Записки научных
семинаров ПОМИ
Том 272, 2000 г.

Alexei Stepanov

SOME CONSTRUCTIONS OF EXACT SEQUENCES

Abstract. The article presents a construction of a 5-term relative exact sequence in pure categorical terms and the Mayer–Vietoris sequence for weak ∞-groupoids of R.Street. In the first half it is shown that several well-known exact sequences can be obtained using *Z-diagram* described at the beginning of the article.

There are a number of categories which are close to the category of topological spaces in the sense of homotopy theory. The objects of such a category can be viewed as weak ∞-groupoids in a very intuitive way. Therefore, the language of weak ∞-groupoids seems to be very convenient to make explicit constructions in categories of this type. The article presents a combinatorial construction of the path space of a weak ∞-groupoid and apply the machinery of Section 1 to get the long exact sequence of homotopy fiber. The construction is quite natural and gives us information on the structure of the relative terms.

In the last section the Mayer–Vietoris sequence of a fibre square of weak ∞-groupoids is obtained under certain conditions. Of course, this construction makes sense for topological spaces or for any category mentioned in the previous paragraph. But the formulation of the conditions is more natural in terms of weak ∞-groupoids. This sequence generalizes the sequence of fibration and in this case the conditions given in the article are equivalent to the homotopy lifting property.

Introduction

We present a construction of a 5-term relative exact sequence in pure categorical terms and the Mayer–Vietoris sequence for weak ∞-groupoids.

The first two sections have rather methodical than scientific nature. They show that several well-known exact sequences can be obtained using *Z-diagram* 1.6 described in Section 1. The main idea of this construction was taken from lectures of A.Bak at the Bielefeld University.

There are a number of categories which are close to the category of topological spaces in the sense of homotopy theory. Not all of them are

A part of the article was written in the Bielefeld University under the support of *INTAS 93–436* and *DFG–RFBR 96-01-00092G*.

closed model categories. For example, it is shown in [5] that the category of global actions of A.Bak [1] is not a closed model category. The objects of such a category can be viewed as weak ∞-groupoids (see [4]) in a very intuitive way. Therefore, the language of weak ∞-groupoids of Street [6] seems to be very convenient to make explicit constructions in categories of this type. In Section 3 we present a combinatorial construction of the path space of a weak ∞-groupoid and apply the machinery of Section 1 to get the long exact sequence of homotopy fiber. The construction is quite natural and gives us information on the structure of the relative terms. However, the proof that this construction is self contained is rather technical. For the proof one can use either explicit formulae based on the axioms of a weak ∞-groupoid or the pasting theorem of Jonson [3]. In any case the proof seems to be too long to be included in the article which displays conceptions rather than results.

In the last section we obtain under certain conditions the Mayer–Vietoris sequence of a fibre square of weak ∞-groupoids. Of course, this construction makes sense for topological spaces or for any category mentioned in the previous paragraph. But the formulation of the conditions is more natural in terms of weak ∞-groupoids. This sequence generalizes the sequence of fibration. In fact, if one places a zero object (one-point topological space) in the right-bottom corner of the square 4.1, then the condition of Corollary 4.5 will be just the homotopy lifting property.

1. 5-Term relative exact sequence

Let $\mathcal{C}$ be a pointed category with pullbacks (fibre products). Suppose that the square

$$\begin{tikzcd} R \arrow[r] \arrow[d] & B \arrow[d, "\varphi"] \\ C \arrow[r, "\psi"] & D \end{tikzcd}$$

is a pullback. Then we denote R by $B \times_D C$ or by *pullback*(ψ, φ) (i.e. *pullback* denotes the functor from the category of diagrams in $\mathcal{C}$ of the form $\bullet \to \bullet \leftarrow \bullet$ to the category $\mathcal{C}$). We start with some easy properties of this functor.

Consider a commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r, "\psi'"] \arrow[d] & C \arrow[d, "\varphi"] \\ D \arrow[r] & E \arrow[r, "\psi"] & F \end{tikzcd}$$

The following lemma is an easy exercise in diagram chasing.

1.1. Lemma. *Suppose that the canonical morphism $B \to pullback(\psi, \varphi)$ is a monomorphism (e.g. $B = C \times_F E$). Then the left square is a pullback if and only if the big rectangle is a pullback.*

1.2. Lemma. *Let A and D be the kernels of morphisms ψ' and ψ, respectively. If the right square is a pullback, then $A \cong D$.*

Proof. Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r, "\psi'"] \arrow[d, "\psi'"] & E \arrow[d, "\psi"] \\ 0 \arrow[r] & C \arrow[r, "\varphi"] & F \end{tikzcd}$$

By the conditions of the lemma both squares are pullbacks. By the previous lemma the rectangle is a pullback as well, i.e. A is kernel of ψ. Hence, $A \cong D$.

1.3. Lemma. *Let A and D be the kernels of morphisms ψ' and ψ, respectively, and let φ be an isomorphism. Then the left square is a pullback.*

Proof. Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \arrow[d] \\ B \arrow[r] & E \arrow[r, "\varphi^{-1}\psi"] & C \end{tikzcd}$$

By the conditions of the lemma the right square and the rectangle are pullbacks. By Lemma 1.1 the left square is a pullback as well.

(1.4) Let P be a functor on $\mathcal{C}$ preserving zero objects and τ be a natural transformation from P to the identity functor. Denote by $\Omega A = \Omega^1 A$ the kernel of the morphism $\tau(A)$. Clearly Ω is a functor on $\mathcal{C}$. Let $\mathcal{G}$ be another pointed category with pullbacks and $\pi = \pi_0 : \mathcal{C} \to \mathcal{G}$ be a functor such that $\pi \circ P = 0$, i.e. $\pi(P(A))$ is a zero object for any $A \in \mathcal{O}bj\,\mathcal{C}$. Denote by π_i the composition of the functors π and Ω^i (for a functor $F : \mathcal{C} \to \mathcal{C}$ we always denote by F^i the composition $F \circ F^{i-1}$ where $F^1 = F$). A sequence $G'' \to G \to G'$ in $\mathcal{G}$ is called a *zero sequence* if the composite morphism $G' \to G''$ is zero, i.e. factors through a zero object. A zero sequence is called *exact* if the canonical morphism $G'' \longrightarrow \operatorname{Ker}(G \to G')$ is an epimorphism. A long sequence $\cdots \to G_0 \to G_1 \to G_2 \to \cdots$ is called exact if each sequence $G_{i-1} \to G_i \to G_{i+1}$ is exact. A square

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & G' \arrow[d, "\xi'"] \\
G'' \arrow[r, "\xi''"] & \hat{G}
\end{tikzcd}$$

is called exact if the canonical morphism $G \to \mathit{pullback}(\xi', \xi'')$ is an epimorphism.

The reader can keep in mind the following picture: $\mathcal{C}$ is the category of pointed topological spaces with continuous maps; $\mathcal{G}$ is the category of pointed sets; if $A \in \mathcal{O}bj\,\mathcal{C}$ then $P(A)$ is the space of all paths started from the base point, ΩA is the loop space, and $\pi = \pi_0$.

(1.5) Let $\psi : A \to C$ is a morphism in $\mathcal{C}$. Our aim is to construct 5-term relative exact sequence connecting $\pi_1(A) \to \pi_1(C)$ and $\pi_0(A) \to \pi_0(C)$. Denote by $R = R(\psi)$ the fibre product of the morphisms ψ and $\tau(C)$. Clearly, R is a functor from the category $\mathcal{A}\mathcal{R}\mathcal{R}(\mathcal{C})$ of morphisms in $\mathcal{C}$ to $\mathcal{C}$ itself. Since the diagrams

$$\begin{tikzcd}
P(A) \arrow[r, "P(\psi)"] \arrow[d, "\tau(A)"'] & P(C) \arrow[d, "\tau(C)"] \\
A \arrow[r, "\psi"] & C
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Omega(C) \arrow[r] \arrow[d, "0"'] & P(C) \arrow[d, "\tau(C)"] \\
A \arrow[r, "\psi"] & C
\end{tikzcd}$$

are commutative there exist uniquely determined morphisms $P(A) \to R$ and $\Omega(C) \to R$ making the following diagram commute:

$$\begin{tikzcd}
\Omega(A) \arrow[rr] \arrow[d] & & \Omega(C) \arrow[d] \arrow[dl] \\
P(A) \arrow[r] \arrow[d] & R \arrow[r] \arrow[dl] & P(C) \arrow[d] \\
A \arrow[rr] & & C
\end{tikzcd} \tag{1.6}$$

The next assertion follows directly from lemmas 1.2 and 1.3.

1.7. Lemma. *The following three squares arising from* (1.6) *are pullbacks:*

$$\begin{array}{ccc} R & \longrightarrow & P(C) \\ \downarrow & & \downarrow{\scriptstyle \tau(C)} \\ A & \xrightarrow{\psi} & C \end{array} \qquad \begin{array}{ccc} \Omega(C) & \longrightarrow & R \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} \Omega(A) & \longrightarrow & \Omega(C) \\ \downarrow & & \downarrow \\ P(A) & \longrightarrow & R \end{array}$$

1.8. Proposition. *The sequence in the category $\mathcal{G}$*

$$\pi_1(A) \longrightarrow \pi_1(C) \longrightarrow \pi_0(R) \longrightarrow \pi_0(A) \longrightarrow \pi_0(C)$$

is a zero sequence. It is exact if and only if the functor π maps three pullback squares (1.7) *to exact squares in category $\mathcal{G}$.*

Proof. The functor π takes diagram (1.6) to the following:

$$\begin{tikzcd} \pi_1(A) \arrow[rrr] \arrow[d] & & & \pi_1(C) \arrow[d] \arrow[dll] \\ 0 \arrow[r] & \pi_0(R) \arrow[rr] \arrow[dl] & & 0 \arrow[d] \\ \pi_0(A) \arrow[rrr] & & & \pi_0(C) \end{tikzcd}$$

Hence, the sequence in the proposition is a zero sequence. Exactness follows directly from the definition in (1.4).

2. Examples

In this section we will show that several classical exact sequences can be obtained by the method described above.

(2.1) *Homology.* Let $\mathcal{C} = \mathbf{Ch}\,\mathcal{A}$ be the category of all chain complexes in an abelian category $\mathcal{A}$. Define $\pi = H_0$. For $C \in \mathcal{C}$ let $P(C) = C[1] \oplus C$ with the differential $d_n^{P(C)} : C_{n+1} \oplus C_n \to C_n \oplus C_{n-1}$ given by the formula $d_n^{P(C)}(a', a) = \big(a - d_{n+1}^C(a'), d_n^C(a)\big)$. Let $\tau(C) : P(C) \to C$ be the projection onto the second component. It is easy to check that P defines a functor $\mathcal{C} \to \mathcal{C}$ with an obvious action on morphisms, $\tau(C)$ is a chain map so that τ is a natural transformation $P \to id_{\mathcal{C}}$, and each complex $P(C)$ is acyclic which yields $\pi \circ P = 0$. Clearly, the kernel of $\tau(C)$ is $\Omega(C) = C[1]$ so that $H_0(\Omega(C)) = H_1(C)$.

Let now $\psi : A \to C$ be a chain map. The pullback of the diagram $A \to C \leftarrow P(C)$ is $R(\psi) = C[1] \oplus A$, i.e. this is just the cone of the map f shifted by 1. One checks routinely that H_0 maps pullbacks from

Lemma 1.7 to exact squares. Thus by proposition 1.8 we get the exact sequence

$$H_1(A) \longrightarrow H_1(C) \longrightarrow H_0(R) \longrightarrow H_0(A) \longrightarrow H_0(C)$$

which can be extended to the left in an obvious way.

(2.2) *Derived functors.* Let $L : \mathcal{A} \to \mathcal{A}$ be an additive semiexact functor on an Abelian category $\mathcal{A}$. Suppose that $L(P) = 0$ for any projective P, e.g. $L = L_nF$, $n \geqslant 1$, is the left derived functor of a right exact additive functor F.

For each $C \in \mathcal{A}$ choose an epimorphism $P_C \to C$ with P_C projective and denote its kernel by Ω_C. Given a morphism $\psi : A \to C$ form the pullback $R = R(\psi)$ of the diagram $A \to C \leftarrow P_C$. Since $R(\psi) \to A$ is an epimorphism, there is a map $P_A \to R$ making the following diagram commute

$$\begin{tikzcd} \Omega_A \arrow[rr] \arrow[d] & & \Omega_C \arrow[d] \arrow[dl] \\ P_A \arrow[r] \arrow[d] & R \arrow[r] \arrow[dl] & P_C \arrow[d] \\ A \arrow[rr] & & C \end{tikzcd}$$

Note that since L is semiexact, it maps any biuniversal square to an exact square. It is easy to check that pullback squares (1.7) are biuniversal in this example. Therefore we can use the same arguments as in the proof of Proposition 1.8 to get the exact sequence

$$L(\Omega_A) \longrightarrow L(\Omega_C) \longrightarrow L(R) \longrightarrow L(A) \longrightarrow L(C)$$

(formally we can not apply Proposition 1.8 directly because P_C is not functorial).

If $L = L_nF$ ($n \geqslant 1$) is a left derived functor, then for any $C \in \mathcal{A}$ we have $L_nF(\Omega_C) = L_{n+1}F(C)$. If $0 \to K \to A \to C \to 0$ is exact, then $R \to P_C$ is an epimorphism and hence splits to yield $R \cong P_C \oplus K$. Then we have $L_nF(R) \cong L_nF(K)$. Thus, the sequence above turn into the usual exact sequence

$$\begin{aligned} \cdots \longrightarrow L_{n+1}F(A) \longrightarrow L_{n+1}F(C) \longrightarrow L_nF(K) \\ \longrightarrow L_nF(A) \longrightarrow L_nF(C) \longrightarrow \cdots \end{aligned}$$

(2.3) *Kernels and cokernels.* Let

$$\begin{array}{ccc} A_1 & \longrightarrow & B_1 \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\psi'} \\ A_2 & \longrightarrow & B_2 \end{array}$$

be a commutative square in a pointed category $\mathcal{G}$ with pullbacks and cokernels. We are going to construct a 5-term exact sequence

$$\operatorname{Ker}\psi \longrightarrow \operatorname{Ker}\psi' \longrightarrow \pi(R) \longrightarrow \operatorname{Coker}\psi \longrightarrow \operatorname{Coker}\psi'.$$

Let $\mathcal{C} = \mathcal{A}\mathcal{R}\mathcal{R}(\mathcal{G})$ be the category of morphisms in $\mathcal{G}$, $\pi = \operatorname{Coker} : \mathcal{C} \to \mathcal{G}$. Define a functor $P : \mathcal{C} \to \mathcal{C}$ by the rule $P(A_1 \xrightarrow{\psi} A_2) = (A_1 \xrightarrow{id} A_1)$ with an obvious action on morphisms, and a natural transformation $\tau : P \to id_{\mathcal{C}}$ by commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{id} & A_1 \\ {\scriptstyle id}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ A_1 & \xrightarrow{\psi} & A_2 \end{array}$$

Then $\Omega(\psi') = \operatorname{Ker}\tau(\psi') = (\star \to \operatorname{Ker}\psi')$, where $\star$ denotes a zero object, $\pi{\circ}P(\psi) = \star$, and $R = \big(A_1 \to pullback(B_1 \to B_2 \leftarrow A_2)\big)$. Thus we get the required zero sequence. Unfortunately, the last condition of Proposition 1.8 does not hold in general. However, it does if objects of $\mathcal{G}$ are sets with an extra structure and the categorical images coincide with the set theoretical ones. Let us formulate these conditions more precisely.

2.4. Lemma. *Suppose that there is a functor $F : \mathcal{G} \to \mathcal{S}\varepsilon\mathcal{T}\mathcal{S}_\star$ satisfying the following conditions.*

(i) *F preserves pullbacks and zero objects.*
(ii) *F takes cokernels to surjective maps.*
(iii) *An inverse image of a surjective map under F is epi.*
(iv) *F preserves images, i.e., $F\big(\operatorname{Ker}(coker\varphi)\big) = \operatorname{Ker}\Big(coker\big(F(\varphi)\big)\Big)$.*

Then the functor $\pi = \operatorname{Coker} : \mathcal{A}\mathcal{R}\mathcal{R}(\mathcal{G}) \to \mathcal{G}$ maps three pullbacks of Lemma 1.7 to exact squares.

Proof. We shall prove the assertion only for the first pullback square of Lemma 1.7. The proof for 2 other squares is similar and will be left to the reader.

Consider square 2.3. Let $D = \mathit{pullback}(B_1 \to B_2 \leftarrow A_2)$ and $D' = \operatorname{Ker}\big(\pi(\psi) \to \pi(\psi')\big)$. We have to prove that $\pi(R) \to D'$ is an epimorphism. Since $\pi(R) = \operatorname{Coker}(A_1 \to D)$, in view of condition (iii) it suffices to prove that the map $F(D) \to F(D')$ is surjective. Consider commutative cube

$$\begin{tikzcd}
& D \arrow[rr] \arrow[dl] \arrow[dd] & & B_1 \arrow[dl] \arrow[dd] \\
A_2 \arrow[rr, crossing over] \arrow[dd] & & B_2 & \\
& D' \arrow[rr] \arrow[dl] & & \star \arrow[dl] \\
\pi(\psi) \arrow[rr] & & \pi(\psi') \arrow[from=uu, crossing over] &
\end{tikzcd}$$

with pullbacks in top and bottom surfaces and apply the functor F. Take an element $d' \in F(D')$ and denote by c its image in $F\big(\pi(\psi)\big)$. By condition (ii) there is its preimage $a \in F(A_2)$. Denote by b the image of a in $F(B_2)$. Each element constructed above goes to the base point of $F\big(\pi(\psi')\big)$. In particular,

$$b \in \operatorname{Ker}\Big(F(B_2) \to F\big(\pi(\psi')\big)\Big) =$$
$$F\Big(\operatorname{Ker}\big(B_2 \to \pi(\psi')\big)\Big) = F(\operatorname{Im}\psi') = \operatorname{Im} F(\psi')$$

by conditions (i) and (iv). Thus, there is $b_1 \in B_1$ goes to b under $F(\psi')$. Since the top surface of the cube is a pullback, there is a unique $d \in F(D)$ which maps to a and b_1. Now it is obvious that its image in $F(D')$ is d' because the bottom surface is a pullback.

2.5. Remark. The last condition of the lemma does not hold for the most interesting situation when $\mathcal{G} = \mathcal{GROUPS}$ because the categorical image in this case is the normal closure of the set theoretical image. This was the motivation for A.Bak [2] to introduce a new category $\mathcal{COSETS}$ of coset spaces which is in some sense a universal category containing $\mathcal{GROUPS}$ in which condition (iv) of the lemma is satisfied.

2.6. Proposition. *Let $\mathcal{G}$ be a category with zero objects, pullbacks, and cokernels. Consider a commutative square in $\mathcal{G}$*

$$\begin{array}{ccc} A_1 & \longrightarrow & B_1 \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\psi'} \\ A_2 & \longrightarrow & B_2 \end{array}$$

and denote by $\pi(R)$ the cokernel of the canonical morphism $A_1 \longrightarrow$ pullback$(B_1 \to B_2 \leftarrow A_2)$. If $\mathcal{G}$ satisfies conditions of Lemma 2.4, then the following 5-term sequence is exact.

$$\operatorname{Ker}\psi \longrightarrow \operatorname{Ker}\psi' \longrightarrow \pi(R) \longrightarrow \operatorname{Coker}\psi \longrightarrow \operatorname{Coker}\psi'$$

3. ∞-Groupoids and their path spaces

In this section we introduce an explicit construction of a "path space" for a weak ∞-groupoid and apply the machinery of Section 1 to this construction. This will be the main tool for the next section.

(3.1) Recall [6] that an ∞-category C is a bunch of nonempty sets $C_0, C_1, C_2, \ldots$ together with maps $s_k, t_k : C_{k+1} \to C_k$ and $Id_k : C_{k-1} \to C_k$ and partial binary operations $\underset{p}{*}$ on the sets C_k $(0 \leqslant p < k)$ satisfying the following axioms.

(A0) $s_{k-1}\circ s_k = s_{k-1}\circ t_k$, $t_{k-1}\circ s_k = t_{k-1}\circ t_k$, $s_k\circ Id_{k+1} = t_k\circ Id_{k+1} = id_{C_k}$

We shall denote (somewhat abusively) by $t_p, s_p : C_k \to C_p$, $Id_k : C_p \to C_k$ $(k \geqslant p)$ the composite maps $t_p\circ\cdots\circ t_{k-1}$, $s_p\circ\cdots\circ s_{k-1}$, and $Id_k\circ\cdots\circ Id_{p+1}$.

(ST) The operation $a \underset{p}{*} b$ is defined for $a, b \in C_k$ iff $s_p(a) = t_p(b)$. If so, then $s_{k-1}(a \underset{k-1}{*} b) = s_{k-1}(b)$, $t_{k-1}(a \underset{k-1}{*} b) = t_{k-1}(a)$ and $s_{k-1}(a \underset{p}{*} b) = s_{k-1}(a) \underset{p}{*} s_{k-1}(b)$, $t_{k-1}(a \underset{p}{*} b) = t_{k-1}(a) \underset{p}{*} t_{k-1}(b)$ for $p < k-1$.

(Ass1) $(a \underset{q}{*} b) \underset{q}{*} c = a \underset{q}{*} (b \underset{q}{*} c)$.

(Ass2) $(a \underset{q}{*} b) \underset{p}{*} (a' \underset{q}{*} b') = (a \underset{p}{*} a') \underset{q}{*} (b \underset{p}{*} b')$ provided $p \neq q$, $p, q < k$, $a, b, a', b' \in C_k$, and all the operations above are defined.

(Id) Let $a, b \in C_k$. Then $a \underset{k-1}{*} Id_k\big(s_{k-1}(a)\big) = Id_k\big(t_{k-1}(a)\big) \underset{k-1}{*} a = a$ and
$Id_{k+1}(a \underset{p}{*} b) = Id_{k+1}(a) \underset{p}{*} Id_{k+1}(b)$

A morphism φ between two ∞-categories C and D is a family of maps $\varphi_k : C_k \to D_k$ compatible with all the structures described above. We let φ denote also the map $\coprod \varphi_k : \coprod C_k \to \coprod D_k$. The category of all ∞-categories will be denoted by $\mathcal{CAT}^\infty$.

The sets $C_0, C_1, C_2, \dots$ can be seen as the sets of objects, morphisms, morphisms between morphisms etc. Having this in mind, for $a, b \in C_k$, $s_{k-1}(a) = s_{k-1}(b)$, $t_{k-1}(a) = t_{k-1}(b)$ we denote

$$\operatorname{Mor}(a,b) = \operatorname{Mor}_C(a,b) = \{c \in C_{k+1} \mid s_k(c) = a,\ t_k(c) = b\} \qquad (i > k).$$

(3.2) On the other hand, a topological space can be viewed as an ∞-category in the following way. C_0 is the set of points, C_1 the set of Moore paths, and C_k the set of Moore homotopies (see [4]). From this geometric point of view it is natural to define a relation $\sim$ on C_0 in the following way: $a \sim b \iff \operatorname{Mor}(a,b) \neq \varnothing$. Similarly, we define a relation $\sim$ on the set C_k by the rule: $a \sim b \iff s_{k-1}(a) = s_{k-1}(b)$, $t_{k-1}(a) = t_{k-1}(b)$, and $\operatorname{Mor}(a,b) \neq \varnothing$. Clearly, these relations are not symmetric in general. Therefore, we are forced to restrict our attention to weak ∞-groupoids.

(3.3) For our goal it suffices to use Street's definition [6] of a weak ∞-groupoid. Recall that an ∞-category C is called a weak ∞-groupoid if for any $k \geqslant 0$ given $a \in C_{k+1}$ there is $a' \in \operatorname{Mor}\big(t_k(a), s_k(a)\big)$ such that $a \underset{k}{*} a' \sim Id_{k+1}\big(t_k(a)\big)$ and $a' \underset{k}{*} a \sim Id_{k+1}\big(s_k(a)\big)$.

The category $\mathcal{GRP}^\infty$ of all weak ∞-groupoids is the full subcategory of $\mathcal{CAT}^\infty$. For each ∞-groupoid C we choose a base point $\star = \star_C \in C_0$ and consider only the morphisms preserving the base points. Thus we get the pointed category $\mathcal{GRP}^\infty_\star$ with obvious zero objects.

Clearly, if C is a weak ∞-groupoid, then $\sim$ is an equivalence relation on each C_k. Define $\pi_0(C) = C_0/\!\sim$ to be a pointed set. Then π_0 defines a functor $\mathcal{GRP}^\infty_\star \to \mathcal{SETS}_\star$.

(3.4) Our next aim is to construct a "path space" $P(C)$ for a pointed weak ∞-groupoid C. The elements of $P(C)$ will be pairs of n- and $(n+1)$-dimensional morphisms. To make the definition consistent in some places we will have to add some lower-dimensional factors.

To simplify the notation we shall identify a with $Id_k(a)$ where k will be determined by context. To make easier reading formulae below, we shall use subscripts to determine the dimension of an element, e.g. an element a_n will always belongs to C_n. Note that the properties of the operation $\underset{p}{*}$ on C_n depend on whether $p = n-1$ or not. For $p = n-1$ this operation will be called the *vertical composition* as opposed to the *horizontal composition*. The vertical composition will be denoted by $\underset{p}{*}$ (with boldface star).

Let $\star$ denotes the base point of C_0. Then define

$$P(C)_0 = \{(a_0, \alpha_1) \mid a_0 \in C_0,\ \ \alpha_1 \in \mathrm{Mor}(\star, a_0)\}$$

Now we define the sets $\mathrm{Mor}_{P(C)}\big((b_{n-1}, \beta_n),\, (e_{n-1}, \varepsilon_n)\big)$ by induction on $n = 1, 2, \ldots$. If (a_n, α_{n+1}) is an element of this set and $k < n-1$ we denote

$$\begin{aligned} s_k(b_{n-1}, \beta_n) &= s_k(e_{n-1}, \varepsilon_n) = (b_k, \beta_{k+1}), \\ t_k(b_{n-1}, \beta_n) &= t_k(e_{n-1}, \varepsilon_n) = (e_k, \varepsilon_{k+1}), \ \text{ and} \\ a_n^{(k+1)} &= a_n \underset{0}{*} \beta_1 \underset{1}{*} \beta_2 \cdots \underset{k}{*} \beta_{k+1} \end{aligned}$$

(of course, the definition of $a_n^{(k+1)}$ depends on the low dimensional sources of the pair (a_n, α_{n+1}); to avoid a confusion we shall always have a notation for this sources). For $n = 1$ set $a_1^{(0)} = a_1$.

With the above conventions we define $\mathrm{Mor}_{P(C)}\big((b_{n-1}, \beta_n),\, (e_{n-1}, \varepsilon_n)\big)$ to be

$$\{(a_n, \alpha_{n+1}) \mid a_n \in \mathrm{Mor}_C(b_{n-1}, e_{n-1}),\ \alpha_{n+1} \in \mathrm{Mor}_C(\varepsilon_n,\ a_n^{(n-1)} \underset{n-1}{*} \beta_n)\}$$

Now define $P(C)_n$ to be the disjoint union of all sets $\mathrm{Mor}_{P(C)}\big((b_{n-1},\beta_n),\, (e_{n-1}, \varepsilon_n)\big)$. Then the functions s_k, t_k have an obvious meaning, and we set

$$Id_{n+1}(a_n, \alpha_{n+1}) = \big(Id_{n+1}(a_n), Id_{n+2}(\alpha_{n+1})\big).$$

We continue with the definition of the composition. Let

$$\begin{aligned} &(a_n, \alpha_{n+1}) \in \mathrm{Mor}_{P(C)}\big((b_{n-1}, \beta_n),\, (e_{n-1}, \varepsilon_n)\big) \qquad \text{and} \\ &(\tilde a_n, \tilde\alpha_{n+1}) \in \mathrm{Mor}_{P(C)}\big((\tilde b_{n-1}, \tilde\beta_n),\, (\tilde e_{n-1}, \tilde\varepsilon_n)\big). \end{aligned}$$

We adapt the conventions above and set additionally

$$\begin{aligned} s_k(\tilde b_{n-1}, \tilde\beta_n) &= s_k(\tilde e_{n-1}, \tilde\varepsilon_n) = (\tilde b_k, \tilde\beta_{k+1}) \text{ and} \\ t_k(\tilde b_{n-1}, \tilde\beta_n) &= t_k(\tilde e_{n-1}, \tilde\varepsilon_n) = (\tilde e_k, \tilde\varepsilon_{k+1}). \end{aligned}$$

For the composition $(\tilde a_n, \tilde\alpha_{n+1}) \underset{p}{*} (a_n, \alpha_{n+1})$ to be defined it is necessary that

$$(\tilde{b}_p, \tilde{\beta}_{p+1}) = (e_p, \varepsilon_{p+1}).$$

It follows that for $k < p$

$$(b_k, \beta_{k+1}) = (e_k, \varepsilon_{k+1}) = (\tilde{b}_k, \tilde{\beta}_{k+1}) = (\tilde{e}_k, \tilde{\varepsilon}_{k+1}).$$

In the definition we shall use the following elements:

$$\tilde{a}_n^{(n-1)} = \tilde{a}_n \underset{0}{*} \tilde{\beta}_1 \cdots \underset{n}{*} \tilde{\beta}_{n-1} \text{ when } n = p+1 \text{ and}$$
$$\tilde{e}_{p+1}^{(p)} = \tilde{e}_{p+1} \underset{0}{*} \tilde{\beta}_1 \cdots \underset{p-1}{*} \tilde{\beta}_p \text{ when } n > p+1$$

(of course one can erase the tildes in the right hand sides of the formulae above). Now we define the vertical and horizontal compositions by the formulae:

$$(\tilde{a}_n, \tilde{\alpha}_{n+1}) \underset{n-1}{*} (a_n, \alpha_{n+1}) = (\tilde{a}_n \underset{n-1}{*} a_n,\ \tilde{a}_n^{(n-1)} \underset{n-1}{*} \alpha_{n+1} \underset{n}{*} \tilde{\alpha}_{n+1}),$$
$$(\tilde{a}_n, \tilde{\alpha}_{n+1}) \underset{p}{*} (a_n, \alpha_{n+1}) = (\tilde{a}_n \underset{p}{*} a_n,\ \tilde{e}_{p+1}^{(p)} \underset{p-1}{*} \alpha_{n+1} \underset{p}{*} \tilde{\alpha}_{n+1}).$$

3.5. Remark. Note that in some sense α_{n+1} is contravariant. If one chooses another direction of α_{n+1} he will get "contravariant" α_1 and very complicated formulae in the definition above.

A proof that Definition 3.4 is self contained is rather technical and will be omitted. Note that the construction of $P(C)$ make sense for any ∞-category C. Now we shall prove that $P(C)$ is a weak ∞-groupoid if C is.

3.6. Proof. For a morphism

$$(a_n, \alpha_{n+1}) \in \mathrm{Mor}_{P(C)}\big((b_{n-1}, \beta_n),\ (e_{n-1}, \varepsilon_n)\big)$$

we shall construct a morphism

$$(a'_n, \alpha'_{n+1}) \in \mathrm{Mor}_{P(C)}\big((e_{n-1}, \varepsilon_n),\ (b_{n-1}, \beta_n)\big)$$

and a "homotopy"

$$(c_{n+1}, \gamma_{n+2}) \in \mathrm{Mor}_{P(C)}\big((a'_n, \alpha'_{n+1}) \underset{n-1}{*} (a_n, \alpha_n),\ Id_n(b_{n-1}, \beta_n)\big)$$

In fact this will prove that (a'_n, α'_{n+1}) is a left quasi-inverse for (a_n, α_{n+1}). The proof that there is a right quasi-inverse is similar and will be left to the reader. Then one proves, as in the group theory, that they are homotopic and, hence, our left quasi-inverse is also a right quasi-inverse.

Let $a'_n \in \mathrm{Mor}(e_{n-1}, b_{n-1})$ be a left quasi-inverse for a_n, i.e. there is $c_{n+1} \in \mathrm{Mor}\big(a'_n \underset{n-1}{*} a_n,\ Id(b_{n-1})\big)$. As before, we denote $s_k(a_n, \alpha_{n+1}) = (b_k, \beta_{k+1})$, $t_k(a_n, \alpha_{n+1}) = (e_k, \varepsilon_{k+1})$, and $c^{(n)}_{n+1} = c_{n+1} \underset{0}{*} \beta_1 \cdots \underset{n-1}{*} \beta_n$. Since for $k = 1, \dots, n-2$

$$a'_n \underset{n-1}{*} a_n \underset{k-1}{*} \beta_k \overset{(Id)}{=\!=}$$

$$\big(a'_n \underset{n-1}{*} a_n\big) \underset{k-1}{*} \big(Id_n(\beta_k) \underset{n-1}{*} Id_n(\beta_k)\big) \overset{(Ass2)}{=\!=} (a'_n \underset{k-1}{*} \beta_k) \underset{n-1}{*} (a_n \underset{k-1}{*} \beta_k),$$

we have

$$\begin{aligned} s_n(c^{(n)}_{n+1}) &= a'_n \underset{n-1}{*} a_n \underset{0}{*} \beta_1 \cdots \underset{n-2}{*} \beta_{n-1} \underset{n-1}{*} \beta_n \\ &= a'^{\,(n-1)}_n \underset{n-1}{*} a^{(n-1)}_n \underset{n-1}{*} \beta_n \end{aligned}.$$

By 3.1(Id),

$$t_n(c^{(n)}_{n+1}) = Id(b_{n-1}) \underset{0}{*} \beta_1 \cdots \underset{n-2}{*} \beta_{n-1} \underset{n-1}{*} \beta_n = \beta_n$$

Now it is easy to see that, by 3.1(ST),

$$c^{(n)}_{n+1} \underset{n}{*} (a'^{\,(n-1)}_n \underset{n-1}{*} \alpha_{n+1}) \in \mathrm{Mor}(a'^{\,(n-1)}_n \underset{n-1}{*} \varepsilon_n,\ \beta_n).$$

Since C is a weak ∞-groupoid, there is a right quasi-inverse for the latter $(n+1)$-morphism. Denote it by α'_{n+1} and let γ_{n+2} be a homotopy from

$$c^{(n)}_{n+1} \underset{n}{*} (a'^{\,(n-1)}_n \underset{n-1}{*} \alpha_{n+1}) \underset{n}{*} \alpha'_{n+1} \overset{(Ass1)}{=\!=} c^{(n)}_{n+1} \underset{n}{*} (a'^{\,(n-1)}_n \underset{n-1}{*} \alpha_{n+1} \underset{n}{*} \alpha'_{n+1})$$

to $Id_{n+1}(\beta_n)$. Now it is clear that (a'_n, α'_{n+1}) and (c_{n+1}, γ_{n+2}) constructed above are the required polymorphisms.

(3.7) It is obvious that P is a functor from $\mathcal{GRP}^{\infty}_{*}$ to itself. Define a natural transformation τ from P to the identity functor by the rule $\tau(C)(a_n, \alpha_{n+1}) = a_n$. It is straightforward to verify that the kernel $\Omega(C)$ of $\tau(C)$ has the following structure.

$$\Omega(C)_0 = \mathrm{Mor}_C(\star,\star), \qquad \Omega(C)_k = \{\alpha \in C_{k+1} \mid s_0(\alpha) = t_0(\alpha) = \star\},$$
$$t_k^{\Omega(C)} = s_{k+1}^C, \qquad s_k^{\Omega(C)} = t_{k+1}^C, \qquad Id_k^{\Omega(C)} = Id_{k+1}^C,$$

and the operations $\underset{k}{*}$ in $\Omega(C)$ differs from $\underset{k+1}{*}$ in C by the order of factors. It follows that the compositions $\pi_k = \pi_0 \circ \Omega^k$ coincide (as functors to pointed sets) with the usual homotopy group functors. Namely,

$$\pi_k(C) = C_k^*/\sim, \text{ where } C_k^* = \mathrm{Mor}\big(Id_{k-1}(\star), (Id_{k-1}(\star)\big)$$

and $\sim$ is the equivalence relation defined in (3.2). The multiplication on $\pi_k(C)$ is induced by the operation $\underset{k-1}{*}$. Of course, our definition of the first homotopy group differs from the usual one by the order of factors (the higher homotopy groups are Abelian, so the order of factors does not play any role). To get the usual first homotopy group one may consider $P(C^{op})$ instead of $P(C)$.

Our next purpose is to get the long exact sequence of homotopy fiber using the technique of Section 1. Before proving this result we exhibit some properties of the relative groupoid $R(\psi)$ which will be used in the next section as well. Let $\psi : C \to D$ be a morphism of pointed weak ∞-groupoids. Denote by $R = R(\psi)$ the pullback of the diagram $C \xrightarrow{\psi} D \xleftarrow{\tau(D)} P(D)$.

3.8. Lemma. *R has the following structure.*

$$R_0 = \{(c_0, \delta_1) \mid c_0 \in C_0,\ \delta_1 \in \mathrm{Mor}_D\big(\star, \psi(c_0)\big)\},$$
$$\mathrm{Mor}\big((b_{n-1}, \beta_n),\ (e_{n-1}, \varepsilon_n)\big) =$$
$$= \{(c_n, \delta_{n+1}) \mid c_n \in \mathrm{Mor}_C(b_{n-1}, e_{n-1}),\ \delta_{n+1} \in \mathrm{Mor}_C(\varepsilon_n,\ \psi(c_n)^{(n-1)} \underset{n-1}{*} \beta_n)\}$$

where the superscript $^{(n-1)}$ has the same meaning as in 3.4. Definitions of Id and of the operations $\underset{p}{*}$ are similar to those in 3.4.

An element (c_n, δ_{n+1}) goes to c_n in C and to $(\psi(c_n), \delta_{n+1})$ in $P(D)$.

Let $r, \tilde r \in R_k^*$ and $(c_n, \delta_{n+1}) \in \mathrm{Mor}(r, \tilde r)$. Then $\psi(c_n)^{(k)} = \psi(c_n)$ for any $k < n$.

3.9. Proposition. *Let $\psi : C \to D$ be a morphism of pointed weak ∞-groupoids. Then there is a long exact sequence of pointed sets*

$$\cdots \longrightarrow \pi_{k+1}(C) \longrightarrow \pi_{k+1}(D) \longrightarrow \pi_k(R) \longrightarrow \pi_k(C) \longrightarrow \pi_k(D) \longrightarrow \cdots$$
$$\longrightarrow \pi_1(C) \longrightarrow \pi_1(D) \longrightarrow \pi_0(R) \longrightarrow \pi_0(C) \longrightarrow \pi_0(D)$$

Remark. The forgetful functor from $\mathcal{G}\mathcal{R}\mathcal{O}\mathcal{U}\mathcal{P}\mathcal{S}$ to $\mathcal{S}\mathcal{E}\mathcal{T}\mathcal{S}_*$ preserves exact sequences. Therefore, if we erase the π_0 terms we will get the exact sequence of groups.

Proof. We shall check that each 5-term sequence

$$\pi_{k+1}(C) \longrightarrow \pi_{k+1}(D) \longrightarrow \pi_k(R) \longrightarrow \pi_k(C) \longrightarrow \pi_k(D)$$

is exact. Fix $k \geqslant 0$, and set $\pi = \pi_k$. By Proposition 1.8, we have to show that $\pi \circ P$ is a zero functor and that π maps pullbacks of Lemma 1.7 to exact squares.

Let $(a_k, \alpha_{k+1}) \in P(C)^*_k$. Then $a_k^{k-1} = a_k$, hence, $\alpha_{k+1} \in \mathrm{Mor}_C(\mathit{Id}_k(\star), a_k)$. Thus, $\big(\alpha_{k+1}, \mathit{Id}_{k+2}(\alpha_{k+1})\big)$ is a homotopy from (a_k, α_{k+1}) to the base point of $P(C)_k$. This proves that $P(C)^*_k$ has only one equivalence class.

To prove the exactness statement consider a pullback of weak ∞-groupoids

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ T & \xrightarrow{\ \varphi\ } & Z \end{array}$$

with $\pi(Y) = 0$ (each square in Lemma 1.7 is of this shape). Then the exactness of the image of this square under π is equivalent to the following statement: Given $t \in T^*_k$ such that $\varphi(t) \sim \mathit{Id}_k(\star)$ there are $\tilde{t} \sim t$ and $y \in Y^*_k$ such that $\varphi(\tilde{t}) = \psi(y)$. This statement is obvious if Y^*_k maps onto the connected component of the base point of Z^*_k as in the case of the first square of Lemma 1.7. The statement above for the third square of Lemma 1.7 follows from Lemma 3.8.

It remains to verify the statement above for the square

$$\begin{array}{ccc} \Omega(D) & \longrightarrow & R \\ \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & C \end{array}$$

Since R^*_k is a fibre product of $P(D)^*_k$ and C^*_k over D^*_k, it can be identified with the set

$$\{(c_k, \delta_{k+1}) \,|\, c_k \in C^*_k,\ \ \delta_{k+1} \in \mathrm{Mor}_D\big(\mathit{Id}_k(\star),\ \psi(c_k)\big)\}$$

If $(c_k, \delta_{k+1}) \in R^*_k$ goes to the connected component of $\star_C$, then there exist an element $\gamma_{k+1} \in \mathrm{Mor}_C(\mathit{Id}_k(\star),\ c_k)$. Let γ'_{k+1} be a quasi-inverse for γ_{k+1} and let γ_{k+2} be a homotopy from $\gamma_{k+1} \underset{k}{*} \gamma'_{k+1}$ to $Id_{k+1}(\star)$. Then $\big(Id_k(\star),\ \psi(\gamma'_{k+1}) \underset{k}{*} \delta_{k+1}\big)$ maps to $\star_C$ and $(\gamma_{k+1},\ \gamma_{k+2} \underset{k}{*} \delta_{k+1}) \in R_{k+1}$ is a desired homotopy from (c_k, δ_{k+1}) to $\big(Id_k(\star),\ \psi(\gamma'_{k+1}) \underset{k}{*} \delta_{k+1}\big)$.

4. The Mayer–Vietoris sequence

(4.1) Let now

$$\begin{tikzcd} A \arrow[r, "\psi'"] \arrow[d, "\varphi'"'] & B \arrow[d, "\varphi"] \\ C \arrow[r, "\psi"] & D \end{tikzcd}$$

be a commutative square in $\mathcal{GRP}^{\infty}_{*}$. According to Proposition 3.9 there are two long exact relative sequences corresponding to morphisms ψ and ψ'. By the functorial properties of our construction we have canonical morphism between this two sequences. We shall give certain conditions on the square above which guarantee that the canonical morphism $\pi_k\big(R(\psi)\big) \to \pi_k\big(R(\psi')\big)$ is an isomorphism. The importance of this isomorphism is shown by the following statement.

4.2. Lemma. *If all morphisms $\pi_k\big(R(\psi')\big) \to \pi_k\big(R(\psi)\big)$ are isomorphisms then there is a long exact Mayer–Vietoris sequence*

$$\begin{tikzcd}[column sep=small] & & \pi_{k+1}(B) \arrow[dr] & & & \pi_k(B) \arrow[dr] & & \\ \cdots \arrow[r] & \pi_{k+1}(A) \arrow[ur] \arrow[dr] & & \pi_{k+1}(D) \arrow[r, "\partial_{k+1}"] & \pi_k(A) \arrow[ur] \arrow[dr] & & \pi_k(D) \arrow[r] & \cdots \\ & & \pi_{k+1}(C) \arrow[ur] & & & \pi_k(C) \arrow[ur] & & \end{tikzcd}$$

i.e. each square of the diagram above is exact, the image of the connecting morphism ∂_{k+1} coincides with the intersection of the kernels of $\pi_k(\varphi')$ and $\pi_k(\psi')$, and the images of $\pi_{k+1}(\varphi)$ and $\pi_{k+1}(\psi)$ generate the kernel of ∂_{k+1} (recall that $\pi_{k+1}(G)$ is a group for any weak ∞-groupoid G and $k \geqslant 0$).

Proof. The proof goes by standard diagram chasing and will be left to the reader.

(4.3) Now we formulate conditions on square 4.1 which will be used to guarantee the isomorphisms of the previous lemma. They says that a polymorphism from D can be decomposed to (or replaced by) the product of images of polymorphisms from B and C.

(D'_k) Given $b_k \in B^*_k$, $c_k \in C^*_k$, and $\delta_{k+1} \in \mathrm{Mor}_D\big(\varphi(b_k),\ \psi(c_k)\big)$ there are $\beta_{k+1} \in B_{k+1}$ and $\gamma_{k+1} \in C_{k+1}$ such that $s_k(\beta_{k+1}) = b_k$, $t_k\big(\psi(\gamma_{k+1})\big) = \psi(c_k)$, and $t_k\big(\varphi(\beta_{k+1})\big) = s_k\big(\psi(\gamma_{k+1})\big)$.

(D''_k) The same as (D'_k) but in addition $\delta_{k+1} \sim \psi(\gamma_{k+1}) \underset{k}{*} \varphi(\beta_{k+1})$.

Note that if square (4.1) is a pullback, then $t_k(\beta_{k+1})$ and $s_k(\gamma_{k+1})$ has a unique common preimage in A_k.

4.4. Theorem. *Suppose that square* 4.1 *is a pullback and consider the canonical map*

$$\pi_k\big(R(\psi')\big) \to \pi_k\big(R(\psi)\big).$$

Under condition 4.3*(D''_k) this map is surjective. Under condition* 4.3*(D'_{k+1}) this map is injective.*

Proof. Take an element $(c_k, \delta_{k+1}) \in R(\psi)^*_k$. Since $\delta_{k+1} \in \mathrm{Mor}_D\big(Id_k(\star),\ \psi(c_k)\big)$, by ($D''_k$), there are $\beta_{k+1} \in B_{k+1}$ and $\gamma_{k+1} \in C_{k+1}$ such that $\delta_{k+1} \sim \psi(\gamma_{k+1}) \underset{k}{*} \varphi(\beta_{k+1})$ and $s_k(\beta_{k+1}) = Id_{k+1}(\star)$. Since our square is a pullback, there is a common preimage $a_k \in A^*_k$ of $s_k(\gamma_{k+1})$ and $t_k(\beta_{k+1})$. Now it is easy to see that $(a_k, \beta_{k+1}) \in R(\psi')^*_k$ and that its image in $R(\psi)^*_k$ is homotopic to the original element (c_k, δ_{k+1}). This proves surjectivity.

Let now (a_k, β_{k+1}) and $(\tilde{a}_k, \tilde{\beta}_{k+1})$ be two elements of $R(\psi')^*_k$. Suppose that their images in $R(\psi)^*_k$ are homotopic. Denote this homotopy by (c_{k+1}, δ_{k+2}), so that $c_{k+1} \in \mathrm{Mor}_C\big(\varphi'(a_k),\ \varphi'(\tilde{a}_k)\big)$ and $\delta_{k+2} \in \mathrm{Mor}_D\big(\varphi(\tilde{\beta}_{k+1}),\ \psi(c_{k+1}) \underset{k}{*} \varphi(\beta_{k+1})\big)$ (recall Lemma 3.8 which says that $\psi(c_{k+1})^{(k)} = \psi(c_{k+1})$). Denote by β'_{k+1} a quasi-inverse for β_{k+1}. The existence of δ_{k+2} and the groupoid property imply that $\varphi(\tilde{\beta}_{k+1}) \underset{k}{*} \varphi(\beta'_{k+1}) \sim \psi(c_{k+1})$. By condition ($D'_{k+1}$) and the remark in (4.3), there is $a_{k+1} \in A_{k+1}$ such that its image in B_{k+1} is homotopic to $\tilde{\beta}_{k+1} \underset{k}{*} \beta'_{k+1}$. It follows that $\tilde{\beta}_{k+1} \sim \varphi'(a_{k+1}) \underset{k}{*} \beta_{k+1}$. Denote this homotopy by β_{k+2}. Then (a_{k+1}, β_{k+2}) is a homotopy in $R(\psi')$ from (a_k, β_{k+1}) to $(\tilde{a}_k, \tilde{\beta}_{k+1})$. This means that these two elements belong to the same connected component of $R(\psi')^*_k$ and finishes the proof of the injectivity.

4.5. Corollary. *Suppose that square* 4.1 *is a pullback and that it satisfies* 4.3*(D''_k) for all $k \geqslant 0$. Then there is long exact Mayer–Vietoris sequence* 4.2.

REFERENCES

1. Bak A., *Global Actions: An algebraic counterpart of topological spaces.* — Russian Math. Surveys **52:5** (1997), 955 – 996.
2. Bak A., *Morphisms of coset spaces and linking diagrams.* Unpublished.
3. Jonson M., *The combinatorics of n-categorical pasting.* — J. of Pure Appl. Algebra **62:3** (1989), 211–225.
4. Kapranov M.M., Voevodsky V.A., *∞-Groupoids and homotopy types.* — Cahiers Topologie Géom. Différentielle Catégoriques **32:1** (1991), 29–46.
5. Minian G., *Generalized Cofibration Categories and Global Actions.* — K-Theory (2000) (to appear).
6. Street R., *The algebra of oriented simplexes.* — J. of Pure Appl. Algebra **49:3** (1987), 283–335.

Sankt-Petersburg
Electrotechnical University

Поступило 30 октября 2000 г.