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TWO THEOREMS OF ROKHLIN

ABSTRACT. Two theorems due to V. A. Rokhlin are proved: on the third stable homotopy group of spheres: $\pi_{n+3}(S^n) \approx \mathbb{Z}_{24}$ for $n \geqslant 5$; and on the divisibility by 16 of the signature of a spin 4-manifold. The proofs use immersion theory.

Here, we prove two theorems of V. A. Rokhlin. Our proofs rely mainly on immersion theory due to Smale and Hirsch. This theory, when it was born, was considered as mysterious, giving the ununderstandable result of turning the 2-sphere inside out. But now, 40 years later the basic results of the theory have very simple proofs: see the book by Gromov [2], Thurston's corrugations [10], and finally the simplest and most recent proof given by Rourke and Sanderson [14]. So, our aim here is to show that the positive changes in understanding immersion theory give also a profit in understanding some famous theorems of Rokhlin, which were also considered as very hard 50 years ago.

§1. The theorems

Here are the two theorems we would like to prove.

Theorem 1.1 (Rokhlin). $\pi_{n+3}(S^n) \approx \mathbb{Z}_{24}$ *if* $n \geqslant 5$.

Theorem 1.2 (Rokhlin). *The signature of any spin 4-manifold is divisible by 16.*

We prove Theorem 1.1 and obtain Theorem 1.2 as a byproduct.

Lemma 1.3 (Hirsch lemma). *If $k > 0$, then an immersion $f\colon M^n \to \mathbb{R}^{n+k+1}$ with a nonzero normal vector field is regularly homotopic to an immersion with vertical normal field. As a result, this immersion can be projected down to the horizontal hyperplane, remaining an immersion.*

Proof. See [5], cf. [14]. •

Lemma 1.4. *The cobordism group of immersions of n-manifolds into $\mathbb{R}^{n+k}$ with normal bundle induced from a k-dimensional vector bundle ξ is isomorphic to the cobordism group of immersions of n-manifolds into $\mathbb{R}^{n+k+1}$ with normal bundle induced from $\xi \oplus \varepsilon^1$, where ε^1 denotes the trivial line bundle.*

Proof. This is proved by Wells with the help of the Hirsch theory, see [17]. •

Lemma 1.5. *The cobordism group* $\mathrm{Imm}(2,1)$ *of immersed surfaces in $\mathbb{R}^3$ is isomorphic to $\mathbb{Z}_8$.*

Proof. First this was announced by Wells. For a simple proof see Pinkhall [13]. If the surfaces and the cobordisms are oriented, then the corresponding group – denoted by $\mathrm{Imm}^{SO}(2,1)$ – is $\mathbb{Z}_2$, as follows from Pontryagin's work on the second stable homotopy group of spheres combined with Lemma 1.4. The isomorphism $\mathrm{Imm}^{SO}(2,1) \approx \mathbb{Z}_2$ is given by the Arf invariant associated to an immersed surface. The well-known Brown invariant (see [11], §4 or [13]) extends the Arf invariant to the unoriented case and establishes the isomorphism stated in Lemma 1.5. •

Lemma 1.6. *The third stable homotopy group of spheres $\pi^s(3)$ is isomorphic to the cobordism group* $\mathrm{Imm}^{SO}(3,1)$ *of immersions of oriented 3-manifolds into $\mathbb{R}^4$.*

Proof. This immediately follows from Lemma 1.4 and the usual Pontryagin construction. •

Lemma 1.7. (a) *There is an epimorphism $\pi^s(3) \to \mathbb{Z}_8$.*
(b) *There is an epimorphism $\pi^s(3) \to \mathbb{Z}_3$.*

Proof. (a) Let $f\colon M^3 \to \mathbb{R}^4$ be any oriented immersion. Since f has a normal vector field, the double-point surface (which is also immersed in $\mathbb{R}^4$) also admits a normal vector field – namely the sum of the two normal vector fields to the two branches of f meeting at the given double point. It is not hard to see that this normal vector field can also be defined at the multiple points of the immersed double-point surface. By Lemma 1.4 this surface with normal field in $\mathbb{R}^4$ defines an element of $\mathrm{Imm}(2,1) \approx \mathbb{Z}_8$. This gives a well-defined homomorphism $\Delta_2\colon \pi^s(3) \to \mathbb{Z}_8$.

We must show that Δ_2 is an epimorphism. This follows from Koschorke's "figure-8 construction," which we recall now. Let $g\colon F^2 \looparrowright \mathbb{R}^3$ be an immersion. We fix $x \in F^2$ and denote by $l(x)$ the line in $\mathbb{R}^3$ going through the point $g(x)$ and orthogonal to $dg_x(T_xF^2)$. Let e_4 be the fourth coordinate vector in $\mathbb{R}^4$ orthogonal to $\mathbb{R}^3$. Let $e(x)$ be the line through $g(x)$ in $\mathbb{R}^4$ parallel to e_4. In the plane $N(x)$ generated by the lines $e(x)$ and $l(x)$, we consider a figure 8 symmetric with respect to these lines and standing "vertically" (i.e., thrice intersecting $e(x)$). If we choose such a figure 8 of the same size in each plane $N(x)$ as x runs over F^2, then these figures sweep the image of the immersion of an oriented 3-manifold in $\mathbb{R}^4$. This immersion may not be self-transverse. We make it self-transverse by a small perturbation and denote the resulting immersion by $\tilde{f}$.

If the immersion $g\colon F^2 \looparrowright \mathbb{R}^3$ of the original surface can be made an embedding by a small perturbation in $\mathbb{R}^4$, then g is the double-point surface of $\tilde{f}$. If the immersion g composed with the inclusion $\mathbb{R}^3 \subset \mathbb{R}^4$ (and made self-transverse) has double points, then around each double point the surface of double points of $\tilde{f}$ has some extra components, which are immersed tori. Thus, starting from an immersion $g\colon F^2 \looparrowright \mathbb{R}^3$ and applying the figure-8 construction, we obtain an immersion $\tilde{f}\colon M^3 \looparrowright \mathbb{R}^4$ with double-point surface equal to g plus some immersed tori. Therefore, it is sufficient to show that any framed immersed surface in $\mathbb{R}^4$ belongs to the image of the "double-point map"

$$\Delta_2\colon \pi^s(3) \to \mathbb{Z}_8.$$

The cobordism group of framed immersed surfaces in $\mathbb{R}^4$ is isomorphic to $\pi^s(2) \approx \mathbb{Z}_2$. Here, the generator can be represented by an embedded framed torus in $\mathbb{R}^4$. Applying to it the figure 8 construction, we see that its class belongs to the image of the map Δ_2, hence (a) is proved.

Part (b): This part is not necessary for the proof. The epimorphism $\pi^s(3) \to \mathbb{Z}_3$ was constructed by T. Ekholm [1] as follows. Represent an element of $\pi^s(3)$ by a framed immersion of a 3-manifold in $\mathbb{R}^5$. Along the double curves, we have at each point x four normal vectors. Their order may change (when x runs along a closed double curve and returns), but the sum of these vectors is a well-defined element in the normal 4-space of the double curve, and hence it can be considered as an element in a tubular neighborhood of this curve. We denote this point in $\mathbb{R}^5$ by $u(x)$. As x runs over the double points, the points $u(x)$ form an oriented curve $\tilde{u}$. (The orientation of this curve comes from that of the manifold of double points, which is oriented because the codimension of the immersion is even, and so a possible interchanging of the normal fibers does not change the orientation.) Hence, the linking number of the image of the immersion with the curve $\tilde{u}$ is a well-defined integer. But this number is *not* invariant under cobordism of the original immersion. If the cobordism – which is a map of a 4-manifold into the 6-space – has a triple point, then this number jumps by ± 3. Otherwise, it is unchanged. Hence, taking it mod 3, we obtain a well-defined map $\pi^3(3) \to \mathbb{Z}_3$. Ekholm constructed an example showing that this map is onto. •

Part (a) of Lemma 1.7 implies that the order of the group $\pi^s(3)$ is either infinite or finite of order $8t$, where $t \geqslant 1$.[1] Let $\mathrm{Imm}(S^3, \mathbb{R}^5)$ denote the group of regular homotopy classes of immersions $S^3 \looparrowright \mathbb{R}^5$. (The group operation is given by the connected sum of immersions.) Note that any such immersion has a normal framing, which is unique up to homotopy. Hence, a natural map

$$J\colon \mathrm{Imm}(S^3, \mathbb{R}^5) \to \pi^s(3)$$

arises, which actually can be identified with the classical J-homomorphism.

Lemma 1.8. *The J-homomorphism is an epimorphism.*

Proof. This follows from the fact that $\Omega_3^{\mathrm{Spin}} = 0$, i.e., that each spin 3-manifold bounds a spin 4-manifold. (See [16].) Let $\alpha \in \pi^s(3)$ and let $f\colon M^3 \subset \mathbb{R}^N$ be a framed embedding representing α. The trivialization of the normal bundle defines a stable trivialization of the tangent bundle of M^3. In particular, a spin structure arises on M^3.

Let W^4 be a compact, oriented spin 4-manifold with boundary M^3 such that its spin structure restricted to the boundary gives a spin structure on M^3. So W^4 has a stable parallelization over its 2-skeleton, extending that given on M^3. Since $\pi_2(SO) = 0$, this extends to the 3-skeleton. So there is a stable parallelization of the tangent bundle of the manifold $\tilde{W}$ obtained from W by deleting finitely many disjoint balls. Using the fact that there is a 1-1 correspondence between stable tangent and stable normal framings, and the latter correspond to regular homotopy classes of immersions of S^3 to $\mathbb{R}^5$, we see that the element α can be represented by an immersed sphere. Since α was arbitrary, the map J is onto. •

Lemma 1.9. *Let* $\mathrm{Emb} = \mathrm{Emb}(S^3, \mathbb{R}^5)$ *denote the set of regular homotopy classes of embeddings. Then* $J(\mathrm{Emb}(S^3, \mathbb{R}^5)) = 0$.

Proof. This immediately follows from the existence of the Seifert surfaces. (See [8].) •

[1] Notice that using part (b) we see that $\pi^s(3)$ is actually either infinite or finite of order $24t$ – in what follows, we tactically avoid the usage of part (b).

Hence, a map $\tilde{J}\colon \operatorname{Imm}(S^3, \mathbb{R}^5)/\operatorname{Emb}(S^3, \mathbb{R}^5) \to \pi^s(3)$ arises on the factor group.

Lemma 1.10. *$\pi_3(SO(n)) \approx \mathbb{Z}$ for $n \geqslant 5$.*

Proof. See [7], Proposition I.12.11. $\bullet$

Lemma 1.11. *There is an isomorphism* $\operatorname{Sm}: \operatorname{Imm}(S^3, \mathbb{R}^5) \to \pi_3(SO(5)) = \mathbb{Z}$.

Proof. This is a special case of Smale's theorem. Here, we deduce it from Lemma 1.4. There is map $\operatorname{Imm}(S^3, \mathbb{R}^5) \to \pi_3(SO)$. (Lift the immersion to $\mathbb{R}^N, N >> 1$, with framing. Deform it into the standard embedding and compare the given framing with the standard one.) An inverse map can be defined with the help of the Hirsch lemma 1.3. A given framing of the standard embedding $S^3 \subset \mathbb{R}^N$ gives an immersion into $\mathbb{R}^5$. Its regular homotopy class is well defined. Indeed, Lemma 1.3 can be applied to the immersion of the cylinder, which gives a regular homotopy. As a result, if $k \geqslant 2$, two regularly homotopic framed immersions in $\mathbb{R}^{n+k+1}$ can be pushed down and remain regularly homotopic in $\mathbb{R}^{n+k}$. $\bullet$

Remark 1.12. If the codimension k in the previous proof is equal to 1, then the framed immersions of n-manifolds in $\mathbb{R}^{n+2}$ can still be pushed down to $\mathbb{R}^{n+1}$, but they may fail to be regularly homotopic in $\mathbb{R}^{n+1}$. Therefore, the map $\operatorname{Imm}(S^n, \mathbb{R}^{n+1}) \to \operatorname{Imm}(S^n, \mathbb{R}^{n+2})$ is not monomorphic in general, but it is epimorphic. For $k \geqslant 2$, the regular homotopy classes of framed immersions of S^n into $\mathbb{R}^{n+k}$ are in a 1-to-1 correspondence with those into $\mathbb{R}^{n+k+1}$.

The integer corresponding to the regular homotopy class of an immersion f is called its *Smale invariant*, and denoted – with a slight abuse of notation – by $\operatorname{Sm}(f)$. Next, we meet the following problem: how to find the Smale invariant $\operatorname{Sm}(f)$ for a given immersion f. The following lemma, which is due to Hughes and Melvin (and it has also been independently found by Ekholm) answers this problem in the case of embeddings.

Lemma 1.13 (Hughes–Melvin [6]; cf. [1]). *Let $i\colon S^3 \subset \mathbb{R}^5$ be an embedding. Let S_i be a Seifert surface in $\mathbb{R}^5$ spanning $i(S^3)$. Let $\sigma(S_i)$ be the signature of the 4-manifold S_i. Then $\operatorname{Sm}(i) = \frac{3}{2}\sigma(S_i)$.*

We give a modification of this lemma.

Lemma 1.14. *Let $f\colon S^3 \to \mathbb{R}^5$ be an immersion bounding a framed immersion $g\colon M^4 \looparrowright \mathbb{R}^6_+$. Then $\operatorname{Sm}(f) = \frac{3}{2}\sigma(M^4)$.*

Proof. As is well known, $\pi_3(SO) = \mathrm{Vect}(S^4) = \{\text{stable bundles on } S^4\}$. We denote by ξ_f the stable bundle corresponding to the immersion f.

Denote by p_1 the map $\mathrm{Vect}(S^4) \to \mathbb{Z}, \xi \to \langle p_1(\xi), [S^4]\rangle$. First of all, we show that this map takes only even values. If ξ is a complex bundle, then $p_1(\xi) = -2c_2(\xi)$, and so $p_1(\xi)$ is even. In general, a stable vector bundle over S^4 can be assumed to have $SO(4)$ as its structure group, hence the bundle is determined by the homotopy class of a map $S^3 \to SO(4)$. This map can be lifted to $\mathrm{Spin}(4)$, which is isomorphic to $SU(2) \times SU(2)$. Hence, ξ admits a complex structure and so $p_1(\xi)$ is even.

If $\xi = [\gamma_H^1]$ (= the stable class of the quaternionic projective line bundle), then $\langle p_1(\xi), [S^4]\rangle = -2$. Hence, the image of p_1 is $2\mathbb{Z}$. Composing the above maps, we obtain a map $\mathrm{Imm}(S^3, \mathbb{R}^5) \to \mathbb{Z}$ that maps $\mathbb{Z}$ onto $2\mathbb{Z}$. Therefore,

$$\pm 2\,\mathrm{Sm}(f) = \langle p_1(\xi_f), [S^4]\rangle.$$

Let f (with its framing in $\mathbb{R}^N$) bound a framed 4-manifold (in $\mathbb{R}_+^{N+1}$). Consider the closed manifold $\hat{M} = M \cup D^4$, and let ν be its stable normal bundle. Then $\langle p_1(\nu), [\hat{M}]\rangle = \langle p_1(\xi_f), [S^4]\rangle$. Indeed, it is easy to define a degree-1 map $\hat{M} \to S^4$ inducing ν from ξ_f. Note that $p_1(\hat{M}) = -p_1(\nu)$. Hence, $\langle p_1(\hat{M}), [\hat{M}]\rangle = \pm 2 \cdot \mathrm{Sm}(f)$, i.e., $\mathrm{Sm}(f) = \pm\frac{3}{2}\sigma(\hat{M})$. (Here, we use the signature formula $p_1(\hat{M}) = 3\sigma(\hat{M})$, which is proved in the next paper.) •

Remark 1.15. Note that because of the existence of a Seifert surface, any embedding $S^3 \to \mathbb{R}^5$ is null-cobordant as a framed manifold. So our formulation (Lemma 1.14) is more general than Hughes–Melvins one (Lemma 1.13). Actually, the opposite implication also holds true: a null-cobordant framed immersion is regularly homotopic to an embedding – but this is a nontrivial fact. (We obtain this equivalence as a byproduct of the proof of Theorem 1.1.) By proving Lemma 1.14 (instead of Lemma 1.13) we can avoid the usage of part (b) of Lemma 1.7.

§2. Proofs of the theorems

Proof of Theorem. Now let us consider all Seifert surfaces in $\mathbb{R}^5$, i.e., all embedded compact, oriented 4-manifolds with boundary diffeomorphic to S^3. Then $\frac{3}{2}$ times the signatures of these manifolds form a subgroup of $\mathbb{Z}$ (since these are the Smale invariants of embeddings). We will see below that this group is not trivial. Hence, it has the form $\frac{3}{2}\sigma_0 \cdot \mathbb{Z}$, where σ_0 is the smallest possible positive value for a signature of a Siefert surface.

Then the quotient group $\mathrm{Imm}(S^3, \mathbb{R}^5)/\mathrm{Emb}(S^3, \mathbb{R}^5)$ is isomorphic to the cyclic group of order $\frac{3}{2}\sigma_0$. On the other hand, we have seen that there is an epimorphic map $\mathrm{Imm}(S^3, \mathbb{R}^5)/\mathrm{Emb}(S^3, \mathbb{R}^5) \to \mathbb{Z}_8$, whence $\frac{3}{2}\sigma_0 \geqslant 24$.

Lemma 2.1. *The signature of the $K3$ surface is 16. If a ball is deleted from it, then it has the homotopy type of a 2-complex.*

Proof. See [4], p. 22, for the proof that we sketch here. We recall that a $K3$ surface is given by a generic degree 4 homogeneous polynomial in $\mathbb{C}P^3$. It is a simple exercise to compute the Chern classes of such an algebraic manifold, and to obtain its Pontryagin class. Now, using the formula $\sigma = \frac{1}{3}p_1$, we obtain the signature. The simple connectedness of such an algebraic manifold follows from the Lefschetz hyperplane section theorem (which can be proved by elementary Morse theory, see [12]). Now a simply connected, punctured 4-manifold has the homotopy type of a 2-complex. (For another proof of Lemma 2.1, see [4], pp. 70–72.) •

Therefore, the $K3$ surface with a ball deleted from it embeds in $\mathbb{R}^5$, and so it is a Seifert surface. Hence, $\sigma_0 \leqslant \sigma(K3) = 16$. If follows that all the inequalities above are equalities. In particular, $\sigma_0 = 16$ and $\mathrm{Imm}/\mathrm{Emb} \approx \mathbb{Z}_{24}$. Therefore, both epimorhisms

$$\mathrm{Imm}(S^3, \mathbb{R}^5)/\mathrm{Emb}(S^3, \mathbb{R}^5) \to \pi^s(3) \to \mathbb{Z}_{24} = \mathbb{Z}_3 \times \mathbb{Z}_8$$

are isomorphisms, whence $\pi^s(3) \approx \mathbb{Z}_{24}$. We have also proved that the signature of any Seifert surface is divisible by 16.

Proof of Theorem 1.2. To prove Theorem 1.2, it remains to note that if $w_1 = w_2 = 0$ for a manifold, then it is spin, i.e., its tangent bundle over the 2-skeleton is trivial and trivialized. Further, any spin 4-manifold is spin cobordant to a simply connected one, and a simply connected spin 4-manifold with a ball deleted has the homotopy type of a 2-complex, hence embeds in $\mathbb{R}^6$. (Indeed by Hirsch theory it admits an immersion in $\mathbb{R}^5$, and since it has the homotopy type of a 2-complex, any map of it in $\mathbb{R}^5$ can be approximated by a topological embedding. A sufficiently close approximation of an immersion remains immersion, so there is an embedding of it in $\mathbb{R}^5$ in the sense of differential topology.)

So this 4-manifold with a ball deleted is a Seifert surface, hence its signature is divisible by 16. •

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