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THE FOURTH ORIENTED COBORDISM GROUP Ω_4 IS ISOMORPHIC TO $\mathbb{Z}$

Abstract. V. A. Rokhlin theorem indicated in the title is proved with using generic maps.

Introduction

Using generic singular maps, we prove the following theorem of Rokhlin:

Theorem 0.1. *The fourth oriented cobordism group Ω_4 is isomorphic to the group of integers; in short, $\Omega_4 \approx \mathbb{Z}$.*

This theorem implies the following signature formula for 4-manifolds:

Corollary 0.2 (signature formula of Rokhlin). *For any smooth, closed 4-manifold M, the following formula holds:*

$$\sigma(M) = \frac{1}{3}\langle p_1(M), [M]\rangle.$$

Proof of Corollary 1.2. Both sides define homomorphisms $\Omega_4 \to \mathbb{Z}$. Since they coincide for the cobordism class of $\mathbb{C}P^2$, they coincide everywhere. •

§1. Proof of Theorem 1.1

Any 4-manifold M^4 admits a generic map g in $\mathbb{R}^6$. Such a map has maximal rank at each point except a 1-manifold $\Sigma(g) \subset M^4$, where the map has Whitney umbrella points, i.e., it has the local form

$$(x, y, z) \mapsto (x^2, xy, y, z), \quad x, z \in \mathbb{R}^1,\ y \in \mathbb{R}^2.$$

Furthermore, g has a double-point surface $\Delta(g)$ immersed in $\mathbb{R}^6$ and isolated triple points $\Delta_3(g)$, which are (nongeneric) triple points of $\Delta(g)$ in $\mathbb{R}^6$. $\Delta(g)$ is a surface with boundary $\partial\Delta(g) = \Sigma(g)$. The surface $\Delta(g)$ is oriented, and the triple points are also oriented (because the codimension of the map is even). Let us denote the algebraic number of triple points of the map g by $t(g)$.

Definition 1.1 (definition of $l(g)$). Let $n(g)$ be a short normal outward pointing vector field of $\Sigma(g)$ in $\Delta(g)$. Let $\tilde{\Sigma}$ be the curve formed by the endpoints of the vector field $n(g)$. Define $l(g) \in \mathbb{Z}$ as the linking number of $\tilde{\Sigma}$ with the image of the map g.

Lemma 1.2 [4]**.** *The difference $t(g)-l(g)$ is invariant under any generic cobordism of generic maps.*

Proof. Let $g_0\colon M_0 \to \mathbb{R}^6$ and $g_1\colon M_1 \to \mathbb{R}^6$ be two cobordant generic maps. Then there exists a 5-manifold W^5 with $\partial W^5 = M_1 - M_0$ and there is a generic map $G\colon W^5 \to \mathbb{R}^6 \times [0,1]$ with $G|M_i = g_i$ (into $\mathbb{R}^6 \times \{i\}$). We must show that

$$t(g_1) - l(g_1) = t(g_0) - l(g_0).$$

We denote by $A(G)$ the algebraic number of the points in $\mathbb{R}^6 \times [0,1]$ that have exactly two preimages, one of which is singular, the other is nonsingular. (We call such points $\Sigma^{1,0} + \Sigma^0$ type points of the map G.) Note that all these points have signs, and hence we can consider their algebraic number.

Claim 1. *We have $t(g_1) - t(g_0) = A(G)$.*

Indeed, the triple lines of G may end only at a triple point of g_0, at a triple point of g_1, or at a $\Sigma^{1,0} + \Sigma^0$ type point of G. $\bullet$

Claim 2. *We have $l(g_1) - l(g_0) = A(G)$.*

To prove this, we extend the short normal vector fields $n(g_1)$ and $n(g_0)$ to a short normal field $n(G)$ along the singular points of the map G. Denote the surface formed by the endpoints by $\tilde{\Sigma}(G)$. This is a surface with boundaries $\tilde{\Sigma}(g_1)$ and $\tilde{\Sigma}(g_0)$. This surface is disjoint with the image of G except one intersection point near each $\Sigma^{1,0} + \Sigma^0$ point. (The signs of these intersection points coincide with the signs of the nearby $\Sigma^{1,0} + \Sigma^0$ points.) The difference of the linking numbers $l(g_1) - l(g_0)$ is equal to the intersection number of $\tilde{\Sigma}(G)$ and $G(W)$, i.e., to $A(G)$. $\bullet$

Corollary 1.3. *The expression $t(g) - l(g)$ is an invariant of the cobordism class of the map g. Hence, it defines a homomorphism $\Phi\colon \Omega_4(\mathbb{R}^6) = \Omega_4 \to \mathbb{Z}$.* $\bullet$

Proposition 1.4. *The homomorphism $\Phi(g) = t(g) - l(g)$ is*

(a) *nontrivial and*

(b) *a monomorphism.*

Consequently, Ω_4 is isomorphic to $\mathbb{Z}$.

Remark 1.5. Actually, the homomorphism $\Phi(g)$ coincides with the signature of the domain manifold of g, and, therefore, $\Phi(g)$ is an isomorphism (but we do not use this fact).

Proof of assertion (a). Quite similarly to the above definitions, we can give the mod 2 versions of all the above notions, which also make sense for nonorientable manifolds. In particular, the mod 2 number of triple points $t_2(g)$ and the mod 2 linking number $l_2(g)$ can be defined, and we put $\Phi_2(g) = t_2(g) + l_2(g)$. Then $\Phi_2(g)$ is a homomorphism from $\mathcal{N}_4(\mathbb{R}^6) = \mathcal{N}_4$ into $\mathbb{Z}_2$. The composition

$$\Omega_4(\mathbb{R}^6) \xrightarrow{\text{unorientation}} \mathcal{N}_4(\mathbb{R}^6) \xrightarrow{\Phi_2} \mathbb{Z}_2$$

is just the mod 2 reduction of Φ. We show that its value on the cobordism class of $\mathbb{C}P^2$ is nonzero, hence Φ is nonzero either. The image of the class $[\mathbb{C}P^2] \in \Omega_4$ in $\mathcal{N}_4$ is equal to the class $[\mathbb{C}P^2 \times \mathbb{C}P^2]$. (The conjugation on $\mathbb{C}P^2$ and the order changing on $\mathbb{C}P^2 \times \mathbb{C}P^2$ have diffeomorphic fixed-point sets and the tubular neighborhoods of these sets are also diffeomorphic. It easily follows that $\mathbb{C}P^2$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ are unorientedly cobordant (see [2]).) Now the square of the Boy surface immersion $B\colon \mathbb{C}P^2 \to \mathbb{R}^3$ is an immersion $\mathbb{C}P^2 \times \mathbb{C}P^2 \to \mathbb{R}^6$, which – when made self-transverse – has an odd number of triple points, by Herbert's theorem (see [6]). •

§2. Proof of assertion (b) of Proposition 1.4

The rest of the paper is devoted to this proof. Let $g\colon M^4 \to \mathbb{R}^6$ be a generic map and assume that $\Phi(g) = 0$. We must show that M^4 is an oriented boundary. According to Corollary 1.3, we can choose any representative $\tilde{M}$ in the cobordism class of M^4 and any generic map $\tilde{g}$ in $\mathbb{R}^6$; the value of $\Phi(\tilde{g})$ is zero, independently from these choices.

Lemma 2.1. *The cobordism class of M^4 contains a manifold $\tilde{M}$ that can be embedded in $\mathbb{R}^7$.*

Remark 2.2. For the map $\tilde{g}\colon \tilde{M} \to \mathbb{R}^6$, we choose the projection of the embedding $\tilde{M} \subset \mathbb{R}^7$ into $\mathbb{R}^6$.

Proof of Lemma 2.1. Take an arbitrary generic immersion $g\colon M^4 \to \mathbb{R}^7$. Since the double points form a 1-dimensional, hence orientable manifold, the two branches can be globally separated. (They cannot interchange going along a loop in the double points, because this would change the orientation of the normal fiber of the double-point set, so it would change also the orientation of the double-point manifold, too. However, in an orientable manifold, there are no orientation-reversing loops.) Hence, a tubular neighborhood of a component of the double-point set in $\mathbb{R}^7$ can be identified with $S^1 \times D^3 \times D^3$ in such a way that the sets

$$\{(\varphi, x, 0) \mid 0, x \in D^3, \varphi \in S^1\} \text{ and } \{(\varphi, 0, y) \mid 0, y \in D^3, \varphi \in S^1\}$$

correspond to the two branches of the map. Here, D^3 is the unit ball in $\mathbb{R}^3$. We choose a number $\varepsilon \in (0, \frac{1}{2})$ and denote by D^3_ε the ε-ball in $\mathbb{R}^3$, and by S_ε its boundary. We define the following two subsets of a tubular neighborhood T of the double curve of the immersion g:

$$\begin{aligned} N_1 = &\{(\varphi, x, 0) \mid \varphi \in S^1, 2\varepsilon \leqslant \|x\| \leqslant 1\} \cup \{(\varphi, 0, y) \mid \varphi \in S^1, 2\varepsilon \leqslant \|y\| \leqslant 1\} \\ &\cup \{(\varphi, -tx, (1-t)y) \mid \ \|x\| = \|y\| = 2\varepsilon, t \in [0,1]\}, \\ N_2 = &\{(\varphi, x, 0) \mid \varphi \in S^1, \|x\| \leqslant \varepsilon\} \cup \{(\varphi, 0, y) \mid \varphi \in S^1, \leqslant \|y\| \leqslant \varepsilon\} \\ &\cup \{(\varphi, -tx, (1-t)y) \mid \ \|x\| = \|y\| = \varepsilon, t \in [0,1]\}. \end{aligned}$$

After an obvious smoothing, the set N_2 becomes the image of a smooth immersion of $S^1 \times S^3$. This manifold obviously bounds. After smoothing, the set $N_1 \cup (g(M^4) \setminus T)$ becomes the image of a smooth embedding of a 4-manifold, which we denote by $\tilde{M}$. It is not hard to see that $\tilde{M}$ is cobordant to M^4, hence Lemma 2.1 is proved. •

We denote again by M^4 the manifold $\tilde{M}$, and by g the projection of the embedding $M^4 \subset \mathbb{R}^7$ into $\mathbb{R}^6$. The omitted coordinate of $\mathbb{R}^7$ is denoted by h and called the *height function* on M^4. Note that the two branches of g at the double-point surface $\Delta(g)$ are globally separated (i.e., they never interchange as we go along a closed curve in $\Delta(g)$). The normal vectors to $\Delta(g)$ at these two branches give two 2-dimensional vector bundles over $\Delta(g)$. We denote them by ξ_1 and ξ_2.

Step 1. One can deform the map g in such a way that the vector field $n(g)$ (see the definition of $l(g)$) is directed vertical upward. (By an isotopy of the image of the curve of the singular points and of the vector field the field can be made vertical upward. This isotopy extends to an isotopy of $\mathbb{R}^6$.)

Step 2. Apply a deformation that changes the map g only in a neighborhood of the curve $\Sigma(g)$ as follows. Push the image of this singularity curve very far away in the vertical upward direction and follow this deformation by a deformation of the map so that the new, pushed-away curve is the image of the singularity curve of the deformed map. Finally, we obtain a map, which we still denote by g, for which $l(g) = 0$. Then, certainly, $t(g) = 0$ too, since $\Phi(g) = t(g) + l(g)$ remains unchanged.

Step 3. Any pair of triple points of opposite signs can be eliminated, similarly to the way described by Banchoff [1] for surfaces in $\mathbb{R}^3$. One can perform these surgeries so that the two branches at the double points still remain globally separated. We obtain a new map g with no triple points and $l(g) = 0$.

Step 4. On this map, we want to perform a surgery similar to that performed in the proof of Lemma 2.1. This surgery cuts off the double-point surface. More precisely, the manifold M^4 is replaced by a manifold $\tilde{M}^4$ consisting of the disjoint union of two manifolds $\tilde{M}_1$ and $\tilde{M}_2$. The new map is an embedding on $\tilde{M}_1$, and the image of $\tilde{M}_2$ lies in a small tubular neighborhood of the double-point set $\Delta(g)$. The images of $\tilde{M}_1$ and $\tilde{M}_2$ are disjoint.

The bundles ξ_1 and ξ_2 coincide over the boundary $\partial\Delta_2(g) = \Sigma(g)$. Note that the two orientations naturally arising on the fibers of this bundle over Σ (one as restriction of the bundle ξ_1, the other one as the restriction of ξ_2) are different. In order to see this, we track back how the orientations on different parts arise.

First of all, $\mathbb{R}^6$ and M^4 are oriented. Hence, an orientation arises on each normal fiber of g, where such a fiber exists, i.e., over the points of $M^4 \setminus \Sigma(g)$. This induces an orientation on $\Delta(g)$, the double-point set in $\mathbb{R}^6$, which defines an orientation on its boundary, i.e., on the image $g(\Sigma(g))$ of singular points. We denote by $\tilde{\Delta}(g)$ the closure of the preimage of the double points in M^4. There is an involution τ on $\tilde{\Delta}(g)$ with fixed-point set $\Sigma(g)$. We divide $\tilde{\Delta}(g)$ into two parts $\tilde{\Delta}^+(g)$ and $\tilde{\Delta}^-(g)$ according to the sign of $h(x) - h(\tau(x))$. The singularity curve $\Sigma(g)$ is their common boundary. The normal fiber of $\tilde{\Delta}(g)$ in M^4 at a point $x \notin \Sigma(g)$ can be identified with the normal bundle of g in $\mathbb{R}^6$ at the point $\tau(x)$. Hence, the normal fibers of $\tilde{\Delta}(g) \setminus \Sigma(g)$ in M^4 are oriented. So the manifold $\tilde{\Delta}(g) \setminus \Sigma(g)$ is oriented. The restrictions of g to both parts of $\tilde{\Delta}(g)$ are orientation preserving. (The involution τ corresponds to interchanging the two normal fibers at the image point in $\mathbb{R}^6$, but this does not change the orientation of the fiber normal to $\Delta(g)$ in $\mathbb{R}^6$.) The involution τ near the common boundary Σ is just a reflection in this boundary. Hence, the orientations of $\tilde{\Delta}^+$ and $\tilde{\Delta}^-$ conflict at Σ. Therefore, the induced orientations on the normal fibers over Σ of $\tilde{\Delta}$ in M^4 are different when we approach $\Sigma(g)$ from $\tilde{\Delta}^+(g)$ or from $\tilde{\Delta}^-(g)$. Hence, an anti-isomorphism (i.e., an orientation reversing isomorphism) $\psi\colon \xi_1|\Sigma \approx \xi_2|\Sigma$ arises.

Lemma 2.3. *There is an orientation reversing isomorphism* $\Psi\colon \xi_1 \to \xi_2$

extending ψ.

Assuming Lemma 2.3, we show how to perform the promised surgery on the map g. Let Riemannian metrics be fixed on the bundles ξ_1 and ξ_2, such that the map Ψ is an isometry on the fibres. Let ε be a small positive number and let $D_\varepsilon(\xi_i)$ be the ε-ball bundle of ξ_i, and $S_\varepsilon(\xi_i)$ its boundary. We join by a segment each point $x \in S_\varepsilon(\xi_i)$ with its image $\Psi(x)$ in the 4-dimensional normal fiber of $\Delta_2(g)$ in which they lie. The segments sweep a 3-manifold C_ε as x runs over $S_\varepsilon(\xi_1)$. We perform the surgery along the (appropriately thickened) manifold C_ε. More precisely, we do the following: Choose two small numbers $0 < \varepsilon_1 < \varepsilon_2$ and define C_{ε_1} and C_{ε_2} as above. Let T_{ε_2} be the tubular ε_2-neighborhood of $\Delta_2(g)$.

The set $C_{\varepsilon_2} \cup \{g(M^4) \setminus T_{\varepsilon_2}\}$ can be smoothened so that it will be the image of an embedding of a closed smooth oriented 4-manifold; this is $\tilde{M}_1$. Since this manifold is embedded, it is null-cobordant. The other part $\tilde{M}_2$ of the new manifold has the following image:

$$D_{\varepsilon_1}(\xi_1) \cup D_{\varepsilon_1}(\xi_2) \cup C_{\varepsilon_1}.$$

$\tilde{M}_2$ is the union of the total space of an S^2-bundle over $\Delta_2(g)$ and of the total space of a D^3-bundle over the boundary curve $\Sigma(g)$. These spaces are glued along the S^2-bundle over $\Sigma(g)$. This manifold $\tilde{M}_2$ is clearly the boundary of the associated D^3-bundle over $\Delta(g)$. So we have shown that the cobordism class $[M^4] \in \Omega_4$ on which Φ vanishes. This concludes the proof of Proposition 1.4 (b) (assuming Lemma 2.3). $\bullet$

§3. Proof of Lemma 2.3

For simplicity, we assume that $\Sigma(g)$ is connected. Otherwise, the following procedure is performed at each component. Let T and $\tilde{T}$ be tubular neighborhoods of $\Sigma(g)$ and $g(\Sigma(g))$, respectively. They can be identified with

$$S^1 \times (-2, 2) \times D^2 \quad \text{and} \quad S^1 \times (-4, 4) \times D^2 \times D^2$$

respectively so that the map g has the form $g(\varphi, x, y) = (\varphi, x^2, y, xy)$, where $\varphi \in S^1, x \in (-4, 4), y \in D^2$ and $(\varphi, x, y) \in \tilde{\Delta}^+(g)$ if and only if $0 \leqslant x$, and $(\varphi, x, y) \in \tilde{\Delta}^-(g)$ if and only if $x \leqslant 0$. Furthermore, assume that the pushed-off singularity $\tilde{\Sigma}(g)$ corresponds to the points $(\varphi, -2, 0, 0)$. The anti-isomorphism ψ corresponds to the identity map of the bundle $\xi_1|\Sigma = \xi_2|\Sigma$. Let s be a nowhere – zero section of the latter bundle. (Such a section exists, because the base is 1-dimensional and the fiber is 2-dimensional. Actually this bundle is trivial.) Extend this section s to sections s_1 and s_2 of the bundles ξ_1 and ξ_2 respectively, which are transverse to the zero sections and do not vanish over points corresponding to each other (i.e., either $s_1(x) \neq 0$ or $s_2(\tau(x) \neq 0)$. Finally, we also assume that in a neighborhood of Σ both sections s_1 and s_2 have zero derivatives in the directions normal to Σ in $\tilde{\Delta}$. More precisely, in the coordinates of T and $\tilde{T}$ given above, we assume that if $g(s(\varphi, 0, 0)) = (\varphi, 0, s_\varphi, 0)$, then

$$g(s_1(\varphi, x, 0)) = (\varphi, x^2, s_\varphi, s_\varphi \cdot x) \quad \text{for} \quad 0 \leqslant x \leqslant 1$$

and

$$g(s_2(\varphi, x, 0)) = (\varphi, x^2, s_\varphi, s_\varphi \cdot x) \quad \text{for} \quad -1 \leqslant x \leqslant 0.$$

Sublemma 3.1. *There is an embedding j of the cylinder $S^1 \times [0, 2]$ into $\tilde{T}$ such that $j(S^1 \times \{2\}) = \tilde{\Sigma}$ and $j(\varphi, 0) = (\varphi, 1, s_\varphi, 0)$.*

In other words, one boundary circle of the cylinder is mapped by j onto $\tilde{\Sigma}$, the other one onto the image of the map

$$S^1 \to \mathbb{R}^6, \quad \varphi \mapsto g(s_1(\varphi, 1, 0)) + g(s_2(\varphi, -1, 0)).$$

Assuming Sublemma 3.1, we finish the proof of Lemma 1.10. One can assume that the sections s_1 and s_2 are so close to the zero section at any point that $||s_1(p)|| \leqslant \varepsilon$ and $||s_2(p)|| \leqslant \varepsilon$ for any $p \in \Delta(g)$ and hence the sum $s_1(p) + s_2(p)$ can be considered as a point in the tubular neighborhood of $\Delta(g)$. Let $\Delta'(g)$ be the surface that remains from $\Delta(g)$ if we delete those points of $\tilde{T} \cap \Delta(g)$ for which $x^2 < 1$. As the point p runs over $\Delta'(g)$, the points $s_1(p) + s_2(p)$ form a surface F with boundary $\partial F = j(S^1 \times \{0\})$. Now consider the surface $F^* = F \cup j(S^1 \times [0, 2])$. This is a surface with boundary $\tilde{\Sigma}$. By the definition of $l(g)$, the intersection number of F^* with the image of g is $l(g)$. In the present case, this is zero. On the other hand, the intersection number of F^* with the image of g is the same as the intersection number of F with the image of g. The latter is the algebraic sum of the zeros of the sections s_1 and s_2. Hence, the relative Euler classes of the bundles ξ_1 and ξ_2 are opposite ($\chi(\xi_1) = -\chi(\xi_2) \in H^2(\Delta(g), \partial\Delta(g); \mathbb{Z})$), and so the bundles are anti-isomorphic. Now the proof of Lemma 2.3 (assuming Sublemma 3.1) is complete. $\bullet$

Proof of Sublemma 3.1. We give the map j by the following formulas:

(a) For $(\varphi, t) \in S^1 \times [0,2]$, $0 \leqslant t \leqslant 1$ put

$$j(\varphi, t) = (\varphi, 1, \frac{1}{2}s_\varphi(e^{i\pi t}+1), \frac{1}{2}s_\varphi(e^{i\pi t}-1)).$$

(Here, we identify the oriented fibers of ξ_i with the complex plane provided with metrics so that the multiplication by i is the usual rotation in this metric. Then $s_\varphi \in C$ can be multiplied by the complex numbers $e^{i\pi t} \pm 1$.) Note that

$$j(\varphi, 0) = (\varphi, 1, s_\varphi, 0) = g(s_1(\varphi, 1, 0)) + g(s_2(\varphi, 1, 0)),$$

which is the boundary of F^*. Furthermore, we have $j(\varphi, 1) = (\varphi, 1, 0, -s_\varphi)$.

(b) For $(\varphi, t) \in S^1 \times [0,2]$, $1 \leqslant t \leqslant 2$, we put

$$j(\varphi, t) = (2-t) \cdot j(\varphi, 1) + (t-1) \cdot (\varphi, -2, 0, 0).$$

For $t = 1$ the two definitions of $j(\varphi, t)$ coincide. For $t = 2$ we obtain the circle $\tilde{\Sigma}$. It is not hard to check, that the image of the map j is disjoint from that of g. Sublemma 3.1 is proved. •

This final argument now completes the proof of Theorem 0.1. •

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