Записки научных
семинаров ПОМИ
Том 267, 2000 г.

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TRIANGULATIONS OF MANIFOLDS AND COMBINATORIAL BUNDLE THEORY: AN ANNOUNCEMENT

Abstract. For a given compact PL-manifold X, studied is the category $\mathbf{CM}(X)$ of combinatorial-manifold structures on X, whose objects are abstract simplicial complexes with geometric realization PL-homeomorphic to X, and the morphisms are "combinatorial subdivisions." The geometric realization $B\,\mathbf{CM}(X)$ of the nerve of $\mathbf{CM}(X)$ is announced to be homotopy equivalent to the classifying space $B\,\mathrm{PL}(X)$ of the simplicial group $\mathrm{PL}(X)$: $B\,\mathbf{CM}(X) \approx B\,\mathrm{PL}(X)$.

The relationship between different notions of piecewise-linear (PL) topology is a question that dates back to the earliest days of topology. Alexander made a fundamental conjecture of the subject, which still remains open:

Conjecture. *If X and Y are two geometric simplicial complexes with the same underlying space, then X and Y have a common stellar subdivision.*

In the absence of a proof of this conjecture, much hard work has been done to prove weaker results in the combinatorial theory of subdivision. Alexander [1] proved that if the underlying space of X and Y is a compact PL manifold, then X can be transformed to Y by a sequence of stellar subdivisions and inverse stellar subdivisions. Whitehead [8] showed that any compact combinatorial manifold has a stellar subdivision which is a *stellar manifold* (defined in the following section), a result of importance in smoothing theory.

In 1996, Morelli [5] proved the Strong Oda conjecture from the theory of toric varieties. An immediate corollary is a proof of Alexander's conjecture for *rational* triangulations:

[1] Partially supported by a grant from the National Science Foundation.
[2] Partially supported by a grant from the Russian Foundation for Basic Research.

Theorem. *If X and Y are finite geometric simplicial complexes in $\mathbb{R}^n$ with the same underlying space and all vertices of X and Y are rational, then X and Y have a common stellar subdivision.*

Using Morelli's result, we can greatly generalize Alexander's and Whitehead's work. We consider subdivisions not only of individual combinatorial manifolds, but of families of combinatorial manifolds, related by subdivision. For any compact PL manifold M, we define a *category of combinatorial manifold structures on M* to be a category whose objects are abstract simplicial complexes with geometric realization PL homeomorphic to M (not necessarily all such complexes) and whose morphisms are combinatorial types of PL subdivision maps between them (not necessarily all such types), subject to certain closure conditions. The categories satisfying these conditions include many of the types of triangulation and subdivision that have been of greatest importance in PL topology. We show that for a given M, all such categories are homotopy equivalent, and have classifying space $BPL(M)$.

We apply this result to get a new theory of combinatorial bundles. For any category $\mathcal{C}$ of combinatorial manifold structures on M, we define a combinatorial bundle theory with classifying space $B\mathcal{C}$ and show this theory to be equivalent to the theory of PL M-bundles. We hope this combinatorial approach will have further applications in PL bundle theory.

§1. Combinatorial manifolds and PL subdivision

Triangulations of piecewise-linear (PL) manifolds, other PL cell complex decompositions of manifolds, and various PL notions of subdivision have been of interest in both classical and recent contexts. A central question has been the relationship between more rigid (and often more useful) PL notions of manifold and subdivision versus more general (and wild) ones. Most generally, one can consider the category of all *combinatorial manifolds* (i.e., abstract simplicial complexes in which the star of every simplex can be geometrically realized as a PL ball) and all combinatorial types of PL subdivisions. This category is problematic for several reasons: it lacks a combinatorial characterization, its objects are not algorithmically recognizable [7], and it lacks sufficient structure to carry out many of the most interesting applications. Some more restrictive notions that have been considered are:

- In [1], Alexander considered the equivalence relation on simplicial complexes generated by stellar subdivisions. He defined an n-

dimensional "manifold" to be an abstract simplicial complex in which the link of any vertex is equivalent to the boundary of an n-simplex. For such manifolds, he developed an elegant algebraic theory that shows that any two such manifold structures on a given compact PL manifold are equivalent – i.e., are related by a sequence of stellar subdivisions and inverse stellar subdivisions. Finally, he showed that this notion of equivalence coincides with the modern notion of PL equivalence on geometric simplicial complexes.

- Cairns [3] developed a smoothing theory for n-dimensional *Brouwer manifolds*, abstract simplicial complexes in which the star of any simplex can be realized as a rectilinear subdivision of an n-simplex. He also showed that not every combinatorial manifold is a Brouwer manifold [4], but was not able to show that every PL manifold has a Brouwer manifold structure. However,
- Whitehead [8] later proved that any compact combinatorial manifold has a stellar subdivision which is a Brouwer manifold. In fact, his proof shows that any n-dimensional combinatorial manifold has a stellar subdivision which is a *stellar manifold*, i.e., the star of any simplex is simplicially isomorphic to a stellar subdivision of an n-simplex.

We announce here results that put all of these notions into a common categorical framework and generalize Alexander's and Whitehead's results.

An abstract simplicial complex X with vertex set S has a canonical geometric realization in $\mathbb{R}^S$, which we denote by $|X|$. A *subdivision* of $|X|$ is a homeomorphism of simplicial complexes $f\colon |Y| \to |X|$ such that

- f maps each simplex of $|Y|$ into a simplex of $|X|$,
- f is linear on each simplex, and
- for every 0-simplex x of $|X|$, $f^{-1}(x) = x$.

A subdivision $f\colon |Y| \to |X|$ determines a poset map $\tilde{f}\colon Y \to X$ sending each simplex $\sigma \in Y$ to the smallest $\tau \in X$ such that $f(|\sigma|) \subseteq |\tau|$. We call any map arising in this way a *subdivision map*. Let M be a compact PL manifold. A *category of combinatorial manifold structures on M* is a category with objects Obj and morphism Mor satisfying the following conditions:

1. Every $X \in \mathrm{Obj}$ is a combinatorial manifold structure on M. Obj contains all stellar manifold structures on M, and if $X \in \mathrm{Obj}$, then

any simplicial complex obtained from X by a sequence of truncations and stellar subdivisions is also in Obj.

2. Mor is a set of subdivision maps which includes all stellar subdivision maps between elements of Obj.

Recall that the **classifying space** $B\mathcal{C}$ (cf. [Q]) of a small category $\mathcal{C}$ is the simplicial cell complex defined as follows. For every chain $c = \{s_0 \xrightarrow{f_1} s_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} s_n\}$ in $\mathcal{C}$ with each $f_i \neq \mathrm{id}$, let Δ_c be an n-dimensional simplicial cell with vertices labelled by the s_i and each edge $\{s_i, s_j\}$ with $s_i < s_j$ labelled by $f_j \circ \cdots \circ f_{i+1}$. $B\mathcal{C}$ is then the cell complex obtained from $\coprod_c \Delta_c$ by identifying faces with identical labelled 1-skeleta. Thus, Alexander's result can be interpreted as a result on connectedness of certain categories of combinatorial manifold structures, and Whitehead's result says $\pi_0(B\mathcal{C}, B\mathcal{D}) = 0$ for certain pairs $\mathcal{D} \subset \mathcal{C}$ of categories of combinatorial manifold structures. Our main theorem generalizes these results to entire homotopy types.

Theorem. *Let M be a compact PL manifold.*

1. *If $\mathcal{C}$ is a category of combinatorial manifold structures of M then $B\mathcal{C} \simeq BPL(M)$, the classifying space for the group of PL automorphisms of M.*
2. *If $\mathcal{C} \subset \mathcal{D}$ are two categories of combinatorial manifold structures on M then the inclusion $B\mathcal{C} \hookrightarrow B\mathcal{D}$ is a homotopy equivalence.*

A key tool in the proof is Morelli's proof of the Strong Oda Conjecture [5, 2]. One corollary of Morelli's result is:

Theorem. *If X and Y are finite geometric simplicial complexes in $\mathbb{R}^n$, $|X| = |Y|$, and all vertices of X and Y are rational, then X and Y have a common stellar subdivision.*

This allows us to apply stellar subdivisions to reduce many general simplicial PL spaces and maps to more tractible ones.

The idea of the proof is as follows. For any chain $c = \{X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} X_k\}$ in $\mathcal{C}$, consider the **Grothendieck construction** over c:

$$E(c) = \{(X_i, s) | X_i \in c, s \in X_i\}.$$

$E(c)$ is a poset under the order that $(X_i, s_i) \leq (X_j, s_j)$ if $i \leq j$ and $f_j \circ f_{j-1} \circ \cdots \circ f_{i+1}(s_i) \subseteq s_j$.

Theorem. *The map of geometric simplical complexes*

$$\begin{array}{rccc} p(c): & BE(c) & \rightarrow & \Delta_c \\ & (X_i, s) & \rightarrow & X_i \end{array}$$

is a PL M-bundle.

We construct $BPL(M)$ as the geometric realization of a Kan Δ-set. The k-simplices of this Δ-set are simplicial PL M-bundles over Δ_k, modulo certain equivalences. The above theorem allows us to embed $B\mathcal{C}$ into this Δ-set by sending each chain c to $p(c)$. We use Morelli's result to retract the entire Δ-set to this embedding.

§2. Combinatorial bundle theories

For any category $\mathcal{C}$ of combinatorial manifold structures on M, we define a *combinatorial bundle theory* associated to $\mathcal{C}$ and show this theory to be closely related to PL M-bundles.

View a simplicial complex as a category, with a morphism $b_1 \rightarrow b_2$ for every relation $b_1 \supseteq b_2$.

Definition. *Let $\mathcal{B}$ be a locally finite simplicial complex. A pre-$\mathcal{C}$-bundle over $\mathcal{B}$ is a stellar subdivision map $s : \mathcal{B}' \rightarrow \mathcal{B}$ and a functor $F : \mathcal{B}' \rightarrow \mathcal{C}$.*

Thus, a pre-$\mathcal{C}$-bundle over $\mathcal{B}$ is a subdivision map, an assignment to each $b \in \mathcal{B}'$ of a combinatorial manifold $F(b)$ with geometric realization PL homeomorphic to M, and, for every relation $b_1 > b_2$ in $\mathcal{B}'$, a way of viewing $F(b_1)$ as a subdivision of $F(b_2)$. We view $|F(b)|$ intuitively as the "fiber" over b. Indeed, we can use the Grothendieck construction to construct a PL M-bundle

$$\pi_1 : B\{(b, s) | b \in \mathcal{B}', s \in F(b)\} \rightarrow B\mathcal{B}'.$$

The base space of this bundle is the barycentric subdivision of $\mathcal{B}'$, and the fiber over each vertex b of this barycentric subdivision is the barycentric subdivision of $|F(b)|$.

We generate an equivalence relation on pre-$\mathcal{C}$-bundles as follows: if $s' : \mathcal{B}'' \rightarrow \mathcal{B}'$ is a stellar subdivision map, then the pre-$\mathcal{C}$-bundles $(s : \mathcal{B}' \rightarrow \mathcal{B}, F : \mathcal{B}' \rightarrow \mathcal{C})$ and $(s \circ s' : \mathcal{B}'' \rightarrow \mathcal{B}, F \circ s' : \mathcal{B}' \rightarrow \mathcal{C})$ are equivalent.

Definition. *A $\mathcal{C}$-bundle is an equivalence class of pre-$\mathcal{C}$-bundles.*

For any representative $(s : \mathcal{B}' \rightarrow \mathcal{B}, F : \mathcal{B}' \rightarrow \mathcal{C})$ for a $\mathcal{C}$-bundle ξ, we view F intuitively as a "classifying map". We can make this last piece of intuition precise by considering the induced topological map of classifying spaces $BF \circ (Bs)^{-1}\colon B\mathcal{B} \to B\mathcal{C}$. Equivalent pre-morphisms give homotopic maps. We call the resulting homotopy class of maps the *classifying map* for ξ. Under an appropriate definition of bundle morphism, we find:

Theorem. *For every locally finite simplicial complex $\mathcal{B}$, let $\mathcal{C}^{\mathcal{B}}$ be the set of strong isomorphism classes of $\mathcal{C}$-bundles over $\mathcal{B}$. Then there exists a bijection, natural in $\mathcal{B}$,*

$$\mathcal{C}^{\mathcal{B}} \longleftrightarrow [B\mathcal{B}, B\mathcal{C}].$$

The map from $\mathcal{C}^{\mathcal{B}}$ to $[B\mathcal{B}, B\mathcal{C}]$ is obtained by sending each equivalence class $[F\colon B \to \mathcal{C}]$ to the equivalence class of BF.

Thus, $B\mathcal{C}$ is a classifying space for $\mathcal{C}$-bundles and the maps BF are classifying maps.

Since $B\mathcal{C} \simeq BPL(M)$, it follows that:

Corollary. *Let $\mathcal{B}$ be a locally finite simplicial complex. There is a bijection, natural in $\mathcal{B}$,*

$$\mathcal{C}^{\mathcal{B}} \longleftrightarrow \{\textit{strong isomorphism classes of } PLM - \textit{bundles over } \mathcal{B}\}.$$

The forward map takes the strong isomorphism class of a pre-$\mathcal{C}$-bundle $(s : \mathcal{B}' \to \mathcal{B}, F : \mathcal{B}' \to \mathcal{C})$ to the strong isomorphism class of the PL bundle obtained from F by the Grothendieck construction.

For appropriately restrictive choices of $\mathcal{C}$, the category of $\mathcal{C}$-bundles seems substantially more tractible than the category of PL M-bundles. In particular, when $M = S^{n-1}$ and $\mathcal{C}$ is the category of all complexes isomorphic to stellar subdivisions of the boundary of Δ_n, the category of $\mathcal{C}$-bundles is a purely combinatorial model for the category of PL sphere bundles.

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Поступило 29 октября 1999 г.

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