Записки научных
семинаров ПОМИ
Том 272, 2000 г.

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UNITS IN GROUP ALGEBRAS OF FINITE p-GROUPS

Abstract. Some properties of the factors of the upper central series and a subgroup of the group of outer automorphisms of the normalized group of units are described.

Dedicated to Professor Anatolii Jakovlev on his 60th birthday

Let $\mathbb{F}_p G$ be the group algebra of a finite p-group G over a field of p elements, and $V(\mathbb{F}_p G)$ the subgroup of units of augmentation 1. Then $V(\mathbb{F}_p G)$ is a finite p-group of order $p^{|G|-1}$. One of the hardest and most important problems for modular group algebras consists in describing the structure of $V(\mathbb{F}_p G)$ and the group automorphism of the normalized group of units.

First Coleman and Passman [4] proved if G is nonabelian then $V(\mathbb{F}_p G)$ is not a p-regular group and has exponent at least p^2. Shalev [10] and Kurdics [7] pointed out that $V(\mathbb{F}_p G)$ is not metabelian for $p > 3$ and Shalev and Mann [8] showed that the wreath product $C_p \wr A$ for $|A| = p^2$ is a section of $V(\mathbb{F}_p G)$ under weak hypotheses on G.

In this paper we describe some properties of the factors the upper central series and one subgroup of the group outer automorphism of the normalized group of units.

Central series of the unit group.

Let G be a finite p-group and let $A(\mathbb{F}_p G)$ be an augmentation ideal of $\mathbb{F}_p G$. Then the augmentation ideal $A(\mathbb{F}_p G)$ is nilpotent and the adjoint group $(A(\mathbb{F}_p G), \circ)$ is isomorphic to $V(\mathbb{F}_p G)$. Therefore $V(\mathbb{F}_p G)$ is a circle group.

Research was supported by Hungarian National Fund for Scientific Research grant Nos. T 025029 and T 029525.

Let $L(A)$ be the associated Lie algebra of the augmentation ideal $A(\mathbb{F}G)$ and denote by C_k the k-th term of the upper central series of $L(A)$. Then

$$C_k = \{a \in A(\mathbb{F}G) \mid [a, x] \in C_{k-1} \quad \text{for all} \quad x \in A(\mathbb{F}G)\}.$$

Du's theorem [5] described the k-th term $\zeta_k(V)$ of the upper central series of $V(\mathbb{F}_pG)$ as

$$\zeta_k(V) = 1 + C_k.$$

It is well-known that $L(A)$ is a restricted Lie algebra, thus it satisfies the identity

$$[x, y, p] = [x, y, y, \dots, y] = [x, y^p]. \tag{1}$$

Thus if $u = 1+a \in \zeta_k(V)$ and $v = 1+b \in V$ then $a \in C_k$ and $b \in A(\mathbb{F}_pG)$. As a consequence (1), we obtain the assertion: $[a^p, b] \in C_{k-p}$ for all $b \in A(\mathbb{F}_pG)$ and $a^p \in C_{k-p+1}$. By Du's theorem $v^p = 1 + a^p \in \zeta_{k-p+1}(V)$ and the factor group $\zeta_k(V)/\zeta_{k-p+1}(V)$ has exponent p.

We obtain the following assertion:

Theorem 1. *If G is a finite p-group, then the factors $\zeta_{i+1}(V)/\zeta_i(V)$ of the upper central series, with the exception of the centre of $V(\mathbb{F}_pG)$, are elementary abelian p-groups and for $k > p$ the factors*

$$\zeta_k(V)/\zeta_{k-p+1}(V)$$

are groups with exponent p.

Note that according to Kruze's theorem [5] $V(\mathbb{F}_pG)$ has a central series

$$V(\mathbb{F}G) = B_1 \supset B_2 \supset B_3 \cdots \supset B_{c+1} = \langle 1 \rangle$$

with factors B_i/B_{i+1} $(1 \leq i \leq c-2)$ of order divisible by p^2, where c is the nilpotency class of $V(\mathbb{F}_pG)$.

Question 1 Probably the conjecture is true that the order of almost all factors $\zeta_{i+1}(V)/\zeta_i(V)$ of the upper central series of $V(\mathbb{F}_pG)$ is divisible by p^2.

Recall that by Coleman's theorem [3] the intersection of the commutator subgroup V' of $V(\mathbb{F}_pG)$ with G is the commutator subgroup G' of G. Therefore, G/G' is isomorphic to a subgroup of V/V' and the factors of the lower central series of V do not need to be elementary abelian subgroups.

Centre of the unit group.

Let C_g be the conjugacy class of G containing the element g. The element $\widehat{C_g} = \sum_{g \in C_g} g$ of the group algebra $\mathbb{F}_p G$ will be called a *class sum*. Clearly, in the group algebra $\mathbb{F}_p G$ all class sums form a basis of the centre of the group algebra.

Definition. *Let G be a finite p-group and let C_g be the conjugacy class of G containing the element $g \notin \zeta(G)$ ($\zeta(G)$ is the centre of G). The smallest nonnegative integer d for which $C_G(g) = C_G(g^{p^d})$ holds and $C_G(g^{p^d})$ is a proper subgroup of the group $C_G(g^{p^{d+1}})$, will be called the depth of the conjugacy class C_g.*

Recall that as in modular theory of representation group the Sylow p-subgroup of the centralizer of any element of a conjugacy class is the *defect group of the conjugacy class*. In the case of p-group it coincides with the centralizer. Clearly, the defect group of a conjugacy class is determined uniquely disregarding conjugacy.

Lemma 1. *Let $\mathbb{F}_p G$ be the group algebra of the finite p-group G over a field of p elements. Then:*

(i) *if the depth of the conjugacy class C_g is d then $\widehat{C_g}^{p^i} = \widehat{C_{g^{p^i}}}$ for $i = 1, \dots, d$ and $\widehat{C_g}^{p^{d+1}} = 0$;*

(ii) *if the depth of the conjugacy class C_g is d, $r \le d$ and c is an element of order p^r central in $C_G(g)$ then the conjugacy classes C_g and C_{gc} are distinct and the depth of the conjugacy class C_{gc} is d;*

(iii) *in case $D = C_G(g) = C_G(h)$ we have $\widehat{C_g}^{p^i} = \widehat{C_h}^{p^i} \neq 0$ if and only if there exist the elements $u \in N_G(D)$ and $c \in \zeta(D)$ central in D, such that $c^{p^i} = 1$ and $g = u^{-1}huc$, where i is not greater than the depths of the conjugacy classes C_g and C_h;*

(iv) *if the conjugacy classes C_g and C_h are such that $\widehat{C_g}^{p^{l+1}} = \widehat{C_h}^{p^{l+1}} \neq 0$, then the defect groups of C_g and C_h coincide;*

(v) *let $\widehat{C_g}^{p^i} \neq \widehat{C_h}^{p^i}$; then $\widehat{C_g}^{p^{i+1}} = \widehat{C_h}^{p^{i+1}} \neq 0$ if and only if $g^{p^i} d = h^{p^i}$ for some $d \in \zeta(D)^{p^i}[p]$, where D is a defect group of C_g.*

Proof.

(i) Let $C_g = \{g = g_1, g_2, \dots, g_s\}$. Obviously,

$$\widehat{C_g}^{p^i} = \sum_{j=1}^{s} g_j^{p^i} + \sum g_{j_1} g_{j_2} \cdots g_{j_{p^i}}, \tag{2}$$

where $g_{j_k} \in C_g$ and any summand $g_{j_1} g_{j_2} \cdots g_{j_{p^i}}$ of the second sum contains at least two distinct elements. We shall prove that in this sum each summand occurs p times, i.e. the sum equals zero in p characteristic.

If the elements $g_{j_1}, g_{j_2}, \dots, g_{j_{p^i}}$ commute then the statement is immediate. Now suppose that there are noncommuting elements among them. We claim $\cap_{k=1}^{p^i} C_G(g_{j_k})$ is a proper subgroup of $C_G(g_{j_1} g_{j_2} \cdots g_{j_{p^i}})$. Indeed, in the opposite case if $b = g_{j_1} g_{j_2} \cdots g_{j_{p^i}}$ then $C_G(b) = \cap_{k=1}^{p^i} C_G(g_{j_k})$ and $b \in C_G(g_{j_k})$ for any k. It follows $g_{j_k} \in C_G(b)$, and hence from $C_G(b) = \cap_{k=1}^{p^i} C_G(g_{j_k})$ we conlcude $g_{j_k} \in C_G(g_{j_l})$ for any indices k, l, contradicting the assumption. Therefore there exists $v \in C_G(b)$ which is not in the subgroup $\cap_{k=1}^{p^i} C_G(g_{j_k})$. Then $v^{-1} g_{j_l} v \in C_g$, and we see that $(v^{-1} g_{j_1} v)(v^{-1} g_{j_2} v) \cdots (v^{-1} g_{j_{p^i}} v)$ occurs in the sum (2) and equals $g_{j_1} g_{j_2} \cdots g_{j_{p^i}}$. Clearly, the number of these elements is

$$|C_G(g_{j_1} g_{j_2} \cdots g_{j_{p^i}}) : \cap_{k=1}^{p^i} C_G(g_{j_k})|,$$

and the index of the subgroup is a p-power. We have shown that if $i \le d$ then

$$\sum_{j=1}^{s} g_j^{p^i} = \widehat{C_{g^{p^i}}} \tag{3}$$

is a class sum. It implies $\widehat{C_g}^{p^i} = \widehat{C_{g^{p^i}}}$ for any $i < d+1$.

In the case $i = d+1$ the support of the left hand side of the equality (3) contains g^{p^i}, and $|C_G(g^{p^{d+1}}) : C_G(g)|$ is the coefficient of the element, which is zero in p-characteristic, thus $\widehat{C_g}^{p^{d+1}} = 0$ and (i) has been proved.

(ii) Let $r \le d$, and let c be an element of order p^r central in $C_G(g)$. It is easy to check that

$$C_G(g) \subseteq C_G(gc) \subseteq C_G((gc)^p) \subseteq \cdots \subseteq C_G((gc)^{p^d}) = C_G(g^{p^d}),$$

which follows $C_G(gc) = C_G((gc)^{p^d})$ and

$$C_G((gc)^{p^d}) = C_G((g)^{p^d}) \subset C_G((g)^{p^{d+1}}) = C_G((gc)^{p^{d+1}}).$$

Hence the depth of the conjugacy class C_{gc} is d. Assume $C_g = C_{gc}$. Then $v^{-1}gv = gc$ and $v^{-1}g^{p^d}v = (gc)^{p^d} = g^{p^d}$. Therefore $v \in C_G(g^{p^d}) = C_G(g)$, from which we infer $gc = g$, a contradiction, proving (ii).

(iii) Let $D = C_G(g) = C_G(h)$ and $\widehat{C_g}^{p^i} = \widehat{C_h}^{p^i} \neq 0$. Then by the first statement of the Lemma $\widehat{C_{g^{p^i}}} = \widehat{C_{h^{p^i}}}$, and there exists $u \in G$ such that $g^{p^i} = u^{-1}h^{p^i}u$. Since i is not greater than the depths of the conjugacy classes C_g and C_h, it is easy to check that $D = C_G(g^{p^i}) = C_G(h^{p^i})$ and $C_G(g^{p^i}) = u^{-1}C_G(h^{p^i})u = u^{-1}C_G(g^{p^i})u$ and $u \in N_G(D)$, whence the elements g and $u^{-1}hu$ commute. Apparently, $c = u^{-1}h^{-1}ug$ is central in $C_G(g)$ and $c^{p^i} = u^{-1}h^{-p^i}ug^{p^i}$. Since $g^{p^i} = u^{-1}h^{p^i}u$ we conclude that $c^{p^i} = 1$.

The converse is easy to verify by (i),

$$\widehat{C_g}^{p^i} = \widehat{C_{u^{-1}huc}}^{p^i} = \widehat{C_{u^{-1}h^{p^i}u}} = \widehat{C_{h^{p^i}}}.$$

(iv) Let C_g and C_h be conjugacy classes such that $\widehat{C_g}^{p^{l+1}} = \widehat{C_h}^{p^{l+1}} \neq 0$. Then the depths of the conjugacy classes C_g and C_h are not less that $l+1$ and $C_{g^{p^{l+1}}} = C_{h^{p^{l+1}}}$ by (i), and $g^{p^{l+1}} = u^{-1}h^{p^{l+1}}u$ for some $u \in G$. Thus

$$C_G(u^{-1}hu) = C_G(u^{-1}h^{p^{l+1}}u) = C_G(g^{p^{l+1}}) = C_G(g),$$

therefore the defect groups of C_g and C_h coincide up to conjugacy.

(v) Let $\widehat{C_g}^{p^i} \neq \widehat{C_h}^{p^i}$ and let $\widehat{C_g}^{p^{i+1}} = \widehat{C_h}^{p^{i+1}} \neq 0$. By (iv), the defect group D of C_g and C_h coincides and there exist the elements $u \in N_G(D)$ and $c \in \zeta(D)$ such that $c^{p^{i+1}} = 1$ and $g = u^{-1}huc$ according to (iii). Thus $g^{p^i} = u^{-1}h^{p^i}uc^{p^i}$ and the element $d = c^{p^i} \in \zeta(D)^{p^i}[p]$ has order p.

Now we will show the converse statement. If $d \in \zeta(D)^{p^i}[p]$, then $d = c^{p^i}$ for some $c \in \zeta(D)$ and we conclude that

$$\widehat{C_g}^{p^i} = \widehat{C_{g^{p^i}}} = \widehat{C_{u^{-1}h^{p^i}ud}} = \widehat{C_{u^{-1}huc}}^{p^i}.$$

According to (iii) the conjugacy classes $C_{u^{-1}h^{p^i}ud}$ and $C_{u^{-1}h^{p^i}u}$ are different by (ii). Thus $\widehat{C_g}^{p^i} \neq \widehat{C_h}^{p^i}$ and we obtain $\widehat{C_g}^{p^{i+1}} = \widehat{C_h}^{p^{i+1}}$ by (i). □

Let G be a finite p-group G, and denote by P_i $(i = 1, 2, \dots, s)$ the defect groups of the conjugacy classes of G. For an arbitrary integer

$0 \le k \le l$ denote by $R_l(P_i)^{p^k}$ the set of the conjugacy classes of the form $C_{g^{p^k}}$ with defect group P_i are such that the depth of the conjugacy class C_g is not less than l.

Theorem 2. *Let $\mathbb{F}_pG$ be the group algebra of the finite p-group G. Denote by P_i $(i = 1, 2, \dots, s)$ the defect groups of the conjugacy classes of G; let $C = \zeta(G)$ and $q_{i,l}$ the order of $\zeta(P_i)^{p^l}[p]$.*

In the direct decomposition of the centre $\zeta(V)$ of $V(\mathbb{F}_pG)$ the number of the cyclic groups of order p^l is

$$f_l(\zeta(V)) = |C^{p^{l-1}}| - 2|C^{p^l}| + |C^{p^{l+1}}|$$
$$+\sum_{i=1}^{s}\left(|R_{l-1}(P_i)^{p^{l-1}}| - |R_l(P_i)^{p^l}| + \frac{|R_{l+1}(P_i)^{p^l}|}{q_{i,l}} - \frac{|R_l(P_i)^{p^{l-1}}|}{q_{i,l-1}}\right).$$

Proof. Since G is a subgroup of $V(\mathbb{F}_pG)$, all elements of the center $\zeta(V)$ are central in $\mathbb{F}_pG$.

Let $C_{g_{i1}}, C_{g_{i2}}, \dots, C_{g_{ik_i}}$ be all the conjugacy classes with defect group P_i, and $C = \zeta(G)$ the center of G. It is well-know that any element of $\zeta(V)$ can be written as

$$x = \sum_{z_i \in C} b_i z_i + \sum_{i=1}^{s}\sum_{j=1}^{k_i} a_{ij}\widehat{C_{g_{ij}}},$$

where $b_i,\ a_{ij} \in \mathbb{F}_p$. Since $x \in V(\mathbb{F}_pG)$, its augmentation is 1, and the order of the conjugacy class $C_{g_{ij}}$ is a p power, hence $\sum_{z_i\in C} b_i = 1$.

The p^l-powers of the abelian group $\zeta(V)$ form a subgroup $\zeta(V)^{p^l}$, and any element of this subgroup is a p^l-power of some element of $\zeta(V)$. First we determine the order of the subgroup $\zeta(V)^{p^l}[p]$ generated by the elements of order p in $\zeta(V)^{p^l}$.

The characteristic of $\mathbb{F}_pG$ is p, and for its central elements the shorter form of the binomial formula holds. Hence any $x^{p^l} \in \zeta(V)^{p^l}$ can be written as

$$x^{p^l} = \sum_{z_i\in C} b_i z_i^{p^l} + \sum_{i=1}^{s}\sum_{j=1}^{k_i} a_{ij}\widehat{C_{g_{ij}}}^{p^l} =$$
$$= \sum_{z_i\in C} b_i z_i^{p^l} + \sum_{i=1}^{s}\sum_{C_{g_{ij}^{p^l}} \in R_l(P_i)^{p^l}} a_{ij}\widehat{C_{g_{ij}^{p^l}}}, \qquad (4)$$

because according to (i) of Lemma 1 $\widehat{C_{g_{ij}}}^{p^l} = 0$, if the depth of the conjugacy class $C_{g_{ij}}$ is less than l.

Now assume that x^{p^l} is an element of order p. Then

$$\sum_{z_i \in C} b_i z_i^{p^l} \in V(\mathbb{F}_p C^{p^l})[p],$$

and by Lemma 1 the number of these elements is $p^{|C^{p^l}|-|C^{p^{l+1}}|}$. Recall that if C_g and C_h are conjugacy classes such that $\widehat{C_g}^{p^{l+1}} = \widehat{C_h}^{p^{l+1}} \neq 0$, then $C_{g^{p^{l+1}}} = C_{h^{p^{l+1}}}$, and it follows from (iv) of Lemma 1 that the defect groups of C_g and C_h coincide. Thus by (4) and by Lemma 1 we have

$$\sum_{C_{g_{ij}^{p^l}} \in R_l(P_i)^{p^l}} a_{ij} \widehat{C_{g_{ij}^{p^l}}}^p = 0 \tag{5}$$

for any i.

Now let $C_{g_{ij_1}^{p^l}}, C_{g_{ij_2}^{p^l}}, \dots, C_{g_{ij_t}^{p^l}}$ be all those conjugacy classes which satisfy the condition

$$\widehat{C_{g_{ij_1}^{p^l}}}^p = \widehat{C_{g_{ij_2}^{p^l}}}^p = \ldots = \widehat{C_{g_{ij_t}^{p^l}}}^p \neq 0.$$

By (v) of Lemma 1 $C_{g_{ij_r}^{p^l}} = C_{g_{ij_1}^{p^l} c_r}$ for some $c_r \in \zeta(P_i)^{p^l}[p]$ and $t = q_{i,l} = |\zeta(P_i)^{p^l}[p]|$. Then (4) implies $(\sum_{k=1}^t a_{ij_k})\widehat{C_{g_{ij_1}^{p^l}}}^p = 0$, and we obtain the linear equation of form

$$\sum_{k=1}^{t} a_{ij_k} = 0 \tag{6}$$

over the field of p elements.

By arguments of Lemma 1 the set $R_l(P_i)^{p^l}$ can be divided into subsets such that the pth powers of the class sums of the conjugacy classes in each subset coincide, but are distinct for distinct subsets. Then by (i) of Lemma 1 in this division there is a subset containing

$$|R_l(P_i)^{p^l} \setminus R_{l+1}(P_i)^{p^l}| = |R_l(P_i)^{p^l}| - |R_{l+1}(P_i)^{p^l}|$$

elements with pth power zero, and, furthermore, by (v) of the same lemma each of the other subsets contains $q_{i,l} = |\zeta(P_i)^{p^l}[p]|$ elements. Evidently, the number of the latter subsets is

$$\frac{|R_{l+1}(P_i)^{p^l}|}{q_{i,l}}.$$

Combining these facts we see that the number of the elements of the form

$$\sum_{C_{g_{ij}^{p^l}} \in R_l(P_i)^{p^l}} a_{ij} \widehat{C_{g_{ij}^{p^l}}},$$

which satisfies (4) and $x^{p^{l+1}} = 1$ is determined by the system of linear equations of form (6) over the field of p elements. We know the number of the equations and the indeterminates, and the equations contain no common indeterminates. Thus the number of the elements is a p-power with exponent

$$|R_l(P_i)^{p^l}| - |R_{l+1}(P_i)^{p^l}| + |R_{l+1}(P_i)^{p^l}| - \frac{|R_{l+1}(P_i)^{p^l}|}{q_{i,l}} =$$

$$= |R_l(P_i)^{p^l}| - \frac{|R_{l+1}(P_i)^{p^l}|}{q_{i,l}}.$$

We obtain that the order of $\zeta(V)^{p^l}[p]$ is p^{r_l}, where

$$r_l = |C^{p^l}| - |C^{p^{l+1}}| + \sum_{i=1}^{s} \big(|R_l(P_i)^{p^l}| - \frac{|R_{l+1}(P_i)^{p^l}|}{q_{i,l}}\big).$$

Since $f_l(\zeta(V)) = r_{l-1} - r_l$, we obtain the formulas to be proved immediately. $\square$

Let $\zeta(\mathbb{F}_pG)$ be a centre of $\mathbb{F}_pG$. Then

$$\zeta(\mathbb{F}_pG) = \mathbb{F}_p\zeta(G) \oplus W(\mathbb{F}_pG),$$

where every element of $W(\mathbb{F}_pG)$ is a sum of class sums and $W(\mathbb{F}_pG)$ is an ideal of the centre $\zeta(\mathbb{F}_pG)$. It follows that

$$\zeta(V(\mathbb{F}_pG)) = V(\mathbb{F}_p\zeta(G)) \times (1 + W(\mathbb{F}_pG)).$$

For the group $V(\mathbb{F}_p\zeta(G))$ there exists the Sandling basis S [9] and $V(\mathbb{F}_p\zeta(G)) = \prod_{u\in S}\langle u\rangle$. It is a very interesting problem:

Question 2. Determine the basis for $1+W(\mathbb{F}_pG)$.It is very likely that if G has only the conjugacy classes of the depth zero, then

$$\zeta(V(\mathbb{F}_pG)) = \prod_{u\in S}\langle u\rangle \times \prod_{i=1}^{s}(1+\widehat{C}_{a_i}),$$

where $C_{a_i}, (i=1,2,\dots,s)$ are all non-singleton conjugacy classes of G.

Let p^k be the exponent of the centre of a finite p-group G and let r be the greatest depth of the conjugacy classes of G. According to Lemma 1, if the depth of the conjugacy class C_g is d, then $\widehat{C_g}^{p^i} = \widehat{C_{g^{p^i}}}$ for $i = 1,\dots,d$ and $\widehat{C_g}^{p^{d+1}} = 0$. From these we can conclude that the exponent p^t of the center of $V(\mathbb{F}G)$ is equal to $\max\{p^k, p^{r+1}\}$.

According to Theorem 1 we conclude that

Corollary. *Let G be a finite p-group and $\mathbb{F}$ be a field of characteristic p. Denote by c and p^n the nilpotency class and the exponent of the group $V(\mathbb{F}G)$ respectively. Then $p^n \leq p^{[\frac{c-1}{p}]+t}$, where $[\frac{c-1}{p}]$ is the upper integral part of $\frac{c-1}{p}$.*

p-Central group of units.

Definition. *A p-group G is called p-central if all elements of order p lie in the centre of G for odd prime p and for $p=2$ all elements of order 2 and 4 are central in G.*

V. Bovdi and M. Dokuchaev [2] give answer to the question when all elements of order two in $V(\mathbb{F}_pG)$ generate an elementary subgroup. From this result it follows if the 2-group $V(\mathbb{F}_2G)$ would be 2-central then

$$G = \langle a, b \mid a^{2^n} = b^{2^m} = 1, a^b = a^{1+2^{n-1}}, n, m \geq 3\rangle.$$

It is easy to see that

$$u = a^{2^{n-2}} + (1+a^{2^{n-2}})(a+ab)$$

is a non central element of order 4. Therefore the nonabelian 2-group $V(\mathbb{F}_2G)$ is not 2-central. Similarly as before as a consequence of Baginski's result [1] is easy to obtain that G is also abelian.

Theorem 3. *If G is a finite p-group, then $V(\mathbb{F}G)$ is p-central if and only if G is an abelian group.*

Automorphism of the unit group.

Let G be a finite p-group G and denote by $\mathrm{Aut}\,(V)$ the group of automorphism of the group of normalized units $V(\mathbb{F}_pG)$. We know very little about group $\mathrm{Aut}\,(V)$. It is a well-known result of Gaschütz, according to which the finite p-group V possesses an outer automorphism ψ of p-power order, but it is an open question whether there is an outer automorphism ψ of order p.

First, we remark that every outer automorphism of G can be extended to an outer automorphism of V. If ψ is an outer automorphism of G and $x = \sum_{g\in G} \alpha_g g \in V(\mathbb{F}_pG)$ then the mapping $\widetilde{\psi}$ defined by $\widetilde{\psi}(x) = \sum_{g\in G} \alpha_g \psi(g)$ is an outer automorphism of $V(\mathbb{F}_pG)$. Indeed, if $v^{-1}xv = \widetilde{\psi}(x)$ for every $x \in V(\mathbb{F}_pG)$ then by the property of $\widetilde{\psi}$ we have $v^{-1}Gv = G$, and by Coleman's theorem [3] $v = hz$ for some $z \in \zeta(V)$ and $h \in G$. It implies that $h^{-1}gh = \psi(g)$ for any $g \in G$, contradicting the assumption that ψ is outer automorphism.

Theorem 4. *Let G be a finite p-group of order $p^n > 2$ and p^m is the index of the Frattini subgroup of G. Then the group $\mathrm{Aut}\,(V)$ contains an elementary abelian subgroup of order p^m, which consists of the outer automorphism of V.*

Proof. Let $A(\mathbb{F}_pG)$ be an augmentation ideal and $g \in G \setminus D_2(\mathbb{F}_pG)$. The subspace $\langle \lambda(g-1) \mid \lambda \in \mathbb{F}_p \rangle$ of $A(\mathbb{F}_pG)$ has a complementary subspace V such that $A(\mathbb{F}_pG)^2 \subseteq V$. Since $xV, Vx \subseteq A(\mathbb{F}_pG)^2$ for all $x \in A(\mathbb{F}_p(G)$, the subspace V is the ideal of $A(\mathbb{F}_pG)$. Clearly, every element $x \in A(\mathbb{F}_pG)$ can be uniquely written as

$$x = v + k(g-1),$$

where $v \in V$ and $k \in \mathbb{F}_p$. We define a linear transformation w_g on $\mathbb{F}_pG$ such that

$$w_g(v + k(g-1)) = v + k(g-1) + k\widehat{G},$$

where $\widehat{G} = \sum_{h\in G} h$. Because $|G| > 2$, the element $\widehat{G}$ belongs to $A(\mathbb{F}_pG)^2$. The equality

$$w_g(v + k(g-1) - k\widehat{G}) = v + k(g-1))$$

indicates that w_g is surjective. Let

$$x_1 = v_1 + k_1(g-1) \quad \text{and} \quad x_2 = v_2 + k_2(g-1).$$

By the property that V is ideal of $A(\mathbb{F}_pG)$, $(g-1)^2 \in A^2(\mathbb{F}_pG)$ and $\widehat{G}$ belongs to the annihilator of $A(\mathbb{F}_pG)$. Therefore, we conclude that

$$w_g(x_1x_2) = x_1x_2 = w_g(x_1)w_g(x_2)$$

and w_g is the automorphism of $A(\mathbb{F}_pG)$. From

$$w_g^p(x) = w_g^p(v + k(g-1)) = v + k(g-1) + kp\widehat{G} = v + k(g-1) = x$$

it follows that w_g has order p.

It is easy to see that $V(\mathbb{F}_pG)$ isomorphic to the adjoint group $(A(\mathbb{F}_p(G), \circ)$ of a nilpotent augmentation ideal $A(\mathbb{F}_pG)$. If $x_1, x_2 \in A(\mathbb{F}_pG)$, then

$$w_g(x_1 \circ x_2) = w_g(w_1) \circ w_g(x_2)$$

and w_g is an automorphism of $(A(\mathbb{F}_p(G), \circ)$.

Suppose that w_g is an inner automorphism of $(A(\mathbb{F}_p(G), \circ)$. Then there exists $y \in A(\mathbb{F}_p(G)$ such that $w_g(x) = y' \circ x \circ y$, where $y' \circ y = y \circ y' = 0$. The following equality follows:

$$(1+y)^{-1}(g-1)(1+y) = w_g(g-1) = (g-1) + \widehat{G}.$$

Thus $\widehat{G} = (g-1)y - y(g-1) = [g, y]$. The trace of the Lie commutator $[g, u]$ is zero, which is impossible.

Let

$$D_1(\mathbb{F}_pG)/D_2(\mathbb{F}_pG) = \prod_{i=1}^{m} \langle g_i D_2(\mathbb{F}_pG)\rangle.$$

It is well-known that $D_2(\mathbb{F}_pG)$ is a Frattini subgroup of G. Then

$$A(\mathbb{F}_pG) = \langle (g_1-1), (g_2-1), \ldots, (g_m-1), A^2(\mathbb{F}_pG)\rangle.$$

Let

$$\begin{aligned} V_i = &\langle (g_1-1), (g_2-1), \ldots, (g_{i-1}-1), \\ &(g_{i+1}-1), \ldots, (g_m-1), A^2(\mathbb{F}_pG)\rangle. \end{aligned}$$

Then

$$\mathbb{F}_pG = \{\lambda(g_i - 1) \mid \lambda \in \mathbb{F}_p\} \oplus V_i$$

and w_{g_i} is an automorphism of order p and $w_{g_i}w_{g_j} = w_{g_j}w_{g_i}$. Therefore the subgroup generated by $w_{g_i}, i = 1, 2, \ldots, m$ is an elementary abelian group of order p^m. $\square$

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Поступило 6 июля 2000

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