

Abstract. On the set of all smooth Riemannian manifolds, a number of different uniform structures are studied, some of which are defined for the more general class of proper metric spaces, while the others take into account the smooth and Riemannian structures. Appropriate bordism theories are presented. Several homology theories that do not distinguish manifolds in one component of a given uniform structure, but can distinguish manifolds in different components are discussed. The proofs are omitted.

§1. Introduction

For closed manifolds there exists an effective highly elaborated classification approach the main steps of which are the Thom–Pontrjagin construction, bordism theory, surgery, Wall groups, the exact sequence of Browder–Novikov–Sullivan–Wall. All this can be expressed in an algebraic language of e.g. $\mathbb{Z}\pi$-, unitary $\mathbb{Z}\pi$-, $\mathbb{Q}\pi$-modules and their K-theory, etc. Moreover, there exist many number valued invariants like classical characteristic numbers, signature, higher signature, analytic and Reidemeister torsion, η-invariants.

For open manifolds, absolutely nothing of this remains, at least at the first glance. We have the following simple

Proposition 1.1. *Let $\mathcal{M}^n$ be the set of all smooth oriented manifolds and V a vector space or an Abelian group. There does not exist a nontrivial map $c\colon \mathcal{M} \to V$ such that*

1. *$M \cong M'$ (orientation preserving diffeomorphic) implies $c(M) = c(M')$ and*
2. *$c(M\#M') = c(M) + c(M')$.*

Proof. Assume at first $M^n \not\cong \Sigma^n$, fix two points at M^n, then $M_\infty = M_1\#M_2\#\ldots$, $M_i \cong M$ has a well-defined meaning. We can write $M_\infty = M_1\#M_{\infty,2}$, $M_{\infty,2} = M_2\#M_3\#\ldots$ and get $c(M_\infty) = c(M) + c(M_{\infty,2}) = c(M) + c(M_\infty)$, $c(M) = 0$.

Assume $M^n = \Sigma^n$ and $\operatorname{ord}\Sigma^n = k > 1$, which yields

$$c(\Sigma^n \# \cdots \# \Sigma^n) = k \cdot c(\Sigma^n) = c(S^n), \quad c(\Sigma^n) = \tfrac{1}{k} c(S^n),$$

$$c(\Sigma^n) = c(\Sigma^n \# S^n) = (1 + \tfrac{1}{k}) c(S^n), \quad c(S^n) = 2c(S^n), \quad c(S^n) = 0,$$

$$c(\Sigma^n) = 0. \quad \bullet$$

The only number valued invariant defined for all connected manifolds M^n and known to the author is the dimension n. If one characterizes orientability/nonorientability by ± 1, then there are two number-valued invariants. That is all.

Denote by $\mathcal{M}^n([cl])$ the set of all diffeomorphism classes of closed n-manifolds. Then we have

Proposition 1.2. $\#\mathcal{M}^n([cl]) =$ *alef zero.*

Proof. According to Cheeger, there are only finitely many diffeomorphism types for (M^n, g) with $\operatorname{diam}(M^n, g) \leqslant D$, $r_{\text{inj}}(M^n, g) \leqslant i$, $|\text{sectional curvature}(M^n, g)| \leqslant K$, where $r_{\text{inj}}(M^n, g)$ denotes the injectivity radius. Setting $D_\nu = K_\nu = i_\nu = \nu$ and considering $\nu \to \infty$, we count all diffeomorphism types of closed Riemannian n-manifolds, in particular all diffeomorphism types of closed manifolds. $\bullet$

On the other hand for open manifolds holds

Proposition 1.3. *The cardinality of $\mathcal{M}([open])$ is at least that of the continuum, $n \geqslant 2$.*

Proof. Assume $n \geqslant 3$, n odd, let $2 = p_1 < p_2 < \dots$ be an increasing sequence of prime numbers and let $L^n(p_\nu) = S^n/\mathbb{Z}_{p_\nu}$ be the corresponding lens space. Consider $M^n := d_1 \cdot L(p_1) \# d_2 \cdot L(p_2) \# \dots$, $d_\nu = 0, 1$. Then any $0,1$-sequence $(d_1, d_2, \dots)$ defines a manifold and different sequences define nondiffeomorphic manifolds. If $n \geqslant 4$ is even, multiply with S^1. For $n = 2$ the assertion follows from the classification theorem in [20]. $\bullet$

There are simple methods to construct from only one closed manifold $M^n \neq \Sigma^n$ infinitely many nondiffeomorphic manifolds. This, Proposition 1.3 and other considerations support the naive imagination that "measure of $\mathcal{M}([open])$: measure $\mathcal{M}([cl]) = \infty : 0$." We understand this as an additional hint how difficult would be any classification of open manifolds.

A certain requirement of what should be the goal of such a classification comes from global analysis. The main task of global analysis is the solution of linear and nonlinear differential equations in dependence of the underlying geometry and topology. Examples for open manifolds are discussed and solved in [8–10]. To each open manifold there are attached certain functional spaces and admitted maps which enter into the classification should induce maps between the functional spaces, i.e., $f\colon M \to M'$ induces a map $(f.\ sp.)(M') \to (f.\ sp.)(M)$. A very simple example shows that this is a reasonable requirement. Consider the diffeomorphism $f = \left(\tan\left(\frac{\pi}{2}\cdot\right)\right)^{-1} :]0,\infty[\to]0,1[$. Then $1 \in L_2(]0,1[)$ but $1 = f^*1 \notin L_2(]0,\infty[)$. Our first conclusion is that one should classify pairs (M,g) and maps should be adapted to the Riemannian metrics under consideration. A pure differential topological classification will be to difficult, not handable and as we shortly indicated, will be less important for applications. Manifolds which appear in applications are endowed with a Riemannian metric. Accepting this, the corresponding algebraic topology should be analytic and simplicial L_p-(co)homology, bounded (co)homology and others. Consider

$$(\mathbb{R}^n, g_{\text{standard}}) = (\mathbb{R}^n, dr^2 + r^2 d\sigma^2_{S^{n-1}})$$

and a canonical uniform triangulation $K_{\mathbb{R}^n}$ of $\mathbb{R}^n$. Then

$$\overline{H}^{*,2}(\mathbb{R}^n) = H^{*,2}(\mathbb{R}^n) = H^{*,2}(K_{\mathbb{R}^n}) = \overline{H}^{*,2}(K_{\mathbb{R}^n}) = 0.$$

But if we endow $\mathbb{R}^n$ with the hyperbolic metric $g_H = dr^2 + (\sinh r)^2 d\sigma^2_{S^{n-1}}$, then for $n = 2k$

$$\dim \overline{H}^{k,2}(\mathbb{R}^n, g_H) = \dim H^{k,2}(\mathbb{R}^n, g_H) = \infty$$

and for $n = 2k+1$

$$\dim \overline{H}^{k,2}(\mathbb{R}^n, g_H) = 0, \quad \dim H^{k,2}(\mathbb{R}^n, g_H) = \dim H^{k+1,2}(\mathbb{R}^n, g_H) = \infty.$$

Nevertheless, $(\mathbb{R}^n, g_{\text{standard}})$ and $(\mathbb{R}^n, g_H)$ are canonical isomorphic by the best possible map $\mathrm{id}_{\mathbb{R}^n}$. We come to our preliminary conclusions,

1. admitted maps $(M,g) \to (M',g')$ must be strongly adapted to g, g',
2. the classification approach will from a certain step on be connected with spectral properties.

Our approach can be characterized as follows. We first decompose the set of all (M,g)'s into the set of components of a certain uniform structure and then classify (up to a certain amount) the manifolds in the component under consideration, i.e., we have to define two classes of invariants, one for the components and one for the manifolds inside a component. Moreover, we consider several uniform structures which become finer and finer. This implies that the (arc) components become smaller and smaller. Since the whole approach is sufficiently extensive, we can present here only the main steps and sketch some proofs.

The paper is organized as follows. In Sec. 2, we define those uniform structures which consider (M^n, g) only as proper metric spaces. The uniform structures in Sec. 4 take into account the smooth and Riemannian structure of (M^n, g). Section 5 is devoted to bordism theories, adapted in a certain sense to uniform structures under consideration. In the concluding Sec. 6 we define and discuss classes of invariants in the two senses sketched above. Details and complete proofs are contained in [8–10]. All three papers will be presented for publication.

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§2. Uniform structures of proper metric spaces

Let $Z = (Z, d_Z)$ be a metric spaces, $X, Y \subset Z$ subsets, $\epsilon > 0$, define $U_\epsilon(X) = \{z \in Z \mid \operatorname{dist}(z, X) < \epsilon\}$, analogously $U_\epsilon(Y)$. Then the Hausdorff distance between X and Y is defined as

$$d_H(X, Y) = d_H^Z(X, Y) = \inf\{\epsilon \mid X \subset U_\epsilon(Y), Y \subset U_\epsilon(X)\}.$$

If there is no such $\epsilon > 0$, then we set $d_H^Z(X, Y) = \infty$. d_H^Z is an almost metric on the set of all closed subsets, i.e., it has values in $[0, \infty]$ but satisfies all other conditions of a metric. If Z is compact, then d_H^Z is a metric on the set of all closed subsets. A metric space (X, d) is called *proper* if the closed balls $\overline{B_\epsilon(x)}$ are compact for all $x \in X$, $\epsilon > 0$. This implies that X is separable, complete and locally compact. Any complete Riemannian manifold (M^n, g) is a proper metric space. In the sequel we restrict to proper metric spaces. Let (X, d_X), (Y, d_Y) be metric spaces, $X \sqcup Y$ their disjoint union. A metric on $X \sqcup Y$ is called *admissible* if d restricts to d_X and d_Y, respectively. The Gromov–Hausdorff distance $d_{GH}(X, Y)$ is defined as

$$d_{GH}(X, Y) = \inf\{d_H^{X \sqcup Y}(X, Y) \mid d \text{ admissible on } X \sqcup Y\}.$$

Note that the Gromov–Hausdorff distance can be infinite. Gromov defined d_{GH} originally as

$$\begin{aligned} d_{GH}(X, Y) = \inf\{d_G^Z(i(X), j(Y)) \mid i\colon X \to Z, j\colon Y \to Z \text{ isometric} \\ \text{embeddings into a metric space } Z\}. \end{aligned} \tag{2.1}$$

Lemma 2.1. *If X and Y are compact metric spaces and $d_{GH}(X,Y)=0$, then X and Y are isometric.*

Proof. This follows from the definition and an Arzela–Ascoli argument. •

Denote by $\mathcal{M}$ the set of all isometry classes $[X]$ of proper metric spaces X and $\mathcal{M}_{GH}=\mathcal{M}/\sim$, where $[X]\sim[Y]$ if $d_{GH}([X],[Y])=0$.

Proposition 2.2. *d_{GH} defines an almost metric on $\mathcal{M}_{GH}$.* •

In the sequel, we denote $X=[X]$ if this leads to no confusion.

Now we define the uniform structure. Let $\delta>0$ and set

$$V_\delta=\{(X,Y)\in\mathcal{M}^2_{GH}\mid d_{GH}(X,Y)<\epsilon\}.$$

Lemma 2.3. *$\mathcal{L}=\{V_\delta\}_{\delta>0}$ is a basis for a metrizable uniform structure $\mathcal{U}_{GH}(\mathcal{M}_{GH})$.*

Proof. $\mathcal{L}$ is locally defined by a metric. Hence it satisfies all desired conditions. •

Let $\overline{\mathcal{M}}_{GH}$ be the completion of $\mathcal{M}_{GH}$ with respect to $\mathcal{U}_{GH}$ and denote the metric in $\overline{\mathcal{M}}_{GH}$ by $\overline{d}_{GH}$.

Lemma 2.4. *$\mathcal{M}_{GH}=\overline{\mathcal{M}}_{GH}$ as sets and d_{GH} and $\overline{d}_{GH}$ are locally equivalent.*

Proposition 2.5 [8]. *$\mathcal{M}_{GH}=\overline{\mathcal{M}}_{GH}$ is locally arcwise connected.* •

The proof is quite elementary but rather lengthy.

Corollary 2.6. *In $\mathcal{M}_{GH}$ components and arc components coincide. Moreover, each component is open and $\overline{\mathcal{M}}_{GH}=\mathcal{M}_{GH}$ is the topological sum of its components,*

$$\mathcal{M}_{GH}=\sum_{i\in I}\operatorname{comp}_{GH}(X_i). \quad \bullet$$

Proposition 2.7. *Let $X\in\mathcal{M}_{GH}$. Then $\operatorname{comp}_{GH}(X)$ is given by*

$$\operatorname{comp}_{GH}(X)=\{Y\in\mathcal{M}_{GH}\mid d_{GH}(X,Y)<\infty\}. \quad \bullet$$

We call a map $\Phi\colon X\to Y$ *metrically semilinear* if it satisfies the following two conditions:

1. It is uniformly metrically proper, i.e., for each $R > 0$ there is $S > 0$ such that the inverse image under Φ of a set of diameter $\leqslant R$ is a set of diameter $\leqslant S$.

2. There exists a constant $C_\Phi \geqslant 0$ such that for all $x_1, x_2 \in X$ $d(\Phi(x_1), \Phi(x_2)) \leqslant d(x_1, x_2) + C_\Phi$.

Two metric spaces X and Y are called *metrically semilinear equivalent* if there exist metrical semilinear maps $\Phi\colon X \to Y$, $\Psi\colon X \to Y$ and constants D_X, D_Y such that for all $x \in X$, $y \in Y$

$$d(x, \Psi\Phi x) \leqslant D_X, \quad d(\Psi\Phi y, y) \leqslant D_Y. \tag{2.2}$$

Proposition 2.8 [8]**.** *$Y \in \mathrm{comp}_{GH}(X)$, i.e., $d_{GH}(X, Y) < \infty$ if and only if X and Y are metrically semilinear equivalent.* $\bullet$

At this general level there are still some important further classes of maps in the category of proper metric spaces.

We call a map $\Phi\colon X \to Y$ *coarse* if it is

1. metrically proper, i.e., for each bounded set $B \subseteq Y$ the inverse image $\Phi^{-1}(B)$ is bounded in X, and

2. uniformly expansive, i.e., for $R > 0$ there is $S > 0$ such that $d(x_1, x_2) \leqslant R$ implies $d(\Phi x_1, \Phi x_2) \leqslant S$.

A coarse map is called *rough* if it is additionally uniformly metrically proper. X and Y are called *coarsely* or *roughly equivalent* if there exist coarse or rough maps $\Phi\colon X \to Y$, $\Psi\colon Y \to X$, respectively, satisfying (2.2). A metrically proper map $\Phi\colon X \to Y$ is called *Lipschitz* if there holds $d(\Phi(x_1), \Phi(x_2)) \leqslant C_\Phi \cdot d(x_1, x_2)$. Lipschitz maps are continuous. X and Y are called *coarsely Lipschitz equivalent* if there are Lipschitz maps $\Phi\colon X \to Y$, $\Psi\colon Y \to X$ satisfying (2.2). If additionally $\Psi\Phi$ and $\Phi\Psi$ are homotopic to id_X, id_Y by means of a Lipschitz homotopy, respectively, then X and Y are called *coarsely Lipschitz homotopy equivalent.*

The following is immediately clear from the definitions.

Proposition 2.9. *Metrically semilinear maps are rough and rough maps are coarse, whence the inclusions*

$$\{\textit{metrically semilinear equivalence class of } X \ (\equiv \mathrm{comp}_{GH}(X))\}$$

$$\subseteq \{\textit{rough equivalence class of } X\} \subseteq \{\textit{coarse equivalence class of } X\}$$

and

$$\{\textit{coarse Lipschitz homotopy equivalence class of } X\}$$

$$\subseteq \{\textit{coarse Lipschitz equivalence class of } X\}$$

$$\subseteq \{\textit{coarse equivalence class of } X\}. \quad \bullet$$

For later applications to open manifolds (M^n, g) we will sharpen the Lipschitz notions by requiring not only (2.2) but additionally controling the Lipschitz constants.

From now on we define a Lipschitz map as defined above additionally to be uniformly metrically proper.

Define for a Lipschitz map $\Phi \colon X \to Y$

$$\operatorname{dil} \Phi := \sup_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{d(\Phi x_1, \Phi x_2)}{d(x_1, x_2)}.$$

Set

$$\begin{aligned} d_L(X, Y) := \inf\{ &\max\{0, \log \operatorname{dil} \Phi\} + \max\{0, \log \operatorname{dil} \Psi\} + \sup_{x \in X} d(\Psi\Phi x, x) \\ &+ \sup_{y \in Y} d(\Phi\Psi y, y) \;\mid\; \Phi \colon X \to Y, \Psi \colon Y \to X \text{ Lipschitz maps}\}, \end{aligned}$$

if $\{\dots\} \neq \emptyset$ and $\inf\{\dots\} < \infty$ and set $d_L(X, Y) = \infty$ otherwise. Then $d_L \geqslant 0$, symmetric and $d_L(X, Y) = 0$ if X and Y are isometric. Set $\mathcal{M}_L = \mathcal{M}/\sim$, where $X \sim Y$ if $d_L(X, Y) = 0$.

Let $\delta > 0$ and define

$$V_\delta = \{(X, Y) \in \mathcal{M}_L^2 \mid d_L(X, Y) < \infty\}.$$

The proofs of the following assertions are already more technical and lengthy. Hence we must refer to [8].

Proposition 2.10. *$\mathcal{L} = \{V_\delta\}_{\delta > 0}$ is a basis for a metrizable uniform structure $\mathcal{U}_L(\mathcal{M}_L)$* $\bullet$

Denote by $\mathcal{M}_L(nc)$ the (class of) noncompact proper metric spaces.

Proposition 2.11. *$\mathcal{M}_L(nc)$ is complete with respect to $\mathcal{U}_L(\mathcal{M}_L)$.* $\bullet$

Proposition 2.12. *$\mathcal{M}_L$ and $\overline{\mathcal{M}}_L(nc) = \mathcal{M}_L(nc) \subset \mathcal{M}_L$ are locally arcwise connected. Hence components coincide with arc components, components are open and $\mathcal{M}_L$ and we have the following topological sum representations*

$$\begin{aligned} \mathcal{M}_L &= \operatorname{comp}_L(\textit{point}) + \sum_{i \in I} \operatorname{comp}_L X_i, \\ \mathcal{M}_L(nc) &= \sum_{i \in I} \operatorname{comp}_L X_i, \quad \textit{and} \\ \operatorname{comp}_L(X) &= \{Y \in \mathcal{M}_L \mid d_L(X, Y) < \infty\}. \end{aligned}$$

In particular all compact spaces lie in the component of the 1-point space. •

A sharpening of this uniform structure is given if we restrict the maps to Lipschitz homeomorphisms.

Define

$$d_{L,\mathrm{top}}(X,Y) = \inf\{\max\{0, \log \operatorname{dil}\Phi\} + \max\{0, \log \operatorname{dil}\Phi^{-1}\} \\ \mid \Phi\colon X \to Y \text{ bi-Lipschitz homeomorphism}\}$$

if there exists such a Φ and define $d_{L,\mathrm{top}}(X,Y) = \infty$ if they are not bi-Lipschitz homeomorph. Then $d_{L,\mathrm{top}}$ is $\geqslant 0$ and symmetric, and $d_{L,\mathrm{top}}(X,Y) = 0$ if X and Y are isometric. Set $\mathcal{M}_{L,\mathrm{top}} = \mathcal{M}/\sim$, $X \sim Y$ if $d_{L,\mathrm{top}}(X,Y) = 0$, and set

$$V_\delta = \left\{(X,Y) \in \mathcal{M}^2_{L,\mathrm{top}} \;\mid\; d_{L,\mathrm{top}}(X,Y) < \delta\right\}.$$

Proposition 2.13. *$\mathcal{L} = \{V_\delta\}_{\delta>0}$ is a basis for a metrizable uniform structure $\mathcal{U}_{L,\mathrm{top}}(\mathcal{M}_{L,\mathrm{top}})$.* •

Proposition 2.14.

a) $\overline{\mathcal{M}}^{\mathcal{U}_{L,\mathrm{top}}}_{L,\mathrm{top}} = \mathcal{M}_{L,\mathrm{top}}$.

b) *$\mathcal{M}_{L,\mathrm{top}}$ is locally arcwise connected. Hence components coincide with arc components and components are open.*

c) *$\mathcal{M}_{L,\mathrm{top}}$ has a decomposition as a topological sum,*

$$\mathcal{M}_{L,\mathrm{top}} = \sum_{i\in I} \operatorname{comp}(X_i).$$

d) $\operatorname{comp}(X) = \operatorname{comp}_{L,\mathrm{top}}(X) = \{Y \in \mathcal{M}_{L,\mathrm{top}}(X,Y) < \infty\}$. •

Remark. We see that 2.14 a) is valid in $\mathcal{U}_{L,\mathrm{top}}$ without restriction to noncompact spaces as we due in the $\mathcal{U}_L$-case. •

Finally, we define three further uniform structures which measure or express the homotopy neighborhoods and, secondly, admit only compact deviations of the spaces inside one component.

Define

$$d_{L,h}(X,Y) :=$$

$$\inf\Big\{\max\{0, \log \operatorname{dil}\Phi\} + \max\{0, \log \operatorname{dil}\Psi\} + \sup_X d(\Psi\Phi x, x) + \sup_Y d(\Phi\Psi y, y)$$

$$\mid X \xrightarrow{\Phi} Y \text{ and } Y \xrightarrow{\Psi} X \text{ are mutually inverse (uniformly proper)}$$

Lipschitz homotopy equivalences$\}$

if there are such homotopy equivalences and set $d_{L,h}(X,Y)=\infty$ otherwise. Here and in the sequel we require from the homotopies to id_X or id_Y, respectively, that they are uniformly proper and Lipschitz.

$d_{L,h}\geqslant 0$, $d_{L,h}$ is symmetric and $d_{L,h}(X,Y)=0$ if X and Y are isometric. Define $\mathcal{M}_{L,h}=\mathcal{M}/\sim$, $X\sim Y$ if $d_{L,h}(X,Y)=0$ and set

$$V_\delta=\{(X,Y)\in\mathcal{M}^2_{L,h}\mid d_{L,h}(X,Y)<\delta\}.$$

Proposition 2.15. $\mathcal{L}=\{V_\delta\}_{\delta>0}$ *is a basis for a metrizable uniform structure* $\mathcal{U}_{L,h}\mathcal{M}_{L,h}$. •

Proposition 2.16. a) $\overline{\mathcal{M}}^{\mathcal{U}_{L,h}}_{L,h}(nc)=\mathcal{M}_{L,h}(nc)$.

b) $\mathcal{M}_{L,h}$ *is locally arcwise connected. Hence components coincide with arc components and components are open.*

c) $\mathcal{M}_{L,h}$ *has a representation as a topological sum,*

$$\mathcal{M}_{L,h}=\sum_{i\in I}\mathrm{comp}(X_i).$$

d) $\mathrm{comp}(X)\equiv\mathrm{comp}_{L,h}(X)=\{Y\in\mathcal{M}_{L,h}\mid d_{L,h}(X,Y)<\infty\}$. •

Let $\Phi\colon X\to Y$, $\Psi\colon Y\to X$ be Lipschitz maps. We say Φ and Ψ are stable Lipschitz homotopy equivalences at ∞ inverse to each other it there exists a compact set $K^0_X\subset X$ such that for any $K^0_X\subset K_X$ there exists $K_Y\subset Y$ such that $\Phi_{X\setminus K_X}: X\setminus K_X\to Y\setminus K_Y$ is a Lipschitz h. e. with homotopy inverse $\Psi|_{Y\setminus K_Y}$ and Ψ has an analogous property. Set

$$d_{L,h,\mathrm{rel}}(X,Y,\mathrm{rel}):=$$

$$\inf\Big\{\max\{0,\log\mathrm{dil}\,\Phi\}+\max\{0,\log\mathrm{dil}\,\Psi\}+\sup_X d(\Psi\Phi x,x)+\sup_Y d(\Phi\Psi y,y)$$

$$\mid X\xrightarrow{\Phi}Y \text{ and } Y\xrightarrow{\Psi}X \text{ are mutually inverse stable Lipschitz}$$

$$\text{homotopy equivalences at }\infty\Big\}$$

if there exist such a Φ, Ψ and set $d_{L,h,\mathrm{rel}}(X,Y)=\infty$ otherwise. Then $d_{L,h,\mathrm{rel}}\geqslant 0$, symmetric and $d_{L,h,\mathrm{rel}}=0$ if X and Y are isometric. Set $\mathcal{M}_{L,h,\mathrm{rel}}=\mathcal{M}/\sim$, $X\sim Y$ if $d_{L,h,\mathrm{rel}}(X,Y)=0$ and set

$$V_\delta=\{(X,Y)\in\mathcal{M}^2_{L,h,\mathrm{rel}}\mid d_{L,h,\mathrm{rel}}(X,Y)<\delta\}.$$

Proposition 2.17. *$\mathcal{L} = \{V_\delta\}_{\delta>0}$ is a basis for a metrizable uniform structure $\mathcal{U}_{L,h,\mathrm{rel}}$.* •

Proposition 2.18.

a) $\overline{\mathcal{M}}_{L,h,\mathrm{rel}}^{\mathcal{U}_{L,h,\mathrm{rel}}}(nc) = \mathcal{M}_{L,h,\mathrm{rel}}(nc)$.

b) *$\mathcal{M}_{L,h,\mathrm{rel}}$ is locally arcwise connected. In particular components coincide with arc components, components are open and $\mathcal{M}_{L,h,\mathrm{rel}}$ has a representation as topological sum,*

$$\mathcal{M}_{L,h,\mathrm{rel}} = \sum_{i\in I} \mathrm{comp}(X_i),$$

where

$$\mathrm{comp}(X) \equiv \mathrm{comp}_{L,h,\mathrm{rel}}(X) = \{Y \in \mathcal{M}_{L,h,\mathrm{rel}} \mid d_{L,h,\mathrm{rel}}(X,Y) < \infty\}. \quad \bullet$$

The last uniform structure $\mathcal{U}_{L,\mathrm{top},\mathrm{rel}}$ is defined by $d_{L,\mathrm{top},\mathrm{rel}}(X,Y)$, where we require that $\Phi\colon X \longrightarrow Y$, $\Psi\colon Y \longrightarrow X$ are outside compact sets bi-Lipschitz homeomorphisms, inverse to each other. We obtain $\mathcal{M}_{L,\mathrm{top},\mathrm{rel}}$. There holds $\overline{\mathcal{M}}_{L,\mathrm{top},\mathrm{rel}} = \mathcal{M}_{M,\mathrm{top},\mathrm{rel}}$. The other assumptions of 2.16 hold correspondingly.

We finish the section with a scheme which makes clear the achievements.

One coarse equivalence class

$\swarrow$ splits into $\searrow$

many GH-components many L-components,

one L-component

$\swarrow$ splits into $\searrow$

L, top, rel-components L, h, rel-components

$\swarrow$ $\searrow$

L, top-components L, h-components.

It is now a natural observation that the classification of noncompact proper metric spaces splits into two main tasks:

1. "counting" the components at any horizontal level,

2. "counting" the elements inside each component.

A really complete solution of this two problems, i.e., a complete characterization by computable and handable invariants, is now a day hopeless. It is a similar platonic goal as the "classification of all topological spaces." Nevertheless stands the task to define series of invariants which at least permit to decide (in good cases) nonequivalence. This will be the topic of Sec. 5.

Finally we remark that GH-components ($d_{GH}(X,Y) < \infty$) and L-components ($d_L(x,Y)$) are very different. Roughly spoken, d_{GH} is in the small unsharp and in the large relatively sharp, d_L quite inverse. We refer for many geometric examples to [8].

§3. Some materials from nonlinear global analysis on open manifolds

Let (M^n, g) be a Riemannian manifold. We consider the conditions (I) and (B_k),

(I) $\quad r_{\mathrm{inj}}(M^n, g) = \inf_{x \in M} r_{\mathrm{inj}}(x) > 0,$

(B_k) $\quad |\nabla^i R| \leqslant C_i,\ 0 \leqslant i \leqslant k,$

where r_{inj} denotes the injectivity radius, $\nabla = \nabla^g$ the Levi–Civita connection, $R = R^g$ the curvature and $|\cdot|$ the pointwise norm. (M^n, g) has bounded geometry of order k if it satisfies the conditions (I) and (B_k). Every compact manifold (M^n, g) or homogeneous Riemannian space or Riemannian covering $(\tilde{M}, \tilde{g})$ of a compact manifold (M^n, g) satisfies (I) and (B_∞). More general, given M^n open and $0 \leqslant k \leqslant \infty$, there exists a complete metric of bounded geometry of order k (cf. [19]), i.e., the existence of such a metric does not restrict the underlying topological type. (I) implies completeness. Let $(E, h) \to (M^n, g)$ be a Riemannian vector bundle and $\nabla = \nabla^h$ a metric connection with respect to h. Quite analogously we consider the condition

$(\mathrm{B}_k(E, \nabla))$ $\qquad |\nabla^i R^E| \leqslant C_i, \qquad 0 \leqslant i \leqslant k,$

where R^E denotes the curvature of (E, ∇).

Lemma 3.1 [8]. *If (M^n, g) satisfies (B_k) and $\mathcal{U}$ is an atlas of normal coordinate charts of radius $\leqslant r_0$, then there exist constants C_α, C'_β such that*

$$|D^\alpha g_{ij}| \leqslant C_\alpha, \quad |\alpha| \leqslant k, \tag{3.1}$$

$$|D^\beta \Gamma^m_{ij}| \leqslant C'_\beta, \quad |\beta| \leqslant k-1, \tag{3.2}$$

where C_α, C'_β are independent of the base points of the normal charts and depend only on r_0 and on curvature bounds including bounds for the derivations. $\bullet$

The proof is rather long and technical; it uses iterated inhomogeneous Jacobi equations.

This lemma carries over to the case of Riemannian vector bundles. A normal chart U in M of radius $\leqslant r_0$, an orthonormal frame $e_1, \ldots, e_N$ over the base point $p \in U \subset M$, and its radial parallel translation define a local orthonormal frame field e_α, a so-called *synchronous frame* and connection coefficients $\Gamma^\beta_{\alpha i}$ by

$$\nabla_{\frac{\partial}{\partial x_i}} e_\alpha = \Gamma^\beta_{\alpha i} e_\beta .$$

Lemma 3.2 [11]. *Assume $(B_k(M^n, g))$, $(B_k(E, \nabla))$, $k \geqslant 1$, and $\Gamma^\beta_{\alpha i}$ as above. Then*

$$|D^\gamma \Gamma^\beta_{\alpha i}| \leqslant C_\gamma, \quad |\gamma| \leqslant k-1, \quad \alpha, \beta = 1, \ldots, N, \quad i = 1, \ldots, n, \tag{3.3}$$

where the C_γ are constants depending on curvature bounds, r_0, and are independent of U. $\bullet$

We recall for what follows some simple facts concerning Sobolev spaces on open manifolds. Let $(E, h, \nabla^h) \longrightarrow (M^n, g)$ be a Riemannian vector bundle. Then the Levi–Civita connection ∇^g and ∇^h define metric connections ∇ in all tensor bundles $T^u_v \otimes E$. Denote by $C^\infty(T^u_v \otimes E)$ all smooth sections, $C^\infty_c(T^u_v \otimes E)$ those with compact support. In the sequel we write E instead of $T^u_v \otimes E$, keeping in mind that E can be an arbitrary vector bundle. Now we define for $p \in \mathbb{R}$, $1 \leqslant p < \infty$ and r a nonnegative integer

$$\begin{aligned}
|\varphi|_{p,r} &:= \left(\int \sum_{i=0}^{r} |\nabla^i \varphi|^p_x \, d\operatorname{vol}_x(g) \right)^{1/p}, \\
\Omega^p_r(E) &= \{\varphi \in C^\infty(E) \mid |\varphi|_{p,r} < \infty\}, \\
\overline{\Omega}^{p,r}(E) &= \text{completion of } \Omega^p_r(E) \text{ with respect to} |\cdot|_{p,r}, \\
\mathring{\Omega}^{p,r}(E) &= \text{completion of } C^\infty_c(E) \text{ with respect to } |\cdot|_{p,r} \text{ and} \\
\Omega^{p,r}(E) &= \{\varphi \mid \varphi \text{ measurable distributional section with } |\varphi|_{p,r} < \infty\}.
\end{aligned}$$

Furthermore, we define

$$ {}^{b,m}|\varphi| := \sum_{i=0}^{m} \sup_x |\nabla^i \varphi|_x, $$
$$ {}^{b,m}\Omega(E) = \{\varphi \mid \varphi \ C^m - \text{section and } {}^{b,m}|\varphi| < \infty\} \text{ and} $$
$$ {}^{b,m}\mathring{\Omega}(E) = \text{completion of } C_c^\infty(E) \text{ with respect to } {}^{b,m}|\cdot|. $$

${}^{b,m}\Omega(E)$ equals the completion of

$$ {}^{b}_{m}\Omega(E) = \{\varphi \in C^\infty(E) \mid \ {}^{b,m}|\varphi| < \infty\} $$

with respect to ${}^{b,m}|\cdot|$.

Proposition 3.3. *The spaces* $\mathring{\Omega}^{p,r}(E)$, $\bar{\Omega}^{p,r}(E)$, $\Omega^{p,r}(E)$, ${}^{b,m}\mathring{\Omega}(E)$, ${}^{b,m}\Omega(E)$ *are Banach spaces and there are inclusions*

$$ \mathring{\Omega}^{p,r}(E) \subseteq \bar{\Omega}^{p,r}(E) \subseteq \Omega^{p,r}(E), $$

$$ {}^{b,m}\mathring{\Omega}(E) \subseteq \ {}^{b,m}\Omega(E). $$

If $p = 2$, *then* $\mathring{\Omega}^{2,r}(E)$, $\bar{\Omega}^{2,r}(E)$, $\Omega^{2,r}(E)$ *are Hilbert spaces.* •

$\mathring{\Omega}^{p,r}(E)$, $\bar{\Omega}^{p,r}(E)$, $\Omega^{p,r}(E)$ are different in general.

Proposition 3.4 [12]**.** *If* (M^n, g) *satisfies* (I) *and* (B_k), *then*

$$ \mathring{\Omega}^{p,r}(E) = \bar{\Omega}^{p,r}(E) = \Omega^{p,r}(E), \quad 0 \leqslant r \leqslant k+2. \quad \bullet $$

Theorem 3.5. *Let* $(E, h, \nabla^E) \longrightarrow (M^n, g)$ *be a Riemannian vector bundle satisfying* (I), $(B_k(M^n, g))$, $(B_k(E, \nabla))$, $k \geqslant 1$.
a) *Assume* $k \geqslant r$, $r - \frac{n}{p} \geqslant s - \frac{n}{q}$, $r \geqslant s$, $q \geqslant p$. *Then*

$$ \Omega^{p,r}(E) \hookrightarrow \Omega^{q,s}(E) \tag{3.4} $$

continuously.
b) *If* $r - \frac{n}{p} > s$, *then*

$$ \Omega^{p,r}(E) \hookrightarrow \ {}^{b,s}\Omega(E) \tag{3.5} $$

continuously. •

A key role for many calculations and estimates in nonlinear global analysis plays the module structure theorem which asserts under which conditions the tensor product of two Sobolev sections is again a Sobolev section. We do not use this theorem here explicitly and refer to [13].

We consider spaces of metrics in Sec. 4, and now we give a very small review of that. Let M^n be an open smooth manifold, $\mathcal{M} = \mathcal{M}(M)$ the space of all complete Riemannian metrics. Let $g \in \mathcal{M}$. We define

$$^bU(g) = \{g' \in \mathcal{M} \mid {}^b|g - g'|_g := \sup_{x \in M} |g - g'|_{g,x} < \infty,\ {}^b|g - g'|_{g'} < \infty\}.$$

It is easy to see that $^bU(g)$ coincides with the quasi isometry class of g, i.e., $g' \in {}^bU(g)$ if and only if there exist $c, c' > 0$ such that

$$c \cdot g' \leqslant g \leqslant c' \cdot g'.$$

For g, $g' \in \mathcal{M}$, let $\nabla = \nabla^g$ and $\nabla' = \nabla^{g'}$ be the Levi–Civita connections. For $m \geqslant 1$, $\delta > 0$, we set

$$V_\delta = \{(g, g') \in \mathcal{M}^2 \mid {}^b|g - g'|_g < \delta,\ {}^b|g - g'|_{g'} < \delta \quad \text{and}$$
$$^{b,m}|g - g'|_g := {}^b|g - g'|_g + \sum_{j=0}^{m-1} {}^b|\nabla^j(\nabla - \nabla')|_g < \delta\}.$$

Proposition 3.6. *The set $\mathcal{B} = \{V_\delta\}_{\delta > 0}$ is a basis for a metrizable uniform structure on $\mathcal{M}$.* •

Denote by $^{b,m}\mathcal{M}$ the corresponding completed uniform space.

Proposition 3.7. *The space $^{b,m}\mathcal{M}$ is locally contractible.* •

Corollary 3.8. *In $^{b,m}\mathcal{M}$ components and arc components coincide.*

We set

$$^{b,m}U(g) = \{g' \in {}^{b,m}\mathcal{M} \mid {}^b|g - g'|_{g'} < \infty,\ {}^{b,m}|g - g'|_g < \infty\}.$$

Proposition 3.9. *Denote by* $\mathrm{comp}(g)$ *the component of $g \in {}^{b,m}\mathcal{M}$. Then*

$$\mathrm{comp}(g) = {}^{b,m}U(g). \quad \bullet$$

Theorem 3.10. *The space $^{b,m}\mathcal{M}$ has a representation as a topological sum*

$$^{b,m}\mathcal{M} = \sum_{i \in I} {}^{b,m}U(g_i). \quad \bullet$$

Theorem 3.11. *Each component of* ${}^{b,m}\mathcal{M}$ *is a Banach manifold.* •

Denote for given M

$$\mathcal{M}(\mathrm{I},\mathrm{B}_k) = \{g \in \mathcal{M} | g \text{ satisfies (I) and } (\mathrm{B}_k)\}.$$

Remark. (I) always implies completeness. •

Metrics of bounded geometry wear a natural inner Sobolev topology. Let $k \geqslant r < \frac{n}{p} + 1$, $\delta > 0$ and set

$$V_\delta = \Big\{(g, g') \in \mathcal{M}(\mathrm{I},\mathrm{B}_k)^2 \mid {}^b|g - g'|_g < \delta,\ {}^b|g - g'|_{g'} < \delta \quad \text{and}$$
$$|g - g'|_{g,p,r} := \left(\int \left(|g - g'|_{g,x} + \sum_{i=0}^{r-1} |\nabla^i(\nabla - \nabla')|^p_{g,x}\right) d\,\mathrm{vol}_x(g)\right)^{1/p} < \delta\Big\}.$$

Proposition 3.12. *The set* $\{V_\delta\}_{\delta>0}$ *is a basis for a metrizable uniform structure on* $\mathcal{M}(\mathrm{I},\mathrm{B}_k)$. •

Denote by $\mathcal{M}^{p,r}$ the corresponding completed uniform space.

Proposition 3.13. *The space* $\mathcal{M}^{p,r}(\mathrm{I},\mathrm{B}_k)$ *is locally contractible.* •

Corollary 3.14. *In* $\mathcal{M}^{p,r}(\mathrm{I},\mathrm{B}_k)$ *components and arc components coincide.*

Set for $g \in \mathcal{M}^{p,r}(\mathrm{I},\mathrm{B}_k)$

$$U^{p,r}(g) = \{g' \in \mathcal{M}^{p,r}(\mathrm{I},\mathrm{B}_k) :\ {}^b|g - g'|_g,\ {}^b|g - g'|_{g'},\ {}^b|g - g'|_{g,p,r} < \infty\}. \bullet$$

Proposition 3.15. *Denote by* $\mathrm{comp}(g)$ *the component of* $g \in \mathcal{M}^{p,r}(\mathrm{I},\mathrm{B}_k)$. *Then*

$$\mathrm{comp}(g) = U^{p,r}(g). \quad \bullet$$

Theorem 3.16. *Let* M^n *be open,* $k \geqslant r > \frac{n}{p} + 1$. *Then* $\mathcal{M}^{p,r}(I, B_k)$ *has a representation as a topological sum*

$$\mathcal{M}^{p,r}(\mathrm{I},\mathrm{B}_k) = \sum_{i \in I} U^{p,r}(g_i). \quad \bullet$$

We refer to [14] for all proofs.

Finally we give one hint to the manifolds of maps theory. Let (M^n, g), $(N^{n'}, h)$ be open, satisfying (I) and (B_k) and let $f \in C^\infty(M, N)$. Then the differential $df = f_* = Tf$ is a section of $T^*M \otimes f^*TN$. f^*TN is endowed with the induced connection $f^*\nabla^h$. The connections ∇^g and $f^*\nabla^h$ induce connections ∇ in all tensor bundles $T^q_s(M) \otimes f^* T^u_v(N)$. Therefore, $\nabla^m df$ is well defined. Assume $m \leqslant k$. We denote by $C^{\infty,m}(M,N)$ the set of all $f \in C^\infty(M,N)$ satisfying

$$^{b,m}|df| = \sum_{i=0}^{m-1} \sup_{x \in M} |\nabla^i df|_x < \infty.$$

It is now possible to define for $C^{\infty,m}(M,N)$ uniform structures to obtain manifolds of maps $^{b,m}\Omega(M,N)$, $\Omega^{p,r}(M,N)$, manifolds of diffeomorphisms $^{b,m}D(M,N)$, $D^{p,r}(M,N)$ and groups of diffeomorphisms $^{b,m}D(M)$, $D^{p,r}(M)$.

But this approach is extraordinarily complicated and extensive. We refer to [15].

Remark. For closed manifolds all these things are very simple.

§4. Uniform structures of open manifolds

Any complete Riemannian manifold (M^n, g) defines a proper metric space and hence an element of:

$$\mathcal{M},\ \mathcal{M}_{GH},\ \mathcal{M}_L,\ \mathcal{M}_{L,\mathrm{top,rel}},\ \mathcal{M}_{L,\mathrm{top}},\ \mathcal{M}_{L,h,\mathrm{rel}},\ \mathcal{M}_{L,h}.$$

Let $\mathcal{M}^n(mf)$ denote the subset of (classes of) complete Riemannian n-manifolds (M^n, g). The restriction of a uniform structure to a subset yields a uniform structure and we obtain uniform spaces

$$(\mathcal{M}^n_{GH}(mf), \mathcal{U}_{GH}|_{\mathcal{M}^n_{GH}}(mf)),\ (\mathcal{M}^n_L(mf), \mathcal{U}_L|_{\mathcal{M}^n_L}(mf)),$$

etc. These uniform structures do not take into account the smooth and Riemannian structure but only the (distance) metrical structure. Metrically semilinear and Lipschitz maps which enter into the definition of d_{GH}, d_L, etc. are far from being smooth. On the other hand, we defined in [14] and in the last section for one fixed manifold M^n several uniform structures of Riemannian metrics. Roughly spoken, the space $\mathcal{M}(M)$ of complete Riemannian metrics g on M splits into a topological sum $\mathcal{M}(M) = \sum_{i \in I} \mathrm{comp}(g_i)$ depending on the norm in question. Then the fundamental question arises how are $\mathrm{comp}(M, g_i)$ and $\mathrm{comp}_{GH}(M, g_i)$, $\mathrm{comp}_L(M, g_i)$ etc. related? We will give very shortly some partial answers in this section. Moreover, we will generalize the (analytically defined) uniform structures of metrics for one fixed manifold to uniform structures of Riemannian manifolds. Still other questions concern the completions of $\mathcal{M}^n(mf)$ with respect to the considered uniform structures and the completions of certain subspaces of metrics, e. g. Ricci curvature $\leqslant 0$. But these more differential geometric questions are outside the space of this paper and will be studied in [16]. Looking at our general approach, the first main interesting questions are the relations between the components defined until now.

Proposition 4.1 [8]. *Let* $(M^n, g) \in \mathcal{M}^n(mf)$, $g' \in {}^{b,m}\operatorname{comp}(g) \subset {}^{b,m}\mathcal{M}$, $m \geqslant 0$. *Then we have*

a) $(M, g') \in \operatorname{comp}_{L,\text{top}}(M, g)$,

b) $(M, g') \in \operatorname{comp}_L(M, g)$. $\bullet$

Remark. We cannot prove this for d_{GH}. d_{GH} is locally very rough but measures the metric relations in the large relatively exact. But this property does not immediately follow from $g' \in {}^{b,m}\operatorname{comp}(g)$.

Corollary 4.2. *The assertions* a) *and* b) *hold if* $g' \in \operatorname{comp}^{p,r}(g) \subset \mathcal{M}^{p,r}(I, B_k)$, $k \geqslant r > \frac{n}{p} + 1$. $\bullet$

To admit the variation of M in (M^n, g) too, we define still another uniform structure. First we admit arbitrary complete metrics, i.e., we do not restrict to metrics of bounded geometry. Consider complete manifolds (M^n, g) (M'^n, g') and $C^{\infty,m}(M, M')$. A diffeomorphism $f\colon M \longrightarrow M'$ will be called *m-bibounded* if $f \in C^{\infty,m}(M, M')$ and $f^{-1} \in C^{\infty,m}(M', M)$. Sufficient for this is 1. f is a diffeomorphism, 2. $f \in C^{\infty,m}(M, M')$, 3. $\inf_x |\lambda|_{\min}(df)_x > 0$.

Let $\delta > 0$ and set

$$V_\delta = \{((M_1^n, g_1), (M_2^n, g_2)) \in \mathcal{M}^n(mf)^2 \mid \text{There exists an } (m+1)\text{-bibounded diffeomorphism } M_1 \xrightarrow{f} M_2 \text{ with } C^{-1} \cdot g_1 \leqslant f^* g_2 \leqslant C \cdot g_1\},$$

where

$$C = 1 + \delta + \delta\sqrt{2n(n-1)}.$$

Proposition 4.3 [8]. $\mathcal{L} = \{V_\delta\}_{\delta > 0}$ *is a basis for a uniform structure* ${}^{b,m}\mathcal{U}_{\text{diff}}(\mathcal{M}^m(mf))$. $\bullet$

The proof is long and technical.

We let ${}^{b,m}\mathcal{M}^n(mf)$ denote $\mathcal{M}^n(mf)$ endowed with the ${}^{b,m}\mathcal{U}_{\text{diff}}$-topology .

Proposition 4.4. ${}^{b,m}\mathcal{M}^n(mf)$ *is locally arcwise connected.* $\bullet$

Corollary 4.5. *In ${}^{b,m}\mathcal{M}^n(mf)$ components coincide with arc components.* $\bullet$

Theorem 4.6 [8]**.**
a) ${}^{b,m}\mathcal{M}^n(mf)$ *has a representation as topological sum,*

$$ {}^{b,m}\mathcal{M}^n(mf) = \sum_i {}^{b,m}\operatorname{comp}_{\mathrm{diff}}(M_i, g_i). $$

b) ${}^{b,m}\operatorname{comp}_{\mathrm{diff}}(M,g) = \{(M',g') \in \mathcal{M}^n(mf) \mid$ *There exists a diffeomorphism* $f \in C^{\infty,m+1}(M,M')$ *such that* ${}^{b,m}|f^*g' - g|_g < \infty\}$. $\bullet$

Remark. If $f\colon (M_0,g_0) \to (M_1,g_1)$ is a diffeomorphism such that $c_1 \cdot g_0 \leqslant f^*g_1 \leqslant c_2 \cdot g_0$, then $\operatorname{dil}(f) \leqslant c_2$ and $\operatorname{dil}(f^{-1}) \leqslant 1/c_1$.

Corollary 4.7. *If* $(M_1,g_1) \in {}^{b,m}\operatorname{comp}_{\mathrm{diff}}(M_0,g_0)$, *then*

$$ d_{L,\mathrm{top}}((M_0,g_0),(M_1,g_1)), d_L((M_0,g_0),(M_1,g_1)) < \infty. \quad \bullet $$

Remark. If we assume (I), (B_k), $k \geqslant m$, then completing the $(m+1)$-bibounded diffeomorphisms of $C^{\infty,m+1}(M_0,M_1)$ we obtain the space ${}^{b,m+1}\operatorname{Diff}(M_0,M_1)$ and 4.3–4.7 remain valid with C^{m+1}-diffeomorphisms (bibounded) between manifolds of bounded geometry.

Now we define the uniform structures for the Riemannian case which are parallel to them defined at the end of Sec. 2.

Consider pairs $(M_1^n,g_1),(M_2^n,g_2) \in \mathcal{M}^n(mf)$ with the following property:

$$ \begin{gathered} \text{There exist compact submanifolds } K_1^n \subset M_1^n \text{ and } K_2^n \subset M_2^n \\ \text{and an isometry } M_1 \setminus K_1 \xrightarrow{\Phi} M_2 \setminus K_2. \end{gathered} \tag{4.1} $$

For such pairs, we define

$$ \begin{gathered} {}^{b}d_{L,\mathrm{iso,rel}}((M_1,g_1),(M_2,g_2)) := \\ \inf\{\max\{0,\log{}^b|df|\} + \max\{0,\log{}^b|dh|\} + \sup_{x\in M_1}\operatorname{dist}(x,hfx) \\ + \sup_{y\in M_2}\operatorname{dist}(y,fhy) : f \in C^\infty(M_1,M_2),\ g \in C^\infty(M_2,M_1) \text{ and} \\ \text{for some } K_1 \subset K\ f|_{M_1\setminus K_1} \text{ is an isometry and } g|_{f(M_1\setminus K)} = f^{-1}\}. \end{gathered} $$

If (M_1,g_1), (M_2,g_2) satisfy (4.1), then $\{\dots\} \neq \emptyset$ and

$$ {}^{b}d_{L,\mathrm{iso,rel}}((M_1,g_1),(M_2,g_2)) = \inf\{\dots\} < \infty. $$

If (M_1, g_1), (M_2, g_2) do not satisfy (4.1), then we define

$$^b d_{L,\mathrm{iso},\mathrm{rel}}((M_1, g_1), (M_2, g_2)) = \infty.$$

$^b d_{L,\mathrm{iso},\mathrm{rel}}(\cdot, \cdot)$ is $\geqslant 0$, symmetric, and $\leqslant \infty$.

$$^b d_{L,\mathrm{iso},\mathrm{rel}}((M_1, g_1), (M_2, g_2)) = 0$$

if (M_1, g_1) and (M_2, g_2) are isometric.

Remarks. 1) The notions Riemannian isometry and distance isometry coincide for Riemannian manifolds. Moreover, for an isometry f holds $^b|df| = 1$. 2) Any f which enters into the definition of $d_{L,\mathrm{iso},\mathrm{rel}}$ is automatically an element of $C^{\infty,m}(M_1, M_2)$ for all m. The same holds for g.

We denote $\mathcal{M}^n_{L,\mathrm{iso},\mathrm{rel}}(mf) = \mathcal{M}^n(mf)/\sim$ where $(M_1, g_1) \sim (M_2, g_2)$ if

$$^b d_{L,\mathrm{iso},\mathrm{rel}}((M_1, g_1), (M_2, g_2)) = 0.$$

Set

$$V_\delta = \{((M_1, g_1), (M_2, g_2)) \in (\mathcal{M}^n_{L,\mathrm{iso},\mathrm{rel}}(mf))^2 \\ \mid {}^b d_{L,\mathrm{iso},\mathrm{rel}}((M_1, g_1), (M_2, g_2)) < \delta\}.$$

Proposition 4.8. *$\mathcal{L} = \{V_\delta\}_{\delta > 0}$ is a basis for a metrizable uniform structure $^b\mathcal{U}_{L,\mathrm{iso},\mathrm{rel}}$.* $\bullet$

Denote by $^b\mathcal{M}^n_{L,\mathrm{iso},\mathrm{rel}}(mf)$ the corresponding uniform space.

Proposition 4.9. *If $r_{\mathrm{inj}}(M_i, g_i) = r_i > 0$, $r = \min\{r_1, r_2\}$ and*

$$^b d_{L,\mathrm{iso},\mathrm{rel}}((M_1, g_1), (M_2, g_2)) < r,$$

then M_1, M_2 are (uniformly proper) bi-Lipschitz homotopy equivalent. $\bullet$

Corollary 4.10. *If we restrict to open manifolds with injectivity radius $\geqslant r$, then manifolds (M_1, g_1), (M_2, g_2) with $^b d_{L,\mathrm{iso},\mathrm{rel}}$-distance $< r$ are automatically (uniformly proper) bi-Lipschitz homotopy equivalent.* $\bullet$

Remark. If (M_1, g_1) satisfies (I) or (I) and (B_k) and

$$^b d_{L,\mathrm{iso},\mathrm{rel}}(M_1, g_1), (M_2, g_2)) < \infty,$$

then (M_2, g_2) also satisfies (I) or (I) and (B_k).

We cannot show that ${}^b\mathcal{M}^n_{L,\mathrm{iso,rel}}$ is locally arcwise connected, that components coincide with arc components and that

$$ {}^b\operatorname{comp}_{L,\mathrm{iso,rel}}(M,g) = \{(M',g')|{}^b d_{L,\mathrm{iso,rel}}((M,g),(M',g')) < \infty\}. $$

The background for this is the fact that it is impossible to connect non-homotopy-equivalent manifolds by a continuous family of manifolds. A parametrization of nontrivial surgery always contains bifurcation levels where we leave the category of manifolds. A very simple handable case comes from 4.10.

Corollary 4.11. *If we restrict ${}^b\mathcal{U}_{L,\mathrm{iso,rel}}$ to open manifolds with injectivity radius $\geqslant r > 0$, then the manifolds in each arc component of this subspace are bi-Lipschitz homotopy equivalent.*

Proof. This subspace is locally arcwise connected, components coincide with arc components. Consider an (arc) component, elements (M_1,g_1), (M_2,g_2) of it, connect them by an arc, cover this arc by sufficiently small balls and apply 4.10. •

By definition,

$$ {}^b d_{L,\mathrm{iso,rel}}((M_1,g_1),(M_2,g_2)) < \infty \implies d_L((M_1,g_1),(M_2,g_2)) < \infty. $$

Hence $(M_2,g_2) \in \operatorname{comp}_L(M_1,g_1)$, i.e.,

$$ \begin{aligned} \{(M_2,g_2) \in \mathcal{M}^n(mf)|\ {}^b d_{L,\mathrm{iso,rel}}((M_1,g_1),(M_2,g_2)) < \infty\} \subseteq \\ \operatorname{comp}_L(M_1,g_1). \end{aligned} \tag{4.2} $$

For this reason we denote the left hand side $\{\dots\}$ of (4.2) by

$$ {}^b\operatorname{comp}_{L,\mathrm{iso,rel}}(M_1,g_1) = \{\dots\} = \{\dots\} \cap \operatorname{comp}_L(M_1,g_1) $$

keeping in mind that this is not an arc component but a subset (of manifolds) of a Lipschitz arc component.

If one fixes (M_1,g_1), then one has in special cases a good overview on the elements of ${}^b\operatorname{comp}_{L,\mathrm{iso,rel}}(M_1,g_1)$.

Example. Let $(M_1,g_1) = (\mathbb{R}^n, g_{\mathrm{standard}})$. Then ${}^b\operatorname{comp}_{L,\mathrm{iso,rel}}(M_1,g_1)$ is in 1–1-correspondence with relation to $\{(M^n,g)|M^n$ is a closed manifold, g is arbitrary but flat in an annulus contained in disc neighborhood of a point$\}$.

This can be generalized as follows.

Theorem 4.12. *Any component* ${}^b\operatorname{comp}_{L,\mathrm{iso},\mathrm{rel}}(M,g)$ *contains at most countably many diffeomorphism types.*

Proof. Fix $(M,g) \in {}^b\operatorname{comp}_{L,\mathrm{iso},\mathrm{rel}}(M,g)$, an exhaustion $K_1 \subset K_2 \subset \ldots$, $\bigcup K_i = M$, by compact submanifolds and consider $(M',g') \in {}^b\operatorname{comp}_{L,\mathrm{iso},\mathrm{rel}}(M,g)$. Then there exist $K' \subset M'$ and $K_i \subset M$ such that $M \setminus K_i$ and $M' \setminus K'$ are isometric. The diffeomorphism type of M' is completely determined by diffeomorphism type of the pair $(K_1 \underset{\partial K_1 \cong \partial K'}{\cup} K', K_1)$ but there are only at most countably many types of such pairs (after fixing M and $K_1 \subset K_2 \subset \ldots$). $\bullet$

That is, after fixing (M,g), the diffeomorphism classification of the elements in ${}^b\operatorname{comp}_{L,\mathrm{iso},\mathrm{rel}}\,(M,g)$ seems to be reduced to a "handable" countable discrete problem. This is in fact the case in a sense which is parallel to the classification of compact manifolds. This will be carefully and detailed discussed in [8–10]. A key role for this plays the following

Lemma 4.13. *Let* $\sigma_e(\Delta_q)$ *be the essential spectrum of the Laplace operator acting on q-forms and assume* $(M',g') \in {}^b\operatorname{comp}_{L,\mathrm{iso},\mathrm{rel}}(M,g)$. *Then* $\sigma_e(\Delta_q)(M,g) = \sigma_e(\Delta_q)(M',g')$, $0 \leqslant q \leqslant n$. $\bullet$

A further step in this classification approach will be the definition of characteristic numbers for pairs (M,M') and of bordism which will be the content of Sec. 5.

Instead of requiring isometry at infinity we can focus our attention to homotopy porperties and define

$$\begin{aligned} {}^{b,m}d_{L,h}((M_1,g_1),(M_2,g_2)) = \inf\{&\max\{0, \log{}^b|df|\} + \max\{0,\log{}^b|dh|\} \\ &+ \sup_{x\in M_1} \operatorname{dist}(x,hfx) + \sup_{y\in M_2}\operatorname{dist}(y,fhy) \\ &\mid f \in C^{\infty,m}(M_1,M_2) \text{ and } h \in C^{\infty,m}(M_2,M_1) \text{ are mutually inverse} \\ &\text{uniformly proper homotopy equivalences}\}, \end{aligned} \tag{4.3}$$

if $\{\ldots\} \neq \emptyset$ and if $\inf\{\ldots\} < \infty$. Otherwise, we define

$${}^{b,m}d_{L,h}((M_1,g_1),(M_2,g_2)) = \infty.$$

Then $d_{L,h}$ is $\geqslant 0$, symmetric, and $= 0$ if (M_1,g_1) and (M_2,g_2) are isometric. Define $\mathcal{M}^n_{L,h}(mf) = \mathcal{M}^n(mf)/\sim$ where $\sim$ means ${}^{b,m}d_{L,h}$-distance $= 0$. Set

$$V_\delta = \{((M_1,g_1),(M_2,g_2)) \in (\mathcal{M}^n_{L,h}(mf))^2 \mid {}^{b,m}d_{L,h}((M_1,g_1),(M_2,g_2)) < \delta\}.$$

Proposition 4.14. *$\mathcal{L} = \{V_\delta\}_{\delta>0}$ is a basis for a metrizable uniform structure ${}^m\mathcal{U}_{L,h}$.* $\bullet$

Denote by ${}^{b,m}\mathcal{M}_{L,h}$ the corresponding uniform space. Here again we cannot prove that ${}^{b,m}\mathcal{M}^n_{L,h}(mf)$ is locally arcwise connected. Nevertheless,

$$^{b,m}d_{L,h}((M_1,g_1),(M_2,g_2)) < \infty$$

implies

$$d_L((M_1,g_1),(M_2,g_2)) < \infty$$

and hence

$$(M_2,g_2) \in \mathrm{comp}_L(M_1,g_1).$$

$$\{(M_2,g_2) \in \mathcal{M}^n_{L,h}(mf) \mid {}^{b,m}d_{L,h}((M_1,g_1),(M_2,g_2)) < \infty\}$$

$$\subseteq \mathrm{comp}_L(M_1,g_1),$$

and we write ${}^{b,m}\,\mathrm{comp}_{L,h}(M_1,g_1)$ for this set.

Remark. ${}^{b,m}d_{L,h}((M_1,g_1),(M_2,g_2)) = 0 \Rightarrow d_{L,h}((M_1,g_1),(M_2,g_2)) = 0$. The corresponding implication holds in all preceding cases.

In a quite analogous manner as in Sec. 2, Proposition 2.15, and as above, i.e., restricting to maps $\in C^{\infty,m}$, we can define ${}^{b,m}d_{L,h,\mathrm{rel}}$, ${}^{b,m}\mathcal{M}_{L,h,\mathrm{rel}}(mf)$ and ${}^{b,m}\,\mathrm{comp}_{L,h,\mathrm{rel}}(M,g)$, where

$$^{b,m}\,\mathrm{comp}_{L,h,\mathrm{rel}}(M,g) = \{(M',g') \mid {}^{b,m}d_{L,h,\mathrm{rel}}((M,g),(M',g')) < \infty\} \subset \mathrm{comp}_{L,h,\mathrm{rel}}(M,g).$$

All uniform structures defined until now for manifolds are based on the Banach ${}^{b,m}|\ |$-theory. It is possible to construct an extensive theory of uniform structures of manifolds based on the $|\ |_{p,r}$-Sobolev approach. For spaces of metrics and manifolds of maps this has been done e. g. in [14, 15]. We cannot discuss here the Sobolev approach for reasons of space and refer to [8]. An important philosophical hint shall be given. The ${}^{b,m}|\ |$-approach for manifolds is more related to the d_L-uniform structures (as we pointed out), part of the Sobolev approach for manifolds is more related to d_{GH}-uniform structures.

We conclude this section with the hint to [8] where most of the very long details are represented.

§5. Bordism groups for open manifolds

We sketch very shortly our approach to bordism theory for open manifolds. Let (M^n, g), (M'^n, g') be open, oriented, complete. We say (M^n, g) is bordant to (M'^n, g') if there exists an oriented complete manifold (B^{n+1}, g_B) with boundary ∂B such that the following holds.

1) $(\partial B, g_B|_{\partial B}) = (M, g) \cup (-M', g')$. Here $=$ stands for isometry.

2) There exists a uniform Riemannian collar

$$\Phi \colon \partial B \times [0, \delta[\xrightarrow{\cong} \mathcal{U}_\delta(\partial B) \subset B,$$

$$\Phi^*(g_B|_{\mathcal{U}_\delta(\partial B)}) = g_{\partial B} + dt^2.$$

.

3) There exists $R > 0$ such that $B \subseteq \mathcal{U}_R(M)$, $B \subseteq \mathcal{U}_R(M')$.

Remark. Condition 3) looks like $d_H^B(M, M') \leqslant R$, $d_{GH}(M, M') \leqslant R$. But this is not necessarily the case since 1) and 2) do not imply that $\partial B = M \cup M'$ is isometrically embedded as metric length space into the metric length space B. ∂B is isometrically embedded as Riemannian manifold but its inner length metric will not be the induced length metric from B, even not if ∂B is totally geodesic as we assume by 2). •

We denote $(M, g) \underset{b}{\sim} (M', g')$. B^{n+1} is called a *bordism*.

Lemma 5.1. $\underset{b}{\sim}$ *is an equivalence relation.* •

Denote the equivalence = bordism class of (M, g) by $[M, g]$.

Lemma 5.2. $[M \cup M', g \cup g'] = [M \# M', g \# g']$. •

Remark. $(M \# M', g \# g')$ is metrically not uniquely defined but its bordism class is.

Lemma 5.3. *Set* $[M, g] + [M', g'] = [M \cup M', g \cup g'] = [M \# M', g \# g']$. *Then* $+$ *is well defined and the set of all* $[M^n, g]$ *becomes an Abelian semigroup.* •

Denote by Ω_n^{nc} the corresponding Grothendieck group which is the bordism group of all oriented open complete Riemannian manifolds. Here 0 is generated by the diagonal Δ and

$$-[[M, g], [M', g']] = [[M', g'], [M, g]].$$

There is no reasonable approach for a calculation of Ω_n^{nc} known to us. Ω_n^{nc} is much to large. The situation rapidly changes if we consider several refinements of the notion of bordism, combining this with a component in $\mathcal{M}^n(mf)$ and having additional conditions in mind, e. g. geometric conditions as nonexpanding ends or spectral conditions. Moreover, we would be interested to have a geometric realization of 0 and $-[M, g]$.

First we consider bordism with compact support. Here we require as above 1), 2) and aditionally

3) (cs). There exists a compact submanifold $C^{n+1} \subset B^{n+1}$ such that $B \setminus \operatorname{int} C$ is a product bordism, i.e.,

$$(B \setminus \operatorname{int} C, g_B|_{B \setminus \operatorname{int} C}) = ((M \setminus \operatorname{int} C) \times [0,1], g|_{M \setminus \operatorname{int} C} + dt^2).$$

3) (cs) implies 3. (after a compact change of the metric).

Write $\underset{b,cs}{\sim}$ for the corresponding bordism. The corresponding bordism group will be denoted by $\Omega_n^{nc}(cs)$. At the first glance, the calculation of $\Omega_n^{nc}(cs)$ or at last the characterization of the bordism classes seems to be very difficult. But this is not the case as we indicate now. We connect $\Omega_n^{nc}(cs)$ with the components of ${}^b\operatorname{comp}_{L,\text{iso},\text{rel}}(\cdot) \subset {}^b\mathcal{M}^n_{L,\text{iso},\text{rel}}(mf)$.

Remark. If (M_1, g_1), $(M_2, g_2) \in {}^b\operatorname{comp}_{L,\text{iso},\text{rel}}(M, g)$, then in general $(M_1, g_1)\#(M_2, g_2) \notin {}^b\operatorname{comp}_{L,\text{iso},\text{rel}}(M, g)$.

Let

$$\Omega_n^{nc}(cs, {}^b\operatorname{comp}_{L,\text{iso},\text{rel}}(M, g)) \subset \Omega_n^{nc}(cs)$$

be the subgroup generated by

$$\{[M', g']_{cs} | (M', g') \in {}^b\operatorname{comp}_{L,\text{iso},\text{rel}}(M, g)\}.$$

We know $\Omega_n^{nc}(cs)$ completely if we know all $\Omega_n^{nc}(cs, {}^b\operatorname{comp}_{L,\text{iso},\text{rel}}(M, g))$ and we know $\Omega_n^{nc}(cs, {}^b\operatorname{comp}_{L,\text{iso},\text{rel}}(M, g))$ completely if we know the corrsponding generating set. But the elements of this set can be completely characterized by characteristic numbers which we define now.

Fix $(M, g) \in {}^b\operatorname{comp}_{L,\text{iso},\text{rel}}(M, g)$, M oriented. We assume that $(M_1, g_1) \in {}^b\operatorname{comp}_{L,\text{iso},\text{rel}}(M, g)$ and let $\Phi \colon M \setminus K \to M_1 \setminus K_1$ be an orientation preserving isometry. Define the Stiefel–Whitney numbers of the pair (M_1, M) by

$$w_1^{r_1} \dots w_n^{r_n}(M_1, M) := \langle w_1^{r_1} \dots w_n^{r_n}, [K_1] \rangle + \langle w_1^{r_1} \dots w_n^{r_n}, [K] \rangle.$$

Similarly we define for (M_1, M) and $n = 4k$ the Pontryagin numbers

$$p_1^{r_1} \dots p_k^{r_k}(M_1, M) := \int_{K_1} p_1^{r_1} \dots p_k^{r_k}(M_1) - \int_{K} p_1^{r_1} \dots p_k^{r_k}(M_1)$$

and the signature

$$\sigma(M_1, M) := \sigma(K_1) + \sigma(-K).$$

Lemma 5.4 [9]. *The numbers* $w_1^{r_1} \dots w_n^{r_n}(M_1, M)$, $p_1^{r_1} \dots p_k^{r_k}(M_1, M)$, *and* $\sigma(M_1, M)$ *are well defined and*

$$\begin{aligned} w_1^{r_1} \dots w_n^{r_n}(M_1, M) &= \langle w_1^{r_1} \dots w_n^{r_n}(K_1 \cup K), [K_1 \cup K] \rangle, \\ p_1^{r_1} \dots p_k^{r_k}(M_1, M) &= \langle p_1^{r_1} \dots p_k^{r_k}(K_1 \cup -K), [K_1 \cup -K] \rangle, \\ \sigma(M_1, M) &= \sigma(K_1 \cup -K). \end{aligned}$$

Here $K_1 \cup -K := K_1 \cup_{\Phi|_{\partial K}} -K$. $\bullet$

The proof is very simple.

A complete characterization of bordism classes is now given by

Theorem 5.5 [9]. *Fix* $(M_1, g_1), (M_2, g_2) \in {}^b\operatorname{comp}_{L,\mathrm{iso,rel}}(M, g)$. *Then* $(M_1, g_1) \underset{b,cs}{\sim} (M_2, g_2)$ *if and only if all characteristic numbers of* (M_1, M) *coincide with the corresponding characteristic numbers of* (M_2, M). $\bullet$

Corollary 5.6. *The characterization of all elements of* $\Omega_n^{nc}(cs)$ *reduces to "counting" the (generalized) components of* ${}^b\mathcal{M}^n_{L,\mathrm{iso,rel}}(mf)$. $\bullet$

Remark. If we restrict our consideration to closed oriented n-manifolds, then ${}^b\operatorname{comp}_{L,\mathrm{iso,rel}}(S^n)$ contains all closed oriented n-manifolds, independent of any choice of the Riemannian metric, and characteristic numbers defined above of (M^n, S^n) coincide with the characteristic numbers of M^n. This follows from the definition above and the fact that they vanish for S^n or by cutting isometric collared small discs from M^n, S^n and gluing.

Examples. 1) Consider $M' = (P^{2k}\mathbb{C} \setminus$ int (small disc) $\cup$ metric cylinder $S^{2k-1} \times [0, \infty[$, corresponding metric) and $M = (2k$-disc $\cup$ metric cylinder $S^{2k-1} \times [0, \infty[$, corresponding metric). Then $(M', g') \in {}^b\operatorname{comp}_{L,\mathrm{iso,rel}}(M, g)$ but (M', g') is not cs-bordant to (M, g) since we have $\sigma(M', M) = 1$, $\sigma(M', M') = 0$.

2) For any (M, g) there is a map

$$\Phi_n : \Omega_n \longrightarrow \{[M', g']_{cs} \mid (M', g') \in {}^b\operatorname{comp}_{L,\mathrm{iso,rel}}(M, g)\}$$

given by $[N] \to [M \# N, g']_{cs}$ independent of the metric on N. $\bullet$

There are many other types of bordism which are discussed in [9]. We present still a type of bordism where 0 and $-[M, g]$ are geometrically realized. For this sake, we restrict to metrics and bordisms of bounded geometry.

Let (M^n, g), (M'^n, g') be open, oriented, satisfying (I) and (B_k). We say (M^n, g) is (I), (B_k)-bordant to (M'^n, g') if there exists an oriented manifold $(B^{n+1}, g_B$ with boundary ∂B such that the following holds.

1) $(\partial B, g_B|_{\partial B}) = (M, g) \cup ('_M, g')$.

2) There exists $\delta > 0$ such that $\exp\colon U_\delta(0_\nu) \longrightarrow U_\delta(\partial B)$ is a $(k+2)$-bibounded diffeomorphism, i.e., there exists a "uniform collar" of ∂B. Here 0_ν denotes the zero section of the inner normal bundle $\nu = \nu(\partial B)$ of ∂B in B.

3) g_B satisfies (B_k) on B and (I) on $B \setminus U_{\delta/2}(\partial B)$.

4) There exists $R > 0$ such that $B \subseteq U_R(M)$ and $B \subseteq U_R(M')$. We write then $(M, g) \underset{b,bg}{\sim} (M', g')$.

Lemma 5.7 [9]. *Assume* $(M, g) \underset{b,bg}{\sim}$ *via* (B, g_B). *Then there exist* $\delta_1 > 0$ *and* $\tilde{g}_B$ *such that* $(M, g) \underset{b,bg}{\sim}$ *via* $(B, \tilde{g})$, $\tilde{g}_B|_{U_{\delta_1}(\partial B)} \cong g_{\partial B} + dt^2$. $\bullet$

Corollary 5.8. *Without loss of generality we can always assume that the collar of* ∂B *is a metric collar.* $\bullet$

Corollary 5.9. $\underset{b,bg}{\sim}$ *is an equivalence relation.* $\bullet$

Denote by $[M, g]_{bg}$ the equivalence (= bordism) class of (M, g).

Lemma 5.10. $[M \cup M', g \cup g']_{bg} = [M \# M', g \# g']_{bg}$. *Set*

$$[M, g]_{bg} + [M', g']_{bg} := [M \cup M', g \cup g']_{bg} = [M \# M', g \# g']_{bg}.$$

Then $+$ *is well defined and* $\{[M, g]_{bg} | (M, g)$ *open, oriented with* (I) *and* $(\mathrm{B}_k)\}$ *becomes an Abelian semigroup.* $\bullet$

Lemma 5.11. *Let* (M^n, g) *be an open manifold satisfying (I) and let* ϵ *be an end of* M. *Then there exists a geodesic ray* c *tending to* ∞ *in* ϵ *with a uniformly thick tubular neighborhood.* $\bullet$

We say that an end ϵ is nonexpanding (ϵ is a n. e. end), if there exists an $R > 0$ and a ray c in ϵ so that $\epsilon \subseteq U_R(|c|)$, which means that all elements of a neighborhood basis of ϵ are contained in $U_R(|c|)$.

We now restrict to manifolds with finitely many n. e. ends only. Define $chc(r) := (D^n \cup S_r^{n-1} \times [0, \infty[, g_{\text{standard}})$, where $g_{\text{standard}}|_{S_r^{n-1} \times [a,\infty[} = g_{S_r^{n-1}} + dt^2$ and the standard metrics near $\partial D^n \underset{identif.}{=} S_r^{n-1} \times \{0\}$ are smoothed out. Then $chc(r)$ has one end, nonexpanding, and satisfies (I) and (B_∞).

Lemma 5.12. $chc(r_1) \underset{b,bg}{\sim} chc(r_2)$, $\bigcup\limits_{\sigma=1}^{s} chc(r_\sigma) \underset{b,bg}{\sim} chc(r)$. •

Define now

$$\begin{aligned} -[M,g] &:= [-M,g], \\ 0 &:= [chc(1)] \end{aligned} \tag{5.1}$$

and set $\Omega_n^{nc,fe,ne}(I, B_k) := \mathrm{all}[M^n, g]_{bg}$ with finitely many ends (fe), all of them nonexpanding (n. e.).

Theorem 5.13 [9]. *$(\Omega_n^{nc,fe,ne}(\mathrm{I},\mathrm{B}_k), +, -, 0)$ is an Abelian group.* •

The proof contains some delicate geometric constructions.

Examples of manifolds of type f. e., n. e., (I), (B_k) are given by warped product metrics at infinity with C^{k+2}-bounded warping function and e. g. by ends which are an infinite connected sum of a finite number of closed Riemannian manifolds.

We finish with these two (very handable) examples $\Omega_n^{nc}(cs)$ and $\Omega_n^{nc,fe,ne}(I, B_k)$ our short review of bordism theory for open manifolds and refer to [9] for an extensive representation.

§6. Invariants of open manifolds

Consider (M^n, g) with (I) and (B_k), usually $k \geqslant \frac{n}{2} + 1$. (M^n, g) is a proper metric (length) space and we have sequences of inclusions

$$\text{coarse type}(M,g) \supset \text{comp}_{GH}(M,g), \tag{6.1}$$

$$\begin{aligned} \text{coarse type}(M,g) &\supset \text{comp}_L(M,g) \supset \text{comp}_{L,h}(M,g) \supset \\ &\supset \text{comp}_{L,\text{top}}(M,g), \end{aligned} \tag{6.2}$$

$$\begin{aligned} \text{comp}_L(M,g) &\supset {}^{b,m}\text{comp}_L(M,g) \supset {}^{b,m}\text{comp}_{L,h}(M,g) \supset \\ &\supset {}^{b,m}\text{comp}_{\text{diff}}(M,g), \end{aligned} \tag{6.3}$$

$${}^{b}\text{comp}_{L,h,\text{rel}}(M,g) \supset {}^{b}\text{comp}_{L,\text{iso},\text{rel}}(M,g), \tag{6.4}$$

and others. The arising task is to define for any sequence of inclusions invariants depending only on the component and becoming sharper and sharper if we move from left to right. We cannot present here all by us and others defined invariants but present a certain choice. Start with the coarse type. Denote by $C^*(M,g)$ the C^*-algebra obtained as the closure in $\mathcal{B}$ of L_2M of all locally compact, finite propagation operators and by $HX^*(M,g)$ the coarse cohomology. Then we have from [22, 23]

Theorem 6.1. *$HX^*(M,g)$ and the K-theory $K_*(C^*(M,g))$ are invariants of the coarse type, hence invariants of all components right from the coarse type.* $\bullet$

Remarks. 1) We tried to describe the coarse type as the component of some uniform structure but there is still some gap in this approach.

2) Recall that a rough map is a coarse map which is uniformly proper. The rough type is defined like the coarse type, replacing only coarse maps by rough maps.

Block and Weinberger defined in [3] a homology $H_*^{\mathrm{Uf}}(M,g)$ which we here call *rough homology* (since it is functional under rough maps). We prove in [8] that the rough and coarse type coincide.

Theorem 6.2. *The rough homology $H_*^{\mathrm{Uf}}(M,g)$ is an invariant of the coarse type.* $\bullet$

Define the (singular) uniformly locally finite homology $H_*^{\mathrm{Uff}}(M,g)$ as follows. It is the homology of the complex $C_*^{\mathrm{Uff}}(M,g)$. $c = \sum a_\sigma \sigma$ is a chain of C_q^{Uff} if there exists a $K > 0$ depending on c so that $|a_\sigma| \leqslant K$ and the number of simplices σ lying in a ball of given size is uniformly bounded. The boundary is defined to be the linear extension of the singular boundary. The uniformly locally finite cohomology $H^*_{\mathrm{Uff}}(M,g)$ is defined similarly (cf. [1]).

Theorem 6.3. *The groups $H_*^{\mathrm{Uff}}(M,g)$ and $H^*_{\mathrm{Uff}}(M,g)$ are invariants of* $\mathrm{comp}_{L,h}(M,g)$. $\bullet$

Consider the bounded de Rham complex of (M,g),

$$\ldots \longrightarrow {}^{b,1}\Omega^q \xrightarrow{d} {}^{b,1}\Omega^{q+1} \longrightarrow \ldots$$

Its cohomology is the *bounded cohomology* of ${}^bH^*(M,g)$.

Theorem 6.4. *${}^bH^*(M,g)$ is an invariant of* ${}^{b,2}\mathrm{comp}_{L,h}(M,g)$. $\bullet$

Remark. We remember that we required in 5.4 the boundedness of the maps and the homotopies. This is essential. $\bullet$

The ^{b}-case is the smooth L_∞-case and it is quite natural to consider the L_p-case, in particular the L_2-case. But here the point is that an arbitrary bounded map $f\colon (M,g) \to (M',g')$ does not induce an L_2-bounded map of analytic L_2-cohomology. Hence it is far from being clear whether analytic L_2-cohomology is an invariant of ${}^{b,k+1}\mathrm{comp}_{L,h}(M,g)$. This question can be attacked by simplicial L_2-cohomology and L_2-Hodge–de Rham theory.

We will discuss this shortly and recall some simple definitions and facts.

Let K be a locally oriented simplicial complex and $\sigma^q \in K$. We denote

$$I(\sigma^q) = \#\{\tau^{q+1} \in K \mid \sigma < \tau^{q+1}\}, \quad I_q(K) := \sup_{\sigma^q \in K} I(\sigma^q).$$

The complex K is *uniformly locally finite* (u. l. f.) in dimension q if $I_q(K) < \infty$. If this holds for all q, then we call K (in any dimension) u. l. f. The latter is equivalent to $I_0(K) < \infty$. We assume in the sequel K u. l. f. For $1 \leqslant p < \infty$, let

$$C^{q,p}(K) = \left\{ c = \sum_{\sigma^q \in K} c_\sigma \cdot \sigma \;\middle|\; c_\sigma \in \mathbb{R}, \sum_\sigma |c_\sigma|^p < \infty \right\}$$

be the Banach space of all real p-summable q-cochains. Then the linear extension of d, $d\sigma^q = \sum_{\tau^{q+1}} [\tau^{q+1} : \sigma^q]\tau^{q+1}$ is a bounded linear operator

$$d\colon C^{q,p} \to C^{q+1,p}, \quad \sum c_\sigma \sigma \mapsto \sum_{\tau^{q+1}} \left(\sum_{\sigma^q < \tau^{q+1}} [\tau : \sigma] \right) \tau^{q+1},$$

and we obtain a Banach cochain complex $(C^{*,p}, d)$. Its cohomology $H^{*,p}(K)$ is called *simplicial L_p-cohomology*,

$$H^{q,p}(K) = Z^{q,p}/B^{q,p} = \ker(C^{q,p} \xrightarrow{d} C^{q+1,p})/\operatorname{im}(C^{q-1,p} \xrightarrow{d} C^{q,p}).$$

$\overline{H}^{q,p}(K) = Z^{q,p}/\overline{B^{q,p}}$ is called *reduced simplicial L_p-cohomology*. Now, $C^{q,2}(K)$ with $\langle c, c' \rangle = \sum_{\sigma^q} c_\sigma \cdot c'_\sigma$ is a Hilbert space, and we obtain a Hilbert complex $C^{*,2}(K)$. We refer to [17] and [2] for many simple proofs and interesting geometric examples.

Let $(C_{*,p} = C^{*,p}, \partial)$, where

$$\partial\sigma^q = \sum_{\tau^{q-1}} [\sigma^q : \tau^{q-1}]\tau^{q-1}, \quad \partial(\sum c_\sigma \sigma) = \sum_{\tau^{q-1}} (\sum_{\sigma^q > \tau^{q-1}} [\sigma : \tau] c_\sigma) \tau^{q-1},$$

be the Banach complex of p-summable real chains and $H_{*,p}(K)$ or $\overline{H}_{*,p}(K)$ the corresponding L_p-homology or reduced L_p-homology, respectively.

Lemma 6.5. *If $H^{q,p}(K) \neq \overline{H}^{q,p}(K)$, then there exists an infinite number of independent cohomology classes in $H^{q,p}$ whose image in $\overline{H}^{q,p}$ equals to zero, i.e., $\dim\ker H^{q,p} \to \overline{H}^{q,p} = \infty$. The same holds for L_p-homology.* •

Corollary 6.6. *If* $\dim H^{q,p}(K) < \infty$, *then* $H^{q,p}(K) = \overline{H}^{q,p}(K)$. *The same holds for homology.* •

Remark. $H^{q,p}(K)$ is endowed with a canonical topology, the quotient topology. But if $B^{q,p}$ is not closed then points in $H^{q,p}(K)$ are not closed. In particular, any point $0 \neq \overline{b} + B^{q,p} \in H^{q,p}$, $\overline{b} \in \overline{B}^{q,p} \setminus B^{q,p}$, belongs to the closure $\overline{0} \subset H^{q,p}$.

For $p = 2$ $\partial_{q-1} \colon C^{q,2} \to C^{q-1,2}$ is the $\langle\, , \rangle$-adjoint of $d_{q-1} \colon C^{q-1,2} \to C^{q,2}$. Then $\Delta_q(K) := d_{q-1}\partial_{q-1} + \partial_q d_q$ is a well-defined bounded operator $\Delta_q \colon C^{q,2}(K) \to C^{q,2}(K)$ and $\mathcal{H}^q(K) \equiv \mathcal{H}^{q,2}(K) := \ker \Delta_q(K)$ is the Hilbert subspace of harmonic L_2-cochains = harmonic L_2-chains.

Lemma 6.7.

a) $c \in \mathcal{H}^{q,2}(K)$ *if and only if* $dc = \partial c = 0$

b) *There exists an orthogonal decomposition*

$$C^{q,2} = \mathcal{H}^q(K) \oplus dC^{q-1,2} \oplus \partial C^{q+1,2}$$

c) *There are canonical topological isomorphisms*

$$\mathcal{H}^q \cong \overline{H}^{q,2}(K) \cong \overline{H}_{q,2}(K). \quad \bullet$$

Denote by $\sigma(\Delta_q(K))$ the spectrum, by σ_e the essential spectrum.

Lemma 6.8. *The following conditions are equivalent.*

a) $\operatorname{im} \partial_q$ *and* $\operatorname{im} \partial_{q-1}$ *are closed.*

b) $\operatorname{im} d_q$ *and* $\operatorname{im} d_{q-1}$ *are closed.*

c) $\operatorname{im} \Delta_q(K)$ *is closed.*

d) $0 \notin \sigma_e(\Delta_q(K)|_{(\ker \Delta_q)^\perp})$.

e) $H^{q,2}(K) = \overline{H}^{q,2}(K)$ *and* $H^{q+1,2}(K) = \overline{H}^{q+1,2}(K)$.

f) $H_{q,2}(K) = \overline{H}_{q,2}(K)$ *and* $H_{q-1,2}(K) = \overline{H}_{q-1,2}(K)$. •

Example. Let $K = S^1 \times \mathbb{R}$ with a translation invariant u. l. f. triangulation. Then $\overline{H}_{1,2}(K) = (0)$, $\dim H_{1,2}(K) = \infty$, i.e., condition d) is not fulfilled.

For later applications the following simple lemma is quite useful.

Lemma 6.9. *Let* H *be a Hilbert space and* $\Phi \colon H \to H^{q,2}(K)$ *be a vector space isomorphism and homeomorphism. Then* $H^{q,2}(K) = \overline{H}^{q,2}(K)$. *The same holds as homology version.* •

We discuss next the behaviour of functional cohomology under maps and subdivision.

We write $x(y) = \langle x, y\rangle$ for the value of a q-cochain x and a q-chain y. If S is a set of oriented q-simplices, then S can be considered as a q-chain S defined by $S = \sum_{\sigma^q \in S} \sigma$. Then S will be called a *geometric q-chain*. Let K and L be complexes and $T\colon C^{*,p}(K) \to C^{*,p}(L)$ be a linear map. T is called *vicinal* if there exists an N such that for all $\sigma \in K$

$$\#\{\tau \in L | \langle T\sigma, \tau\rangle \neq 0\} \leqslant N$$

and for any $\tau \in L$

$$\#\{\sigma \in K | \langle T\sigma, \tau\rangle \neq 0\} \leqslant N.$$

Lemma 6.10. *If T is vicinal and $|\langle T\sigma, \tau\rangle| \leqslant M$ for all $\sigma \in K$, $\tau \in L$, then T is bounded.* $\bullet$

Let v be a vertex of K,

$$\mathrm{st}(v) = \{\sigma \in K \mid v \text{ is a vertex of } \sigma\},$$

and for $\sigma \in K$

$$N(\sigma) = \overline{\bigcup_{v \in \sigma} \mathrm{st}(v)},$$

for $L \subset K$

$$N(L) = \overline{\bigcup_{\sigma \in L} N(\sigma)}$$

$$N^{(m)}(\cdot) := N(N(\ldots(N(\cdot))\ldots).$$

We recall some definitions from [2].

A linear map $T\colon C^{*,p}(K_1) \to C^{*,p}(K_2)$ is *local* if it is vicinal and there exists a positive integer n such that whenever there is a simplicial bijection $\eta\colon N^n(\sigma) \to N^n(\tau)$ with $\eta(\sigma) = \tau$ then there exists a simplicial bijection $\varepsilon\colon \mathrm{Im}(N^n(\sigma)) \to \mathrm{Im}(N^n(\tau))$ such that $T(\eta_*(\sigma)) = \varepsilon_*(T(\sigma))$. Here $\mathrm{Im}(N^n(\sigma))$ denotes the subcomplex supporting the chain $T(N^n(\sigma))$. In other words, a local map is characterized by the following three properties:

1) The image of a simplex σ is a chain whose support has less than N simplices, N independent of σ.

2) The number of simplices whose image contains a simplex τ is less than N, N independent of τ.

3) The value of the map on a simplex σ depends only on the configuration arround σ.

These conditions are independent of p and purely combinatorial in nature.

Example. d, ∂, and Δ are local operators.

For many applications one can weaken the third condition replacing the existence of the map ε by the weaker condition that there exists a constant c such that

$$|T(\eta_*(\sigma))|_p < c \cdot |T(\sigma)|_p.$$

Such maps will be called *nearly local.*

Lemma 6.11. *If T is nearly local, then it is bounded.*

Now we want to prove the invariance of functional cohomology under subdivisions of finite degree and apply this for duality on open manifolds. For this we need more general complexes than simplicial ones. An absolute complex is a 3-couple $K = (K, <, \dim)$, $\dim : K \to \mathbb{Z}_+$, such that $<$ is a transitive relation, dim is monotone w. r. t. $<$ and for every $x \in K$ there are only finitely many $y \in K$ with $y < x$. If $\dim x = n$, then x is called an *n-cell.* ε: $K \times K \to \mathbb{Z}$ is called a *boundary function* or *incidence number* if $\varepsilon(x, y) \neq 0$ implies $x < y$ and $\dim y = 1 + \dim x$ and for $x, y \in K$, $\dim y = 2 + \dim x$ holds $\sum\limits_{z \in K} \varepsilon(x, z)\varepsilon(z, y) = 0$. We write $K_\varepsilon = (K, <, \dim, \varepsilon)$. Then the definition of u. l. f. K_ε is quite clear and then the spaces

$$C^{*,p}(K_\varepsilon, d),\ H^{*,p}(K_\varepsilon),\ \overline{H}^{*,p}(K_\varepsilon),\ C_{*,p}(K_\varepsilon),\ H_{*,p}(K_\varepsilon),\ \overline{H}_{*,p}(K_\varepsilon)$$

are well defined. Simplicial, cell and CW-complexes produce their (after orientation) K_ε. It is very easy and natural to extend the notions *local* and *nearly local* to this general situation.

Let K be an u. l. f. simplicial complex, K' a subdivision. We say that K' has bounded degree of subdivision if for any q there exists a number m_g such that any $\sigma^q \in K$ is subdivided into $\leqslant m_q$ q-simplices $\sigma'^q \in K'$. Then K' is automatically u. l. f. For $\sigma^q \in K$ let $B(\sigma^q) = \{\sigma' \in K' | |\sigma'| \subset |\sigma^q|\}$. Then $\mathcal{Z}(K') = \{B(\sigma) | \sigma \in K\}$ is a cell decomposition of K' (cf. [21], pp. 528–533 for definitions). $B(\sigma)$ is a finite subcomplex and is called a *block.* There holds

$$H_i(B(\sigma^q), \dot{B}(\sigma^q); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = q \\ 0, & \text{otherwise.} \end{cases}$$

Choose for any $B(\sigma^q)$ an integer orientation b, i.e., a generator of $H_q(B,\dot{B};\mathbb{Z})$. Then we obtain oriented blocks (B,b). b has a representation $B = \sum\limits_{i=0}^{\kappa} \gamma_i {\sigma'}_i^q$, where ${\sigma'}_i^q$ are the q-simplices of $B(\sigma^q)$, $\gamma_i = \pm 1$, $\kappa \leqslant m_q$. Let $(B_1,b_1),\ldots,(B_{q+1},b_{q+1})$ be the $(q-1)$-blocks of $\dot{B}$. Then ∂b has a representation $\partial b = \sum\limits_{\nu=1}^{q+1} \varepsilon_\nu b_\nu$, $\varepsilon_\nu = \pm 1$. $[(B,b):(B_\nu,b_\nu)]' := \varepsilon_n u$ defines incidence numbers and a boundary operator.

Then it is possible by standard procedures to define a cell complex $(\mathcal{Z}(K'),[:]')$ such that the following holds

Lemma 6.12. *After appropriate choice of orientation b, the map $\sigma^q \to B(\sigma^q)$ defines an isomorphism*

$$\zeta\colon (K,[:]) \to (\mathcal{Z}(K'),[:]').$$

This holds even for arbitrary simplicial complexes and simplicial subdivisions. •

In the sequel we abbreviate $\mathcal{Z}(K') = (\mathcal{Z}(K'),[:]')$.

Corollary 6.13. *The map $\zeta\colon \sigma^q \to B(\sigma^q)$ induces canonical topological isomorphisms $\zeta_\#\colon C_{*,p}(K) \to C_{*,p}(\mathcal{Z}(K'))$ and $\zeta_*\colon H_{*,p}(K) \to H_{*,p}(\mathcal{Z}(K'))$. The same holds for $C^{*,p}$ and $H^{*,p}$ and for the reduced case.* •

Let $(B(\sigma^q),b)$ be an elementary q-chain of $C_{q,p}(\mathcal{Z}(K'))$. Then we define

$$\Theta_q(B(\sigma^q),b) := b = \sum_{i=1}^{\kappa} \gamma_i \sigma_i' \in C_{q,p}(K'), \quad \kappa \leqslant m_q.$$

Here ${\sigma'}^q$ are the q-simplices of $B(\sigma^q)$ and $\gamma_i = \pm 1$. We define for

$$c = \sum_{\sigma^q \in K} c_{B(\sigma^q)}(B(\sigma^q),b) \in C_{q,p}(\mathcal{Z}(K'))$$

the value $\Theta_q c$ by linear extension, $\Theta_q c = \sum\limits_{{\sigma'}^q} c_{{\sigma'}^q} {\sigma'}^q$, where $c_{{\sigma'}^q} = \gamma_\nu \cdot c_{B(\sigma^q)}$, ${\sigma'}^q \in B(\sigma^q)$.

Lemma 6.14. *The map $\Theta = (\Theta_q)_q\colon C_{*,p}(\mathcal{Z}(K')) \to C_{*,p}(K')$ is an L_p-chain map. Any $\Theta_q\colon C_{q,p}(\mathcal{Z}(K')) \to C_{q,p}(K')$ is a monomorphism.* •

Lemma 6.15. *$\Theta_q(C_{q,p}(\mathcal{Z}(K')))$ is closed in $C_{q,p}(K')$.* •

Corollary 6.16. $\Theta_q : C_{q,p}(\mathcal{Z}(K')) \to \operatorname{im}(\Theta_q)$ *is a topological isomorphism.* $\bullet$

Define $\eta : K' \to K$ as follows. For a vertex $v' \in K'$ denote by $\operatorname{tr}(e')$ the simplex of smallest dimension which contains v'. Choose for v' a vertex $v = \eta(v')$. η is called a *vertex translation* and defines by extension a simplicial map $\eta : K' \to K$. Then the following lemma immediately follows from the corresponding lemma in classical homology theory (cf. [21], 49.15, p. 537). We identify $C_{q,p}(K')$ and $C_{q,p}(\mathcal{Z}(K'))$ by means of ζ according to 6.13.

Lemma 6.17. $(\eta_{\#q}) \circ \Theta_q : C_{q,p}(K) \to C_{q,p}(K)$ *equals* $\mathrm{id}_{C_{q,p}(K)}$. $\bullet$

Now we can establish

Theorem 6.18. *Let $1 \leqslant p < \infty$ and K be an u. l. f. simplicial complex and K' a simplicial subdivision which is of bounded degree of subdivision in any dimension. Then, after identification of K with $\mathcal{Z}(K')$ by means of ζ, $\Theta : C_{*,p}(K) \to C_{*,p}(K')$ induces topological isomorphisms*

$$\Theta_* : H_{*,p}(K) \to H_{*,p}(K')$$

and

$$\Theta_* : \overline{H}_{*,p}(K) \to \overline{H}_{*,p}(K'). \quad \bullet$$

A first approach to homotopy invariance is given by

Theorem 6.19. *Let K, L be u. l. f. simplicial complexes, $1 \leqslant p < \infty$ and $\varphi_\# : C_{*,p}(K) \to C_{*,p}(L)$, $\psi_\# : C_{*,p}(L) \to C_{*,p}(K)$ chain maps satisfying the following conditions.*

(C1) $\varphi_\#$, $\psi_\#$ *are local.*

(C2) *There are bounded chain homotopies*

$$\psi_\# \varphi_\# \underset{D_1}{\sim} \mathrm{id}_{C_{*,p}(K)}, \qquad \varphi_\# \psi_\# \underset{D_2}{\sim} \mathrm{id}_{C_{*,p}(L)}.$$

Then $\varphi_\#$ and $\psi_\#$ induce topological isomorphisms

$$H_{*,p}(K) \underset{\underset{\varphi_*}{\rightarrow}}{\overset{\overset{\psi_*}{\leftarrow}}{\cong}} H_{*,p}(L), \qquad \overline{H}_{*,p}(K) \underset{\underset{\varphi_*}{\rightarrow}}{\overset{\overset{\psi_*}{\leftarrow}}{\cong}} \overline{H}_{*,p}(L). \quad \bullet$$

Now we apply this to open manifolds (M^n, g) satisfying (I), (B_k), $k > \frac{n}{p} + 1$. Then it is a well known fact that such an M admits a so-called *uniform* triangulation, i.e., a triangulation $t : |K| \to M$ with the following properties.

Let σ^n be a curved n-simplex in M^n. We define the fullness $\Theta(\sigma)$ by $\Theta(\sigma) = \mathrm{vol}(\sigma)/(\mathrm{diam}(\sigma))^n$.

a) The exists a $\Theta_0 > 0$ such that for any curved simplex σ^n the fullness satisfies the inequality $\Theta(\sigma) > \Theta_0$.

b) There exist constants $c_2 > c_1 > 0$ such that for every σ^n we have

$$c_2 \leqslant \mathrm{vol}(\sigma) \leqslant c_1.$$

c) There exists a constant $c > 0$ such that for every vertex $v \in K$ the barycentric coordinate function $\varphi_v \colon M \to \mathbb{R}$ satisfies the condition $|\nabla\varphi_v| \leqslant c$.

If one assumes a), then b) is equivalent to the existence of bounds $d_1 > d_2 > 0$ with $d_2 \leqslant \mathrm{diam}(\sigma) \leqslant d_1$ for all $\sigma \in K$. a) and b) are equivalent to the boundedness of the volumes from below and the diameters from above.

We call triangulations which satisfy the condition a)–c) *uniform*. Uniform triangulations are u. l. f. A connection between combinatorial and analytical theory is given by the following fundamental theorem of Goldstein, Kuzminov, Shvedov (cf. [18]).

Theorem 6.20. *Let $t\colon |K| \to M$ be a uniform triangulation and $1 \leqslant p < \infty$. Then there exists a canonical topological isomorphism w_* (essentially induced by the Whitney transformation),*

$$w_*\colon H^{p,*}(K) \to H^{p,*}(M,g),$$
$$w_*\colon \overline{H}^{p,*}(K) \to \overline{H}^{p,*}(M,g). \quad \bullet$$

Remark. The $p = 2$ version for the reduced case has been established by Dodziuk ([4]).

Consider now (M_1^n, g_1), (M_2^n, g_2) with (I) and (B_k) and a uniformly proper map $f \in C^{\infty,k+1}(M_1, M_2)$.

Proposition 6.21. *Let $t_i\colon |K_i| \to M_i$, $i = 1,2$, be uniform triangulations. Then there exists a uniform triangulation K_1' which has bounded degree of subdivision and a simplicial approximation $f'\colon K^1 \to K_2$ of f which is at L_2-chain level local.* $\bullet$

Theorem 6.22. *Let K^n be an oriented u. l. f. combinatorial homology n-manifold. Then there exist isomorphisms*

$$D\colon H^{*,2}(K^n) \to H_{n-*,2}(K^n),$$
$$\overline{D}\colon \overline{H}^{*,2}(K^n) \to \overline{H}_{n-*,2}(K^n). \quad \bullet$$

Combining 6.19–6.22, we obtain

Theorem 6.23. *Assume* (M_i^n, g_i) *with* (I) *and* (B_k), $k > \frac{n}{2} + 1$. *Let* $f \in C^{\infty,k+1}(M_1, M_2)$, $g \in C^{\infty,k+1}(M_2, M_1)$ *uniformly proper homotopy equivalences, inverse to each other. Then* f *and* g *induce topological isomorphisms*

$$H^{2,*}(M_1, g_1) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} H^{2,*}(M_2, g_2),$$

$$\overline{H}^{2,*}(M_1, g_1) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \overline{H}^{2,*}(M_2, g_2),$$

where Φ, Ψ *are induced by simplicial approximations of* f, g, Θ, η, *and* w. $\bullet$

Corollary 6.24. *Assume* (M_i^n, g_i), f, g *as above,* $0 \leqslant q < n$. *Then*

$$\inf \sigma_e(\Delta_q(M_1)|_{(\ker \Delta_q(M_1))^\perp}) > 0 \Longleftrightarrow \inf \sigma_e(\Delta_q(M_2)|_{(\ker \Delta_q(M_2))^\perp}) > 0.$$

Proof. Apply the analytical version of 5.8. Then the spectral gap exists for $\Delta_q(M_1)$ if and only if

$$H^{2,q}(M_1, g_1) = \overline{H}^{2,q}(M_1, g_1), \quad H^{2,q+1}(M_1, g_1) = \overline{H}^{2,q+1}(M_1, g_1).$$

But this holds if and only if it holds for (M_2, g_2). $\bullet$

We will not discuss here the general theory of characteristic numbers for open manifolds, L_2-intersection theory, and general duality. We only mention that for manifolds of bounded geometry duality holds from H^*_{uff} to H_*^{uff} and L_2-bounded duality in the L_2-case. This yields obstructions for the corresponding components to contain a manifold of bounded geometry.

We also defined homology for the rel-components, but refer to [8–10].

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