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ON THE VANISHING OF THE THIRD SPIN COBORDISM GROUP Ω_3^{Spin}

Abstract. An elementary proof of the theorem given in the title is presented.

Here, we prove the following theorem.

Theorem 1.1. *A spin 3-manifold M is the spin boundary of a spin 4-manifold; in short, $\Omega_3^{\mathrm{Spin}} = 0$.*

The proof given below can be found in various sources (cf. [3, 2]); in our presentation, we try to be as elementary as possible. Preceding the proof, we collect the basics about spin structures and handlebodies, and list their fundamental properties. (For a more detailed treatment of these notions, see [3, 1].) The purpose of this survey is to complete the proofs discussed in [5, 6].

§1. Preliminaries on spin structures and handlebodies

Definition 1.2. A manifold X has a spin structure if its stable tangent bundle $TX \oplus \epsilon^k$ admits a trivialization over the 1-skeleton of X which extends over the 2-skeleton.[1] A *spin structure* is a homotopy class of such trivializations. (Here, ϵ denotes a trivialized bundle. It can also be shown that the definition does not depend on the chosen k if $k \geq 1$.)

An oriented manifold X admits a spin structure if and only if $w_2(X) = 0$. An oriented 3-manifold always admits a spin structure, since its tangent bundle is trivial. Since for a 4-manifold X we have $w_2(X) \cup \alpha \equiv \alpha \cup \alpha \pmod 2$, the existence of a spin structure on X implies that the cup square of every $\alpha \in H^2(X;\mathbb{Z})$ is even. Moreover, if X is simply connected and $\alpha \cup \alpha$ is even for all $\alpha \in H^2(X;\mathbb{Z})$, then $w_2(X) = 0$ – this direction does not hold in the presence of 2-torsion in $H_1(X;\mathbb{Z})$. The different spin

[1] Note that if a trivialization over the 1-skeleton extends to the 2-skeleton, then it extends in a homotopically unique way.

structures of a manifold (of arbitrary dimension) with vanishing $w_2(X)$ can be parametrized by $H^1(X;\mathbb{Z}_2)$.

If $Y \subset X$ is a codimension-0 submanifold of a spin manifold X, then Y inherits a natural spin structure from X. If $N \subset X$ has trivialized normal bundle (e.g., $N = \partial X$), then N inherits a spin structure from X. Now assume that X_1 and X_2 are two spin manifolds with boundary and $N_i \subset \partial X_i$ are codimension-0 submanifolds of the boundaries. If $f: N_1 \to N_2$ is an orientation-reversing diffeomorphism mapping the induced spin structures into each other, then the glued-up manifold $X_1 \cup_{N_2 = f(N_1)} X_2$ admits a canonical spin structure induced from those of X_1 and X_2. Based on the above properties, the spin cobordism group Ω_n^{Spin} of closed spin n-manifolds can be defined. Finally we note that there is an orientation-preserving self-diffeomorphism $f: S^1 \times D^2 \to S^1 \times D^2$ of the solid torus interchanging its two spin structures.

Now we quickly review the part of handlebody theory that is relevant in our subsequent discussions. (For further details, see [1].)

Definition 1.3. For $0 \leqslant k \leqslant n$, an *$n$-dimensional k-handle* h is a copy of $D^k \times D^{n-k}$, attached to the boundary of an n-manifold X along $\partial D^k \times D^{n-k}$ by an embedding $\varphi\colon \partial D^k \times D^{n-k} \to \partial X$.

$D^k \times 0$ is the *core*, $0 \times D^{n-k}$ the *cocore*, φ the *attaching map*, $\partial D^k \times D^{n-k}$ (or its image $\varphi(\partial D^k \times D^{n-k})$) the *attaching region*, and $\partial D^k \times 0$ (or its image) the *attaching sphere* of the handle.

$X \cup h$ is specified by an embedding $\varphi_0: S^{k-1} \to \partial X$ (i.e., a knot in ∂X) with trivial normal bundle together with a *framing* f of $\varphi_0(S^{k-1})$ (i.e., an identification of the normal bundle $\nu\varphi_0(S^{k-1})$ with $S^{k-1} \times \mathbb{R}^{n-k}$).

If ∂X is connected and we restrict ourselves to the study of oriented manifolds, then 1-handles can be attached uniquely (up to isotopy). In order to glue a 2-handle, we need to specify a (1-dimensional) knot in ∂X together with a framing. Homotopically different framings can be parametrized by the group $\pi_1(SO(n-2))$, which is isomorphic to $\mathbb{Z}$ for $n = 4$. In case $X = D^4$ this framing can be identified with the linking number of $\varphi(\partial D^2 \times \{0\})$ with $\varphi(\partial D^2 \times \{p\})$ for some $p \in D^2$ close to 0. The extendability of the spin structure of D^4 to a 2-handle depends on the framing (i.e., the above linking number) mod 2; it is not hard to verify that the spin structure extends if and only if this framing is even.

If we attach a 1-handle to D^4, the resulting 4-manifold is $S^1 \times D^3$, which is diffeomorphic to $(D^2 - \nu\{0\}) \times D^2$, i.e., to D^4 with an open tubular neighborhood of an embedded disk deleted. Consequently, a 1-handle can be symbolized by an unknot in $S^3 = \partial D^4$ with a dot on it, where the circle records the boundary of the 2-disk that (together with its neighborhood) has been deleted. Similarly, the m-component unlink with dots in S^3 denotes the 4-manifold obtained by adding m 1-handles to D^4. (This manifold is obtained as follows: for each component U_i of the unlink delete from D^4 an embedded disk $(D_i, \partial D_i) \subset (D^4, \partial D^4)$ (together with its open tubular neighborhood) which satisfies $\partial D_i = U_i$.) Note that the attaching map of a 2-handle (which maps $S^1 \times D^2$ to the boundary of the union of the 1-handles) can now be regarded as pointing to

$$S^3 - \nu(\text{an } m\text{-component unlink}) \subset S^3$$

(ν denotes a tubular neighborhood), hence, the above framing convention (as the linking number of the attaching circle and a nearby one) extends to all cases. According to the above said, a 4-manifold obtained by adding 1- and 2-handles to D^4 can be represented by a link diagram in S^3, where some knots are framed (they correspond to 2-handles) and others (which form an unlink) are dotted – they correspond to the 1-handles. Such a diagram is usually called a *link diagram* or *Kirby diagram* of the 4-manifold at hand. It is easy to check that the attaching region of a 2-handle passes through a 1-handle if and only if the attaching circle of the former is linked with the dotted circle representing the latter.

Turning the dot on an unknot to 0 framing, we replace the 1-handle represented by the dotted unlink (which is $S^1 \times D^3$) with a 0-framed 2-handle attached along the unknot (which is $S^2 \times D^2$), i.e., we perform a *surgery* along the 1-handle. Conversely, a 0-framed unknot gives rise to an embedded 2-sphere (as the union of the core of the handle with a spanning disk of the unknot) with trivial normal bundle; hence we can perform surgery along this sphere. In the diagram this means that we replace the 0 framing of the unknot with a dot – i.e., we invert the above described surgery. (In this second case, we must be careful, since the dotted circles must form an unlink.)

Definition 1.4. Let X be a compact, oriented n-manifold with boundary ∂X decomposed as a disjoint union $\partial_+ X \sqcup \partial_- X$ of two compact submanifolds, either of which may be empty. (Orient $\partial_\pm X$ so that $\partial X = \partial_+ X \sqcup \overline{\partial_- X}$ in the boundary orientation.) A *handle decomposition* of X (relative to $\partial_- X$) is an identification of X with a manifold obtained from $[0,1] \times \partial_- X$ by adding handles, such that $\partial_- X$ corresponds to $\{0\} \times \partial_- X$ in the obvious way. A manifold X with a given handle decomposition is a *relative handlebody* built on $\partial_- X$, and if $\partial_- X = \emptyset$, then X is a *handlebody*.

A Morse function $f\colon X \to [0,1]$ provides a handle decomposition for X. Without loss of generality, we can assume that the handles are attached in order to increase the index. Considering $1-f$ instead of f, we obtain a new handlebody, and this operation is called *turning the handlebody upside down.*

In what follows, we restrict our attention to the 4-dimensional case. By isotoping the attaching maps of the individual handles we clearly do not change the diffeomorphism type of the manifold. Thus, we can slide a 2-handle over any other without changing the manifold X; after the operation, the attaching circle of the 2-handle we are sliding becomes the connected sum of the two knots (the connected sum is taken with the band along which we slid the 2-handle). To determine the new framing, we orient the two former attaching spheres and compute their *linking number.* The new framing on the connected sum is the sum of the framings $\pm$ twice this linking number – the sign depending on whether the band respects or disrespects the chosen orientations. (In the first case, we say that we *add*, in the second, we *subtract* the corresponding handle.)

It can be shown that if the attaching circle of a 2-handle links a dotted circle geometrically once (i.e., the dotted circle admits a genus-0 Seifert surface intersecting the attaching circle transversally at a single point), then this handle-pair can be canceled without changing the diffeomorphism type of the manifold. In the notation, we simply erase the corresponding knots from our diagram, but to obtain the diagram of the new handlebody, we must slide handles. The reason for this is as follows: if a 2-handle h passes through a 1-handle k which we want to cancel with a 2-handle h', then after cancelation the attaching map of h is changed (since we change the target space of the attaching map). For the cancelling pair (k,h'), this confusion can be resolved in the following way: slide the 2-handle h over h' repeatedly until the resulting 2-handle is disjoint from k. (This can be done because the attaching circle of h' links the dotted circle geometrically once.) After separating the dotted circle (representing the 1-handle k) from all 2-handles (except h'), we cancel the pair (k,h') by deleting the corresponding knots from the diagram – and the remaining (framed) knots provide a diagram of the new handlebody.

Finally, note that if we attach a 2-handle to D^4 along a ± 1-framed unknot, then the boundary of the resulting 4-manifold is the total space of the Hopf bundle, hence diffeomorphic to S^3. Consequently, adding (or deleting) a ± 1-framed unknot to our diagram, we change the 4-manifold but leave its boundary unchanged.

§2. Proof of Theorem 1.1

Now we are ready to prove of Theorem 1.1. We break the proof into several lemmas.

Lemma 2.1. *Every orientable 3-manifold M bounds a 4-manifold which can be built using only one 0-handle and several 2-handles.*

Proof. Since $\Omega_3 = 0$, we know that M bounds a 4-manifold X. Fix a handle decomposition of X and cancel handles until there is a unique 0-handle and there are no 4-handles. Surger the 1-handles and then turn the handlebody upside down. Now the 3-handles of the original decomposition become 1-handles, so by surgery we can turn them into 2-handles. This observation completes the proof. •

Lemma 2.2. *Every spin 3-manifold M is spin cobordant to $S^1 \times \Sigma_g$ for some g (and some spin structure on this latter manifold). Here, Σ_g denotes the Riemann surface of genus g.*

Proof. Consider X^4 with $\partial X = M$ which admits a handle decomposition with a single 0-handle and some 2-handles. Turn the handlebody decomposition upside down, i.e., view X as built on $M \times [0,1]$ by attaching some 2- and a single 4-handle. The spin structure of M extends to some 2-handles and does not extend to the others – depending on the gluing of the individual handles. Assume that h and h' are two 2-handles for which the spin structure does not extend. It is not hard to see that in this case the spin structure extends to the 2-handle obtained by sliding h over h'.

If the spin structure extends to all 2-handles, then M is a spin boundary, and since $S^1 \times S^2$ (for example) can be equipped with a bounding spin structure, the lemma obviously follows. If there are 2-handles for which the spin structure does not extend, then sliding all 2-handles but the last one over the last one, we can assume that there is a unique such handle h. Note that if we delete the cocore of h (resulting in $(D^2 - \{0\}) \times D^2$), then – since both spin structures of $\partial D^2 \times D^2$ extend to $(D^2 - \{0\}) \times D^2$ – any spin structure of the gluing region extends to this complement. Capping off the cocore of h with a Seifert surface of its belt circle in the 4-handle, we obtain an embedded surface $F \subset X$ such that the spin structure of M extends to the complement $X - F$. Consequently, $X - \nu F$ provides a spin cobordism between M and the boundary of the tubular neighborhood νF of F. Next, we show that (by changing X) it can be assumed that the D^2-bundle $\nu F \to F$ is trivial and so we have a spin cobordism between

M and $S^1 \times F = S^1 \times \Sigma_g$. Take the connected sum of X with the complex projective plane $\mathbb{C}P^2$. Since $\mathbb{C}P^2$ contains an embedded sphere (a complex projective line $\mathbb{C}P^1$) having normal bundle with first Chern number 1, it follows that, tubing F to this sphere, we obtain F' having normal bundle with first Chern number 1 higher than F did. (Note that the spin structure obviously extends to the complement of this sphere in $\mathbb{C}P^2$, the complement being a disk.) Now using $\overline{\mathbb{C}P^2}$ (which is $\mathbb{C}P^2$ with the opposite orientation) the same argument lowers the Chern number of the normal bundle of F. Choosing an appropriate orientation and repeating the process, we can now turn the Chern number of the normal D^2-bundle into 0, i.e., we can make it trivial. This completes the proof. •

Lemma 2.3. *$S^1 \times \Sigma_g$ (with any spin structure) is spin cobordant to the disjoint union of several 3-tori (with various spin structures on them).*

Proof. We show that Σ_g (with any spin structure) is spin cobordant to the disjoint union of some 2-dimenional tori (with various spin structures on them). This fact easily implies the above lemma: The spin structure on $S^1 \times \Sigma_g$ is the product of a spin structure on Σ_g and one on S^1. Consider this spin structure on S^1 and multiply the above spin cobordism between Σ_g and $\bigcup_{i=1}^k T^2$ with it; the resulting spin cobordism between $S^1 \times \Sigma_g$ and $\bigcup_{i=1}^k T^3$ proves the lemma.

To show that Σ_g is spin cobordant to $\bigcup_{i=1}^k T^2$, we either use the fact that the group $\Omega_2^{\mathrm{Spin}} = \mathbb{Z}_2$ is generated by the torus T^2 with a particular spin structure (so that Σ_g either is a spin boundary or is spin cobordant to a single copy of T^2), or argue as follows: Σ_g is cobordant to the disjoint union of g tori, and this cobordism can be chosen to be built on $[0,1] \times \Sigma_g$ by attaching 2-handles along the circles separating Σ_g into g copies of T^2. Now, no matter what the spin structure on Σ_g is, these circles inherit the bounding spin structure on S^1, and so the spin structure of $[0,1] \times \Sigma_g$ extends to the cobordism, which proves that Σ_g with the given spin structure is spin cobordant to the disjoint union of g tori (with various spin structures on them). •

The next lemma already completes the proof of our main Theorem 1.1. Since the proof of Lemma 2.4 below involves several steps, we first prove Theorem 1.1 and then return to the proof of the lemma.

Lemma 2.4. *The 3-dimensional torus T^3 is a spin boundary with any of its eight spin structures.*

Proof of Theorem 1.1. Once we have proved this last lemma, the proof of Theorem 1.1 is obviously complete, since by Lemmas 2.2 and 2.3 we have

$$[M, s] = [\Sigma_g \times S^1, \tilde{s}] = \sum_{i=1}^{k} [T^3, s_i] \quad \text{in} \quad \Omega_3^{\mathrm{Spin}}$$

(where s, $\tilde{s}$, and s_i are various spin structures on the corresponding manifolds), and by Lemma 2.4 this expression vanishes. •

Below, we give a proof of Lemma 2.4 using only basic handlebody theory.

§3. Proof of Lemma 2.4

First, we show that T^3 can be given as the boundary of the 4-dimensional handlebody obtained by gluing three 2-handles along the 0-framed Borromean rings in $S^3 = \partial D^4$. This can be verified as follows: T^2 admits a handle decomposition with one 0-, two 1-, and a 2-handle (see Fig. 1(a)).

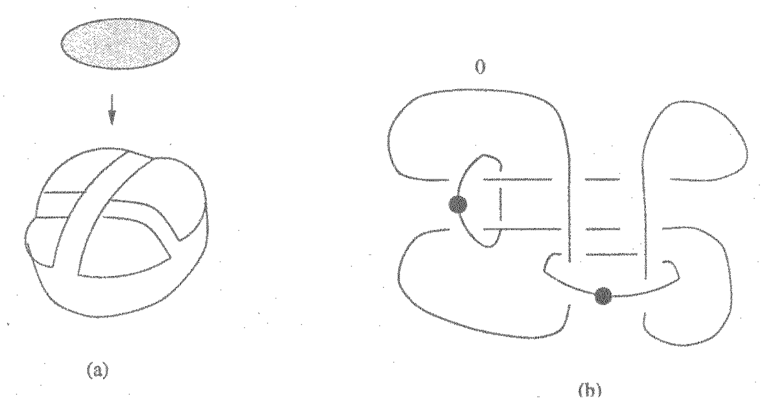

Fig. 1. (a) Handle decomposition of T^2, and (b) Kirby diagram for $T^2 \times D^2$.

Notice that the 2-handle passes through the first 1-handle, then through the second one, then returns to the first one from the opposite direction, and finally passes through the second one (from the opposite direction again). Now, multiplying everything with D^2, we obtain a handle decomposition of $T^2 \times D^2$ with boundary T^3; cf. Fig. 1(b). Surgery

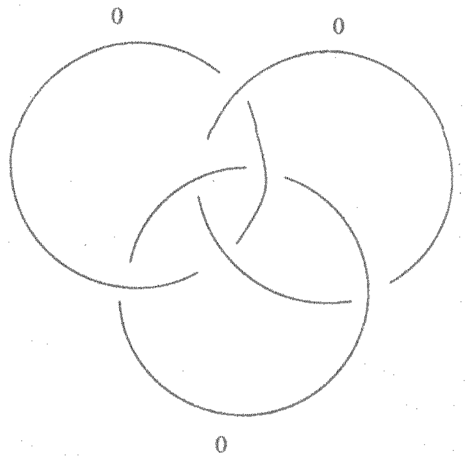

Fig. 2. Presentation of T^3.

along the 1-handles now provides the required presentation of T^3 (see Fig. 2).

Viewing T^3 as above, we see that the small normal circles $\{C_1, C_2, C_3\}$ of the Borromean rings B_1, B_2, and B_3 represent a basis of $H_1(T^3; \mathbb{Z})$. If we want to glue 2-handles to $[0,1] \times T^3$ along the C_i's, the given spin structure of T^3 tells us whether we should use even or odd framing in order to obtain a spin cobordism. (For even framing we always use 0, for odd we use -1.) Note that the eight different choices correspond to the eight possible spin structures on T^3.

Lemma 3.1. *Assume that the given spin structure on T^3 is such that one of the framings on C_1, C_2, and C_3 is 0. Then T^3 (with that given spin structure) is spin cobordant to S^3, hence spin bounds.*

Proof. Consider the 4-manifold Y obtained by gluing 2-handles to D^4 along the 0-framed Borromean rings B_1, B_2, and B_3. As we saw above, $\partial Y = T^3$. We define X as the handlebody obtained from Y by adding 2-handles along the normal circles C_i's (with the framing specified by the spin structure on T^3). Notice that $(X - \operatorname{int} Y)$ provides a spin cobordism between T^3 and ∂X, and so we only need to show that $\partial X = S^3$. (In doing this, we are free to change X as long as ∂X remains fixed.) Assume that the framing of C_1 is 0, and perform surgery on the 2-handle corresponding to C_1 (cf. the text before Definition 1.4). This surgery replaces this latter 2-handle with a 1-handle, which can be canceled by the 2-handle glued along B_1. Now B_2 and B_3 become unlinked, so that we can perform surgery along them. These surgeries result in cancelling pairs (with the 2-handles attached along C_2 and C_3), proving that three surgeries turn X into D^4; consequently, $\partial X = S^3$. Since S^3 admits a unique spin structure (which bounds the unique spin structure on D^4), we see that T^3 with the spin structure described in the lemma spin bounds a spin 4-manifold. •

§4. The end of the proof of Lemma 2.4

Finally, we must analyze the case where all framings on the C_i's are -1. First, we show that the boundary Σ of the handlebody X obtained by gluing handles along the B_i's (with framing zero) and C_i's (with framing -1) is a homology sphere (hence supports a unique spin structure). Then we show that it is the boundary of a spin 4-manifold, which implies that the unique spin structure on $\Sigma = \partial X$ spin bounds. This last step completes the proof of Lemma 2.4.

Lemma 4.1. *If the framings on all C_i's are -1, then the boundary of the above X is diffeomorphic to the boundary of the handlebody Z obtained by gluing a single 2-handle to D^4 along a $+1$-framed trefoil knot, see Fig. 3. Moreover, ∂Z is a homology 3-sphere, hence admits a unique spin structure.*

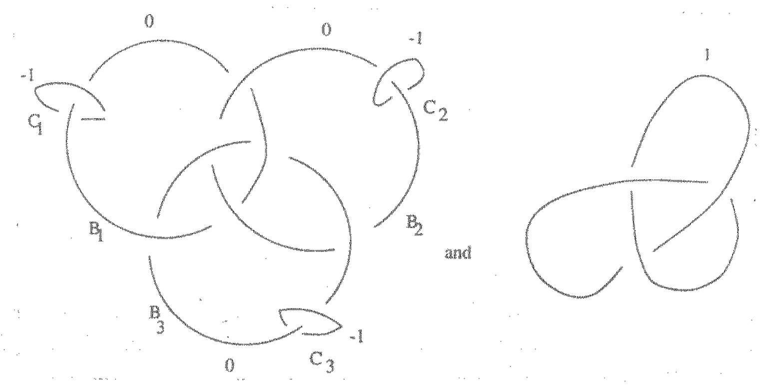

Fig. 3. Handlebodies with diffeomorphic boundaries.

Proof. Slide B_1 over C_2 twice – first use a band resulting an addition, then a subtraction of the two handles. Repeat the same process with B_1 and C_3. In this way, the resulting $\widetilde{B}_1$ is unlinked with B_2 and B_3, i.e., they can be separated by planes. (Note that algebraically B_1, B_2, and B_3 did not link each other – however, this does not mean that they could have been separated by planes.) Now surgeries on B_2 and B_3 (together with the handles glued along C_2 and C_3) create two cancelling pairs. After cancelling these pairs, it is not very hard to verify that $\widetilde{B}_1$ becomes a 0-framed trefoil knot with C_1 a (-1)-framed normal circle to it. Sliding C_1 off of B_1 and deleting it, we obtain the first claim of Lemma 4.1.

Finally, we show that $H_1(\partial Z; \mathbb{Z}) = 0$, i.e., ∂Z is a homology 3-sphere. To prove this, we use the fact that

$$\partial Z = (S^3 - \nu K) \cup_\varphi S^1 \times D^2,$$

where K is a knot and $\varphi\colon \partial(S^1 \times D^2) \to \partial(S^3 - \nu K)$ is specified by the following property: Assume that two circles $m, l \subset S^3 - \nu K$ represent a meridian and a longitude of νK, i.e., m is the boundary of a small normal disk to K and l is a parallel copy of K with linking number $\ell k(l, K) = 0$. Now on the first homology group

$$H_1(\partial(S^1 \times D^2); \mathbb{Z}) = \langle [S^1 \times \{p\}],\ [\{q\} \times \partial D^2] \rangle$$

φ_* must satisfy $\varphi_*([\{q\} \times \partial D^2]) = [m] - [l]$ (because the framing of the trefoil knot in the link diagram of Z is ± 1.)

Since $H_1(S^3 - \nu K; \mathbb{Z})$ is generated by the homology class $[m]$ of the meridian, an easy Mayer–Vietoris-type argument concludes the proof. (Note that the above argument shows that if X^4 is given as D^4 union a single 2-handle with framing n, then $H_1(\partial X; \mathbb{Z})$ is the cyclic group $\mathbb{Z}_n$.) •

Now we return to the proof of Lemma 2.4: The 3-manifold Σ obtained by $+1$-surgery on the trefoil knot can be identified with the link of the singularity $\{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$ (cf., e.g., [4]). As the link of this singularity (being a submanifold of S^5) it obviously bounds a spin 4-manifold: its Seifert surface. Consequently, Σ with its unique spin structure bounds a spin 4-manifold, which completes the proof of Lemma 2.4. (For a detailed proof of this last statement, see Remark 5.2.) We just note here that the above 3-manifold Σ is, in fact, diffeomorphic to the famous Poincaré homology 3-sphere. (For more about this 3-manifold, see [4].) •

§5. Concluding remarks

Remark 5.1. Note that in the above proof we showed that T^3 with a spin structure s is spin cobordant to a 3-manifold M admitting a unique spin structure, then (by finding the diffeomorphism type of M) we showed that M was a spin boundary. (For seven choices of s the corresponding 3-manifold M happened to be S^3, while for the last spin structure we got M to be the Poincaré homology sphere Σ.) The first step of the program, however, can easily be extended to arbitrary spin 3-manifolds, and we see that a spin 3-manifold M is spin cobordant to a 3-manifold N with $H^1(N;\mathbb{Z}_2)=0$, i.e., N admits a unique spin structure.

Remark 5.2. For the sake of completeness, we sketch the proof of the fact that Σ (given as the boundary of the handlebody with a unique 2-handle glued along the $+1$-framed trefoil) bounds a spin 4-manifold. More precisely, we show that the boundary of the 4-manifold X obtained by gluing eight 2-handles along the $+2$-framed knots as given in Fig. 4 is diffeomorphic to Σ.

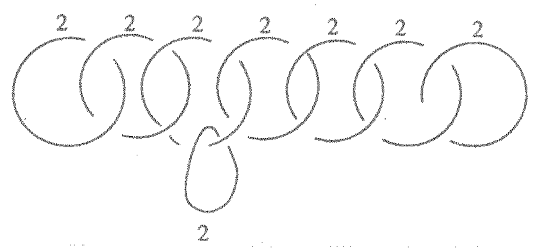

Fig. 4. Spin 4-manifold with boundary Σ.

This X is spin since all framings are even, hence the unique spin structure of D^4 extends over each 2-handle. (Essentially the same handlebody calculation can be found in [4].) Add a 2-handle along a -1-framed unknot to X. (Recall that this operation changes X but leaves its boundary fixed.) By sliding this last added 2-handle over the rightmost $+2$-framed handle we turn our presentation as it is shown by Fig. 5.

Notice that the framing of the $+2$-framed circle becomes $+1$ after this slide; consequently, it can be deleted. When deleting, however, it changes the framings of the linking circles by decreasing the framings by 1, moreover these circles get linked. (Verify this by sliding both handles over the $+1$-framed one and then delete the now separated $+1$-framed unknot.) This means that we can repeat this process and delete the $+1$-framed unknots (which decrease the framings of all linking circles). At the end of the process, we obtain the 4-manifold given by Fig. 6 – with the same boundary Σ.

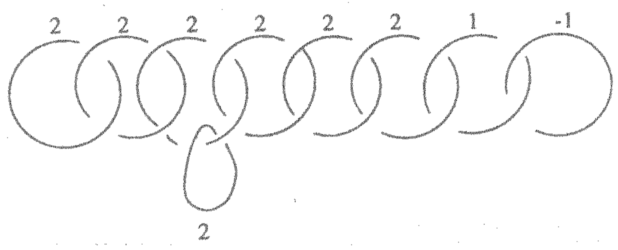

Fig. 5. Result of a handleslide.

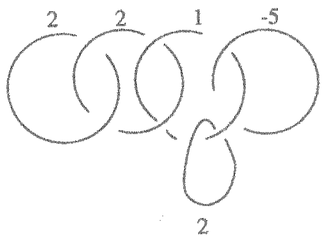

Fig. 6. An intermediate stage.

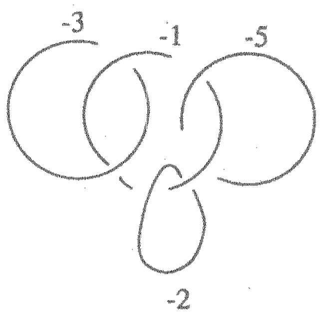

Fig. 7. An intermediate stage.

After the same process has been applied to the leftmost and the bottom $+2$-circles, we obtain Fig. 7. Now sliding the handle corresponding to the -1-framed circle to all three remaining ones, the -1-circle gets separated and can be deleted without changing the boundary of the handlebody. The resulting diagram is shown by Fig. 8(a); repeating the same process with the -1-framed unknot of this diagram, we obtain Fig. 8(b) – representing a handlebody still with the same boundary. Now sliding the

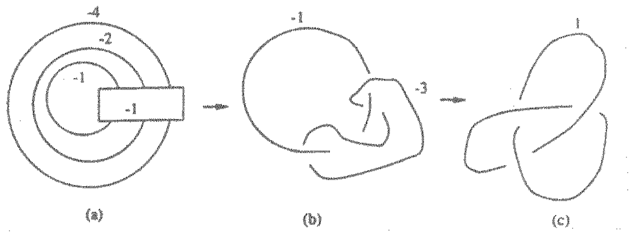

Fig. 8. The final handleslides.

-1-framed unknot of this latter diagram twice over the remaining handle and then deleting it, we arrive at Fig. 8(c) – which proves the assertion.

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