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POLYNOMIAL INVARIANTS OF KNOTS AND LINKS

ABSTRACT. We consider a class of knots corresponding to the so-called d-diagrams with m vertical and n horizontal chords and calculate the set of their invariant polynomials.

§1

We begin with a new point of description of knots, introduced recently in [1, 2].

It has been shown that any knot or link can be encoded by a so-called d-diagram as explained below.

Definition. *A d-diagram is a chord diagram with two sets of non-intersecting chords.*

Each d-diagram generates a knot or a link according to the following rule.

Each chord is determined by its two ends. We connect the chord ends of the first set by continuous lines (arcs) inside the circle and the chord ends of the second set by the lines (arcs) outside the circle. It is obvious that our lines (arcs) can be drawn in such a way that no two lines intersect each other. Let the interior of the circle be white and let the exterior of the circle be black. After that, we consider the homotopy of our circle with constructed lines (arcs) as follows. During the homotopy, the ends of arcs move towards each other along the arcs, involving the small part of neighborhood of each end. At the last moment of homotopy, the arc ends from each pair coincide and there are no intersections except these coincidences.

As a result, we obtain a 4-graph embedded in the plane. And the plane is painted with black-white colors. (Paintings are inherited by the white interior and black exterior of the circle.)

In order to transform the 4-graph with black-white painting into a knot or link, it only remains to establish over- and undercrossing at each cross of the graph.

This can be done as follows. At each vertex, two opposite edges generated by the arcs of the first set determine an overcrossing if the black area is on the right when we move along the edge towards the vertex. Conversely, they determine an undercrossing if the black area is on the left side when we move along the edge towards the vertex. Let the arc belong to the second set. In this case, we determine an overcrossing at the corresponding vertex when white area is on the right on the way along the edge towards the vertex and we determine an undercrossing when the white area is on the left.

Examples. Figure 1 represents a d-diagram. Figure 2 shows the final moment of the homotopy described above. Figure 3 gives the scheme of the knot, i.e., a 4-graph on the plane with over- and undercrossings determined according to the rule above.

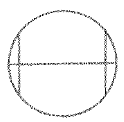


Fig. 1.

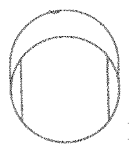


Fig. 2.

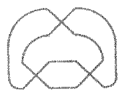


Fig. 3.

Now we are going to consider the following class of d-diagrams. (In our opinion, it is rather natural and simple.)

Definition. *An $S(p,q)$ diagram is a d-diagram with p horizontal chords (the first set) and q vertical chords (the second set) such that each horizontal chord intersects each vertical chord.*

Examples. Figure 4 shows an $S(p,q)$ diagram with $p = 4$ and $q = 2$. The d-diagram in Fig. 5 is not an $S(p,q)$ diagram.

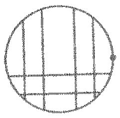


Fig. 4

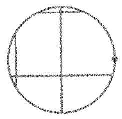


Fig. 5.

Proposition. *If pq is odd, then an $S(p,q)$ diagram defines a link with two components; if pq is even, then an $S(p,q)$ diagram defines a knot. A diagram of the knot (or link with two components) corresponding to the $S(p,q)$ diagram is shown on Fig. 8.*

Proof. Let the circle be oriented counterclockwise. First of all, we consider a d-diagram containing only vertical chords. According to the rule above ,we obtain the following diagram of knot.

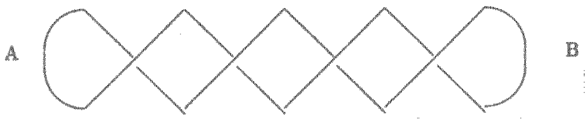


Fig. 6.

If we add q horizontal chords to our p vertical chords on the diagram, we see that the ends of the horizontal chords on the diagram belong to the arc A (the left ends) and the the arc B (the right ends), see Fig 7.

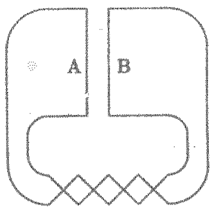


Fig. 7

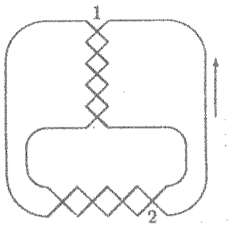


Fig. 8.

This means that the diagram of the knot (or link) corresponding to the d-diagram $S(p,q)$ looks like that in Fig. 8 (including under- and overcrossing on the knot diagram settled by the rule described above; overcrossings are shown by thick lines).

It is easy to check now that when pq is even the scheme defines a knot and when pq is odd the scheme defines a link with two components. •

§2

Now, we calculate some of the invariant polynomials (such as Conway polynomials, Jones polynomials, etc.) for knots with scheme $S(p,q)$.

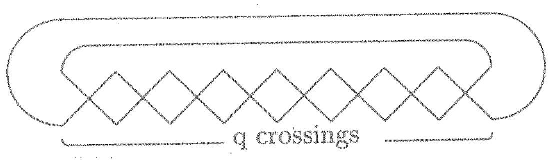
Since the Conway polynomials are defined for oriented knots and links, we establish an orientation by means of arrow in Fig. 8. Taking into consideration the "skein relations," which are as follows:

Fig. 9.

Here, the difference on the left-hand side means the difference between the Conway polynomial of the given knot (or link) and the Conway polynomial of the knot which differs from the given knot only in a neighborhood of the crossing in the way presented in Fig. 9. The right-hand side means the Conway polynomial of the given knot changed in the neighborhood of the crossing as shown in Fig. 9.

We mark each crossing on the scheme of the knot by "+" if a neighborhood of the crossing looks like the first term in Figure 9 and by "−" if it does not. We call these crossings "positive" and "negative," respectively.

We denote by T_q, $q = 1, 2, \dots$, the knot (q is odd) and the two-component link (q is even) described by the diagram:



We choose the orientation of the scheme on Fig. 8 by means of an arrow if pq is even and by two arrows if pq is odd.

It is easy to verify that the upper crossing on Fig. 8 is marked by $(-1)^p$ and the right crossing at the bottom is marked by $(-1)^q$.

To calculate the Conway polynomials $s(p,q)$ of $S(p,q)$, we take the upper crossing on Fig. 8 and apply the skein relations to it.

If pq is even, we have

$$s(p,q) = s(p,q-2) + xt(p), \tag{1}$$

where $t(p,q)$ means the Conway polynomials of $T(p,q)$ with orientation $(-1)^q$.

Equality (1) is a recurrent relation of the second order, so we can find $s(p,q)$ if we know $s(p,0), s(p,1)$, and $t(p)$.

It is easy to see that $t(p)$ satisfies

$$t(p) = t(p-2) + x \tag{2}$$

if p is even and

$$t(p) = t(p-2) - xt(p-1) \tag{3}$$

if p is odd. So, $t(p) = q/2x$ if p is even and $t(p) = \rho(p-1)$ if p is odd. Here, $\rho(m)$ is the mth term of the sequence having two initial terms 0 and 1 and satisfying the recursion $\rho(m) = \rho(m-1) - x\rho(m-1)$, $m = 2, 3, \ldots$. So, for small m we have

$$\begin{aligned}
&\rho(0) = 0, && \rho(1) = 1,\\
&\rho(2) = -x, && \rho(3) = x^2 + 1,\\
&\rho(4) = -x^3 - 2x, && \rho(5) = x^4 + 3x^2 + 1,\\
&\rho(6) = -x^5 - 4x^3 - 3x, && \rho(7) = x^6 + 5x^4 + 6x^2 + 1,\\
&\rho(8) = -x^7 - 6x^5 - 10x^3 - 4x, && \rho(9) = x^8 + 7x^6 + 15x^4 + 10x^2 + 1,
\end{aligned}$$

$$\rho(10) = -x^9 - 8x^7 - 21x^5 - 20x^3 - 5x. \tag{4}$$

It can be shown that

$$\rho(m) = \frac{\lambda_1^m - \lambda_2^m}{\lambda_1 - \lambda_2}, \tag{5}$$

where λ_1 and λ_2 are the roots of the quadratic equation

$$\lambda^2 - x\lambda - 1 = 0. \tag{6}$$

If p is even and q is odd, then

$$s(p,q)=s(p,q-2)-xs(p,q-1),\quad s(p,0)=1,\quad s(p,1)=t(p-1). \quad (7)$$

If p is odd and q is even, we have

$$s(p,q)=s(p,q-2)-xt(p),\quad s(p,0)=1,\quad s(p,1)=t(p-1). \qquad (8)$$

The first equalities in (7) and (8) follow from the skein relation, the second equalities are obvious, and the third equalities follow from the fact that $S(p,1)$ and $T(p)$ determine equivalent knots.

Thus,

$$s(p,q)=\rho(p-2)+\frac{q-1}{2}x\rho(p-1),\quad p \quad \text{even},\quad q \quad \text{odd}, \qquad (9)$$

$$s(p,q)=\rho(q-2)+\frac{p-1}{2}x\rho(q-1),\quad p \quad \text{odd},\ q \quad \text{even}. \qquad (10)$$

If p and q are even, then

$$s(p,q)=s(p,q-2)+xt^+(p),\quad s(p,0)=1,\quad t^+(p)=\frac{p}{2}x,$$

whence

$$s(p,q)=1+\frac{pq}{4}x^2. \qquad (11)$$

Finally, let p and q be odd. Then the scheme in Fig. 8 determines a two-component link. To obtain the oriented link, we must show the orientation of both components. The first component is oriented by means of an arrow in Fig. 8. The second component can be oriented in two ways. First of them makes the upper crossing in Fig. 8 positive and the second makes it negative. As for crossing marked by 2 in Fig. 8, it will be negative with the first orientation and positive with the second orientation.

If crossing 1 is positive, we have

$$s(p,q)=s(p,q-2)+xt(p), \qquad (12)$$

whence

$$s(p,q)=s(p,1)+\frac{q-1}{2}xt(p).$$

Using relations $S(p,1)=T(p-1)$ and $t(p)=\rho(p-1)$, we obtain

$$s(p,q)=\rho(p-2)+\frac{q-1}{2}x\rho(p-1). \qquad (13)$$

Similarly, if crossing 1 is negative, we have

$$s(p,q) = s(p,q-2) - xs(p,q-1), \quad s(p,0) = 1, \quad s(p,1) = t(p-1). \tag{14}$$

whence

$$s(p,q) = \rho(q-2) + \frac{p-1}{2} x\rho(q-1). \tag{15}$$

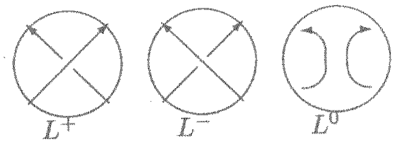
Theorem. *Conway polynomials of* $S(p,q)$ *are described by* (9)–(11), (13), (15). $\bullet$

§3

Now we calculate other polynomial invariants of knots (links) such as Jones polynomials, HOMFLY polynomials, Alexander polynomials, etc. Each of them satisfies skein relations of the form

$$au(L^+) - bu(L^-) = cu(L^0), \tag{16}$$

where L^+, L^-, and L^0 are the small parts of the knot scheme:



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In addition, we have the following relations:

$$u(\mathbf{1}) = 1; \quad u(L \sqcup \mathbf{1}) = u(L)d, \tag{17}$$

where $\mathbf{1}$ means the trivial knot and $\sqcup$ means a trivial link of the knot L and the trivial knot $\mathbf{1}$.

The factors a, b, and c are functions of one variable x or two variables x and t.

In the case of Jones polynomials, $a = x^{-1}$, $b = x$, $c = x^{-1/2} - x^{1/2}$;

In the case of Jones polynomials of two variables, $a = \frac{\sqrt{(xt)}}{}$, $b = -\sqrt{(xt)}$, $c = \frac{\sqrt{(t)}-1}{\sqrt{(t)}}$;

For the HOMFLY polynomials, $a = x$, $b = -t$, $c = 1$.

For the Alexander polynomials, $a =?$, $b = 1$, $c = \frac{1}{1-x^2}$.

For the Kauffman polynomials, $a = x^{-1}$, $b = x$, $c = x^{1/2} - x^{-1/2}$.

In all cases, $d = \frac{a-b}{c}$.

We define

$$\mu(k) = \frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1 - \lambda_2}. \tag{18}$$

It is well known that if a sequence $v(k)$, $k = 0, 1, 2, \ldots$, satisfies the equation $av(k) - bv(k-2) - cv(k-1) = 0$ and the conditions $v(0) = v_0$, $v(1) = v_1$, then

$$v(k) = \delta v_0 \mu(k-2) + v_1 \mu(k-1), \tag{19}$$

where λ_1 and λ_2 are roots of the equation $a\lambda^2 - c\lambda - b = 0$ and $\delta = b/a$.

Let p be odd. We apply (17) to $S(p,q)$ at crossing 1 (see Fig. 8). It is easy to see that crossing 1 is negative. Hence,

$$au(p,q) - bu(p,q-2) = cu(p,q-1) \tag{20}$$

and by (19) (putting $v(k) = u(S(p,k))$ we have

$$\begin{gathered} u(S(p,q)) = \delta u_0 \mu(q-2) + u_1 \mu(q-1), \\ p = 0, 1, 2, \ldots, \quad q = 0, 1, 2, \ldots, \end{gathered} \tag{21}$$

where

$$u_0 = u(S(p,0)) = 1, \quad u_1 = u(S(p,1)) = u(T(p-1)), \quad \delta = a/b.$$

The orientation of $T(p-1)$ is determined by the sign of crossing 2, namely:

if q is even, then crossing 2 is negative;

if q is odd, then $T(p-1)$ is a two-component link and it has two orientations: one with negative crossing 2 and the second with positive crossing 2. We consider the first case in which crossing 2 is negative (the second case will be considered later; we denote by $T(p)^+$ the scheme $T(p)$ with the second orientation). Under our assumption,

$$\begin{gathered} au(T(p-1)) = bu(T(p-3)) - cu(T(p-2)), \\ u_0 = u(T(0)) = d, \qquad u(T(1)) = 1. \end{gathered} \tag{22}$$

Remark. It is clear that $u(T(1)) = 1$.

The schemes $T(0)$, $T(1)$, and $T(2)$ satisfy $au(T(2)) - bu(T(0)) = cu(T(1))$. But $T(0)$ is obtained by changing an overcrossing by undercrossing in the scheme $T(2)$. So, $T(0)$ is a trivial two-component link, whence $u(T(0)) = du(\mathbf{1}) = d$. $\bullet$

From (19) and (22), we have

$$u(S(p,q)) = \delta d\mu(q-2) + \mu(q-1). \tag{23}$$

Now, let p be even. According to (19), taking into account that crossing 1 is positive, we have

$$au(S(p,q)) - bu(S(p,q-2)) - cu(T(p)) = 0$$

or

$$v(q) = \alpha v(q-2) + \beta v(0), \tag{24}$$

where

$$v(q) = u(S(p,q)), \quad \alpha = b/a, \quad \beta = c/a. \tag{25}$$

If q is even, then (23) and (24) imply

$$v(q) = \alpha^{q/2} + \beta\frac{\alpha^{q/2}-1}{\alpha-1}\,v(0) \tag{26}$$

or

$$u(S(p,q) = (a/b)^{q/2} + (c/a)\frac{(a/b)^{q/2}-1}{a/b-1}u(T(p)^{+}). \tag{27}$$

If p is even and q is odd, we have

$$au(S(p,q)) - bu(S(p,q-2) = cu((T(p)),$$
$$u(S(p,0)) = 1, \quad u(S(p,1)) = u(T(p-1)), \tag{28}$$

whence

$$u(S(p,q)) =$$
$$= \left((b/a)^{(q-1)/2}u(T(p-1)) + c/a\frac{(b/a)^{(q-1)/2}-1}{b/a-1}\right)u(T(p)). \tag{29}$$

If p is odd and q is even, we have

$$u(S(p,q)) = \alpha^{q/2} + \beta\frac{\alpha^{q/2}-1}{\alpha-1}T(p-1)^{+}. \tag{30}$$

If p is odd and q is odd, the scheme $S(p,q)$ represents a two-component link and admits two orientations: the first one with negative crossing 1 and positive crossing 2, the second one with positive crossing 1 and negative crossing 2. We denote them by $S(p,q)^{-}$ and $S(p,q)^{+}$, respectively.

Using (21) and (24), we have

$$u(S(p,q)^-) = a/b\mu(q-2) + u(T(p-1)^+)\mu(q-1) \tag{31}$$

and

$$u(S(p,q)^+) = \alpha^{(p-1)/2}u(T(p-1)) + \beta\frac{\alpha^{(p-1)/2}-1}{\alpha-1}u(T(p-1)). \tag{32}$$

§4

Methods of computing invariant polynomials that have been applied to knots or links of type $S(p,q)$ can be applied to a much larger set of knots and links. We describe this set inductively and denote it by W. First, we give some definitions.

Two non-intersecting chords of a d-diagram D are called *closest parallel* if one end of the first chord is adjacent to one end of the second chord and the other end of the first chord is adjacent to the other end of the second chord.

A subset of chords is a *parallel subset* if (after suitable numeration) each next chord is a closest parallel chord to the previous one.

A *proper intersection* of a parallel set $Y = \{A_1B_1, A_2B_2, \dots, A_mB_m\}$ is a chord CD which intersects all chords of Y and one of the arcs between C and D contain only ends of chords belonging to Y (it does not matter whether the chord CD belongs to our diagram).

In order to construct the set W, we start with an $S(p,q)$-knot or link diagram K. We take a subset $V = (AB, A_1B_1, A_2B_2, \dots, A_lB_l)$, $l \geqslant 1$, of parallel chords of K and consider a proper intersection C_1D_1 which is not a chord of the diagram K. We consider the subset $V_1 = (A_1B_1, A_2B_2, \dots, A_lB_l)$ of V and then construct a set V_1 of chords not belonging to K: $C_2D_2, C_3D_3, \dots, C_mD_m$, so that C_1D_1 together with the added chords represents a parallel subset. Denote by K_1 the chord diagram obtained by adding to K all chords of the set V_1; denote by S_1 the set of all diagrams which can be obtained by using the rule above (rule A).

Similarly, starting with S_1 and using rule A, we obtain a set S_2 and then $S_3, S_4, \dots, S_m$. We put

$$S = \bigcup_i S_i.$$

We show how to calculate invariant polynomials of knots and links belonging to S_j if these polynomials are known for knots and links belonging to S_{j-1}, $j = 1, 2, \dots$.

We consider a knot (or a link) K in S_{j-1} and choose a parallel subset $V = (AB,\, A_1B_1,\, A_2B_2, \dots, A_lB_l)$ consisting of $l+1$ chords; we construct a parallel set $\{C_1D_1,\, C_2D_2,\, \dots,\, C_mD_m\}$ so that C_1D_1 is a proper intersection of the subset $(A_1B_1,\, A_2B_2, \dots, A_lB_l)$, add chords of this set to our diagram, and denote the obtained diagram by $K(l, m)$.

For the beginning, we consider the set of diagrams $K(1, m)$, $m = 0, 1, 2, \dots$. It is easy to see that all chords of it (all crossings of the corresponding knots or links) have the same sign, i.e., all crossings are positive or all crossings are negative.

In the case of the plus sign, the invariant polynomials $k(1, m)$ of knots or links with diagram $K(1, m)$ satisfy the equation

$$ak(1, m) - bk(1, m - 2) = c(k(1, 0)). \tag{33}$$

In the case of the minus sign, we have

$$bk(1, m) - ak(1, m - 2) = -ck(1, m - 1). \tag{34}$$

In both cases, we have an equation of the second order in finite differences with respect to $k(1, m)$. It is well known that $k(1, m)$ can be directly expressed via m if we are given $k(1, 0)$ and $k(1, 1)$. The first of the polynomials is connected with the diagram $K(1, 0)$ which is equivalent to K from S_{j-1}. We know this by the inductive hypothesis. In fact, the diagram $K(1, 0)$ differs from the diagram K only by one additional chord β which is the closest parallel to a chord α belonging to K. We notice that the second polynomial $k(1, 1)$ is connected with the diagram $K(1, 1)$ obtained from K by adding the chord β and, furthermore, the chord γ, which is a proper intersection of the set α. We consider the diagram K_β obtained by adding the chord β to K and the diagram $K_{-\beta}$ obtained by adding to K the chord $-\beta$ which is β with opposite sign with respect to α. Let k_β and $k_{-\beta}$ denote the invariant polynomials of K_β and $K_{-\beta}$, respectively. Using the skein relations, we easily check that

$$ak_\beta - bk_{-\beta} = ck(1, 1), \qquad \alpha > 0, \tag{35}$$

$$ak_{-\beta} - bk_\beta = ck(1, 1), \qquad \alpha < 0. \tag{36}$$

But the diagrams K_β and $K_{-\beta}$ belong to $S(j - 1)$. Indeed, the chords α and $-\beta$ can be omitted in the diagram $K_{-\beta}$ because the chords α and β are closest parallel and hence the two corresponding crossings on the diagram of a knot vanish due the second Reidemeister move.

Therefore, $k(1, 1)$ can be found from (35) and (36).

Finally, we show how to calculate $k(l,m)$-invariant polynomials corresponding to the knot (or link) $K(l,m)$.

By the skein relations, we have

$$bk(l,m) - ak(l-2,m) = ck(l-1,m), \qquad \alpha > 0, \tag{37}$$

$$ak(l,m) - bk(l-2,m) = ck(0,m), \qquad \alpha < 0, \tag{38}$$

and, furthermore, $k(0,m) = k$; $k(1,m)$ has just been calculated. This information is sufficient for obtaining explicit formulas for $k(l,m)$.

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