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S. M. Finashin

ROKHLIN'S QUESTION AND QUOTIENTS OF REAL ALGEBRAIC SURFACES BY THE COMPLEX CONJUGATION

Abstract. Complex algebraic surfaces defined over $\mathbb{R}$ are considered. Local and global topological properties of their quotients by the complex conjugation are discussed.

This paper is a brief exposition based mainly on the paper [5] and the preprint [6]. We only sketch the proofs, or even omit them referring to [5, 6] for the details.

Through the paper, we denote by $\mathbb{C}X$ the complex point set of a real algebraic variety X, by $\mathbb{R}X$ its real locus, and by $\widehat{X} = \mathbb{C}X/\operatorname{conj}$ the orbit space of the complex conjugation. We also use in a similar sense the prefixes $\mathbb{C}$ and $\mathbb{R}$ and a hat for conj-invariant subsets of such varieties $\mathbb{C}X$.

§1. Some homotopy properties of the quotients by the complex conjugation

In this section, we present arguments showing that the well-known homotopy properties of complex algebraic varieties, both global (like the Lefschetz hyperplane section theorem) and local (like the Milnor lemma and its generalizations), can be formulated similarly for the quotients by the complex conjugation of the corresponding complex objects, and deduced from the properties of the latter.

As an example, we consider a version of the Milnor lemma for the quotients by the complex conjugation of real isolated complete intersection singularities (ICIS). Given such a singularity $f\colon (\mathbb{C}^{d+k},0)\to(\mathbb{C}^k,0)$, we denote by $\mathbb{C}U$ the intersection of the locus $\{f=0\}$ with a compact small ε-ball around zero, $0<\varepsilon<<1$, $\mathbb{C}B\subset\mathbb{C}^{d+k}$. Given a small real deformation $f^\tau : (\mathbb{C}^{d+k},0)\to(\mathbb{C}^k,0)$, $0<|\tau|<<\varepsilon$, of $f=f^0$, we denote by

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$\mathbb{C}U^\tau = \{f^\tau = 0\} \cap \mathbb{C}B$ the corresponding deformation of $\mathbb{C}U$. If $\mathbb{C}U^\tau$ is assumed to be nonsingular and f^τ to be a morsification (i.e., to be generic), then we call f^τ a *perturbation* (in general, we do not require this). It is well known (and follows, e.g., from [8, p. 121], or [3, p. 25]) that $\mathbb{C}U^\tau$ is homotopy equivalent to a wedge of d-spheres for any deformation f^τ.

1.1. Theorem. *Assume that $\mathbb{C}U^\tau$ is a deformation of a cone-like compact neighborhood $\mathbb{C}U$ for a real ICIS of dimension $d \geqslant 1$. Then the quotient $\widehat{U}^\tau$ is homotopy equivalent to a wedge of d-spheres, provided that $\mathbb{R}U^\tau \neq \varnothing$. If $\mathbb{R}U^\tau = \varnothing$, then $\widehat{U}^\tau$ still has rational homology of a wedge of d-spheres.*

Proof. Note that for $d = 1$ the statement of Theorem 1.1 is trivial, and the condition $\mathbb{R}U^\tau \neq \varnothing$ implies that $\widehat{U}^\tau$ is simply connected for $d \geqslant 2$. Furthermore, $\pi_k(\widehat{U}^\tau) = 0$ for $2 \leqslant k \leqslant d-1$, since a generic mapping of S^k to $\widehat{U}^\tau$ does not intersect $\mathbb{R}U^\tau$ and can be lifted to $\mathbb{C}U^\tau$. In addition, if F is a field of characteristic $\neq 2$, for instance, $\mathbb{Q}$ or $\mathbb{Z}/p$ for a prime $p \neq 2$, then $H_k(\widehat{U}^\tau; F) = 0$ for $k \neq d$ (due to the well-known fact that the projection $\mathbb{C}U^\tau \to \widehat{U}^\tau$ induces an isomorphism between $H^k(\widehat{U}^\tau; F)$ and the conj_*-invariant subspace of $H^k(\mathbb{C}U^\tau; F)$).

It also easily follows from the Smith sequence for conj (cf. [7]) that

$$H_k(\widehat{U}^\tau, \mathbb{R}U^\tau; \mathbb{Z}/2) = H_{k+1}(\widehat{U}^\tau, \mathbb{R}U^\tau; \mathbb{Z}/2) \quad \text{for} \quad k \geqslant d+1.$$

Thus,

$$H_k(\widehat{U}^\tau; \mathbb{Z}/2) = H_k(\widehat{U}^\tau, \mathbb{R}U^\tau; \mathbb{Z}/2) = 0 \quad \text{for} \quad k \geqslant d+1.$$

Furthermore, $H_d(\widehat{U}^\tau)$ is torsion free by the universal coefficients formula, since $H_{d+1}(\widehat{U}^\tau; \mathbb{Z}/p) = 0$ for any prime p. Finally, it remains to apply the Whitehead theorem. •

The same arguments similarly work in a more general setting. For instance, for nonisolated real singularities, they prove that $\widehat{U}^\tau$ and $\partial\widehat{U} \cong \partial\widehat{U}^\tau$ have the same connectedness properties as $\mathbb{C}U^\tau$ and $\partial\mathbb{C}U$, respectively, unless conj acts freely on $\mathbb{C}U^\tau$ and $\partial\mathbb{C}U$, respectively. The connectedness properties of $\mathbb{C}U^\tau$ and $\partial\mathbb{C}U$ can be found, e.g., in [3, p. 76].

A version of the Lefschetz hyperplane section theorem for the quotients $\widehat{X}$ can be formulated as follows.

1.2. Theorem. *Assume that $(\mathbb{C}X, \mathbb{C}A)$ is an n-connected pair of real algebraic varieties, and the complex conjugation in $\mathbb{C}A$ is induced from*

$\mathbb{C}X$. Then the pair $(\widehat{X}, \widehat{A})$ is also n-connected. This is the case, in particular, if $\mathbb{C}X \subset \mathbb{C}\mathrm{P}^N$, $\dim_{\mathbb{C}}(\mathbb{C}X) = n+1$, and $\mathbb{C}A = \mathbb{C}X \cap \mathbb{C}H$, where $\mathbb{C}H \subset \mathbb{C}\mathrm{P}^N$ is a real hyperplane which has a generic intersection with every stratum of a semi-algebraic Whitney stratification of $\mathbb{C}X$.

The proof is similar to the previous one and is left for the reader.

§2. The case of real curves and surfaces: around Rokhlin's question

V. A. Rokhlin emphasized in his seminars and in private conversations that the study of specific properties of the quotients $\widehat{X}$ is important for our knowledge of real algebraic surfaces. He raised several questions involving real plane algebraic curves and the double planes branched along these curves. One of such questions is discussed below.

Assume that $\mathbb{C}X$ is a nonsingular complex surface. Then $\widehat{X}$ is a 4-manifold and the quotient map $q\colon \mathbb{C}X \to \widehat{X}$ is a double covering branched along the real locus $\mathbb{R}X$ of $\mathbb{C}X$. One can endow $\widehat{X}$ with an orientation and a smooth structure making q smooth and orientation preserving, for example, $\widehat{\mathrm{P}}^2 = \mathbb{C}\mathrm{P}^2/\operatorname{conj}$ is well known to be diffeomorphic to S^4.

The curve $\mathbb{C}A \subset \mathbb{C}\mathrm{P}^2$ defined by a real form $f(x,y,z)$ of degree $d = 2k$, with $\mathbb{R}A \neq \varnothing$, divides $\mathbb{R}\mathrm{P}^2$ into two regions,

$$\mathbb{R}\mathrm{P}^2_{\pm} = \{[x:y:z] \in \mathbb{R}\mathrm{P}^2 \mid \pm f \geqslant 0\}.$$

If $\mathbb{R}A$ is nonsingular, then one of these regions is orientable, while the other is not, and for the sake of definiteness it is usually assumed that $\mathbb{R}\mathrm{P}^2_{+}$ is the orientable region.

The unions $\mathfrak{A}_{\pm} = \mathbb{R}\mathrm{P}^2_{\pm} \cup \widehat{A}$ are closed surfaces in $S^4 = \widehat{\mathrm{P}}^2$, called by Rokhlin the *Arnold surfaces*. They consist of two smooth pieces, $\mathbb{R}\mathrm{P}^2_{\pm}$ and $\widehat{A}$, which meet orthogonally along $\mathbb{R}A$, and therefore $\mathfrak{A}_{\pm}$ can be smoothed along $\mathbb{R}A$. Rokhlin asked if the embeddings $\mathfrak{A}_{\pm} \subset S^4$ are *standard*, in the sense that they can be obtained via ambient connected sum from several copies of a standard embedding of a torus $T^2 \subset S^4$ if $\mathfrak{A}_{\pm}$ is orientable, or from standard embeddings of $\mathbb{R}\mathrm{P}^2$ if $\mathfrak{A}_{\pm}$ is not orientable. The surfaces $\mathfrak{A}_{\pm}$ were considered by Arnold [1] in connection with studying the topology of the double planes $p\colon X \to \mathbb{C}\mathrm{P}^2$ branched along $\mathbb{C}A$. There exist 2 liftings $\operatorname{conj}^{\pm}$ of the complex conjugation from $\mathbb{C}\mathrm{P}^2$ to X. The double plane X endowed with an involution $\operatorname{conj}^{\pm}$ is denoted by $\mathbb{C}X^{\pm}$. Thus, if the Arnold surface $\mathfrak{A}_{\mp}$ is standard, then $\widehat{X}^{\pm}$ is decomposable into a connected sum of several copies of $\mathbb{C}\mathrm{P}^2$ and $\overline{\mathbb{C}\mathrm{P}}^2$ if $\mathfrak{A}_{\mp}$ is nonorientable, or copies of $S^2 \times S^2$ if orientable. Such a splitting is called a *complete decomposition.* A weaker form of Rokhlin's question is:

is it true that the $\widehat{X}^\pm$ admits such a decomposition for any double plane $\mathbb{C}X^\pm$ with $\mathbb{R}X^\pm \neq \varnothing$?

I call the weaker question formulated in affirmative form the *CDQ-Conjecture* (CDQ stands for "Complete Decomposability of Quotients").

A weaker version of the CDQ-Conjecture was suggested by Akbulut:

to prove that the Seiberg–Witten invariants of $\widehat{X}^\pm$ vanish if $b_2^+(\widehat{X}^\pm) > 1$.

As an intermediate variation, I present one more conjecture. Let us call 4-manifolds X and Y *blow-up stably equivalent* (BUS-equivalent) if $X \#_n \overline{\mathbb{C}\mathrm{P}}^2$ is diffeomorphic to $Y \#_m \overline{\mathbb{C}\mathrm{P}}^2$ for some $n, m \geqslant 0$. If X is BUS-equivalent to $\#_n \mathbb{C}\mathrm{P}^2$, $n \geqslant 0$, then we call X *BUS-trivial.* Complete decomposability implies BUS-triviality, but not vice versa, as follows from the results of Mandelbaum and Moishezon [9]. On the other hand, the BUS-triviality guarantees vanishing of the Donaldson and SW-invariants (more generally, vanishing of these invariants is a property which is preserved under BUS-equivalence). So, it is natural to suggest an intermediate *BUS-Conjecture* stating that the quotients $\widehat{X}^\pm$ are BUS-trivial for the double planes $\mathbb{C}X^\pm$ with $\mathbb{R}X^\pm \neq \varnothing$. This is weaker than CDQ-conjecture, but stronger than Akbulut's conjecture.

2.1. Theorem [6]. *Assume that a real plane curve $\mathbb{C}A_0$ of degree $2k$ with $\mathbb{R}A_0 \neq \varnothing$ is the union of transversal nonsingular real curves: $\mathbb{C}A_0 = \mathbb{C}B_0 \cup \mathbb{C}C_0$, and a nonsingular real curve $\mathbb{C}A$ is obtained from $\mathbb{C}A_0$ by a small perturbation. Let $\mathbb{C}X^\pm$ denote the double plane branched along $\mathbb{C}A$. Then both quotients $\widehat{X}^+$ and $\widehat{X}^-$ are BUS-trivial. In particular, the SW-invariants of $\widehat{X}^\pm$ vanish if $k > 3$.*

The proof is based on the analysis of the bifurcations which $\widehat{X}^\pm$ can experience as we deform $\mathbb{C}B_0$ and $\mathbb{C}C_0$. A generic deformation of such type may involve simple tangency points, i.e., simple singularities of type A_3, and the corresponding bifurcations of $\widehat{X}^\pm$ can easily be analyzed. It turns out that such bifurcations can be reduced to blow-ups and blow-downs only.

§3. Quotients of real simple singularities

More generally, a property similar to the above one holds not only for A_3, but also for any simple singularity. Namely, using the classification of deformations of real simple singularities [2], one can prove the following result.

3.1. Theorem. *If a real surface $\mathbb{C}X$ has a simple singularity at $s \in \mathbb{R}X$ which is not equivalent to the singularity defined in $\mathbb{C}^3$ by the equation $x^{2n}+y^2+z^2=0$, and $\mathbb{C}X^\tau$ is obtained by a real nonsingular perturbation of $\mathbb{C}X$, then the BUS-equivalence class of $\widehat{X}^\tau$ does not depend on the perturbation. In the case of the above exceptional singularity, the same claim is true for all perturbations except the one for which the real locus $\mathbb{R}X^\tau$ vanishes near s.*

In fact, $\widehat{X}$ has at s a singularity topologically equivalent to an algebraic surface singularity and $\widehat{X}^\tau$ is BUS-equivalent to the manifold $\widehat{X}^{\mathrm{res}}$ obtained from $\widehat{X}$ by resolution of this singularity. This is proved either by elementary means or by using a more sophisticated technique, namely, variation of the complex structure in a neighborhood of a singularity so that the involution of complex conjugation becomes holomorphic (a similar trick was used by Donaldson [4] in the case of real K3 surfaces).

An interesting question is whether the other types of the singularities have a similar property. This property is obviously formulated for the singular points of $\mathbb{C}X$ which become smooth in $\widehat{X}$. Moreover, the Rokhlin Question, as well as its versions, admits a local version for the singularities of that kind. Let us consider such singularities in more detail.

§4. SF-singularities

Note that one of the or both Arnold surfaces $\mathfrak{A}_\pm$ may turn out to be smoothable even if we admit certain types of real singularities on $\mathbb{C}A$. The condition of smoothability is that the links of $\mathfrak{A}_\pm \subset S^4$ at the singular points $x \in \mathbb{R}A$ are unknots. These unknots are piecewise smooth, and their smoothings can easily be extended to smoothings of the embeddings $\mathfrak{A}_\pm \subset S^4$. This gives a smoothing of the double covering $\widehat{X}^\pm \to S^4$ branched along $\mathfrak{A}_\pm$ for the double planes $\mathbb{C}X^\pm \to \mathbb{C}\mathrm{P}^2$ branched along $\mathbb{C}A$.

A singularity at a point $x \in \mathbb{R}X$ of a real algebraic surface will be called *smoothly-folding*, or briefly, SF-singularity, if the link of $\widehat{X}$ at x is a 3-sphere (so, $\widehat{X}$ is a manifold around x).

A cone-like node, which looks like the locus $\{x^2+y^2-z^2=0\} \subset \mathbb{C}^3$, is an example of an SF-singularity, whereas a dot-like node, which looks like the locus $\{x^2+y^2+z^2=0\}$, is not. Another example of an SF-singularity appears on the double plane $\mathbb{C}X^\pm$ if the branching locus $\mathbb{C}A$ splits at

$x \in \mathbb{R}A$ into m nonsingular real and pairwise transversal branches, where m is odd. More examples: any simple surface singularity has a real form which is an SF-singularity; a natural notation would be to mark this form by a superscript "$-$", like A_n^-, D_n^-, E_n^-.

The SF-singularities can be characterized in terms of a minimal resolution $\mathrm{res}\colon \mathbb{C}X^{\mathrm{res}} \to \mathbb{C}X$ and the exceptional curve $\mathbb{C}E \subset \mathbb{C}X^{\mathrm{res}}$ over x. The complex conjugation defines an involution on the resolution graph Γ and we denote by $\widehat{\Gamma}$ the quotient graph with respect to this involution.

4.1. Theorem [6]. *Assume that $x \in \mathbb{R}X$ is an isolated normal real surface singularity, and $\mathbb{C}E$ is the exceptional curve of its minimal resolution. Then the following conditions are equivalent:*

(1) *x is an SF-singularity,*
(2) *$\widehat{E}$ is contractible,*
(3) *$\mathbb{C}E$ splits into real rational irreducible components, $\mathbb{C}E = \mathbb{C}E_1 \cup \dots \cup \mathbb{C}E_n$, where $\widehat{E}_1, \dots, \widehat{E}_m$ are homeomorphic to the 2-disk D^2, and the quotient graph $\widehat{\Gamma}$ is a tree.*

§5. Vanishing of SW-invariants for $\widehat{X}$ after a perturbation of an SF-singularity

Let $\mathbb{C}X$ be a real surface with one singular point $x \in \mathbb{R}X$ and let a nonsingular surface $\mathbb{C}X^\tau$ be obtained from $\mathbb{C}X$ by a real perturbation. More precisely, we assume that there exists a compact regular cone-like neighborhood $\mathbb{C}U \subset \mathbb{C}X$ of x such that the complement $\mathbb{C}X - \mathrm{Int}(\mathbb{C}U)$ is conj-equivariantly diffeomorphis to the complement $\mathbb{C}X^\tau - \mathrm{Int}(\mathbb{C}U^\tau)$ of some compact conj-invariant codimension 0 submanifold $\mathbb{C}U^\tau \subset \mathbb{C}X^\tau$ of a smooth real algebraic surface $\mathbb{C}X^\tau$. In the standard example which I mean, $\mathbb{C}U^\tau$ is a Milnor fiber in the case of a complete intersection singularity. (For instance, $\mathbb{C}X$ may be a double plane branched along a curve $\mathbb{C}A$ with one singular point, and $\mathbb{C}X^\tau$ a double plane branched along a nonsingular curve $\mathbb{C}A^\tau$ obtained by a perturbation of $\mathbb{C}A$.)

5.1. Theorem [6]. *Assume that $\mathbb{C}X$ and $\mathbb{C}X^\tau$ are as above, $x \in \mathbb{R}X$ is an SF-singularity, and $0 < p_g(\mathbb{C}X^{\mathrm{res}}) < p_g(\mathbb{C}X^\tau)$, where p_g is the geometric genus and $\mathbb{C}X^{\mathrm{res}} \to \mathbb{C}X$ is a real resolution. Then the SW-invariants of $\widehat{X}^\tau$ vanish.*

Proof. We have $\partial(\widehat{U}) \cong \partial(\widehat{U}^\tau) \cong S^3$, since x is an SF-singularity. Thus, $\widehat{X}^\tau$ is obtained by taking a connected sum of $\widehat{X}$ with a 4-manifold $\hat{U}^\tau$ obtained by capping $\widehat{U}^\tau$ with a 4-ball.

Standard and well-known calculations imply that $b_2^+(\widehat{X}^\tau) = p_g(\mathbb{C}X^\tau)$ and $b_2^+(\widehat{X}) = p_g(\mathbb{C}X^{\mathrm{res}})$ (cf. Lemma 4.2.2 in [7]). Therefore, $\widehat{X}^\tau$ has trivial SW invariants as a connected sum of manifolds with nonnegative intersection forms. •

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Санкт-Петербургский
электротехнический университет
email serge@metu.edu.tr

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