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LIMIT THEOREMS FOR MAXIMA OF SUMS AND RENEWAL PROCESSES

Abstract. Let $\{X_i\}_{i=1,2,\dots}$ be a sequence of i.i.d. random variables, $S_n = X_1 + \dots + X_n$, and $S_n \to +\infty$ a.s. We discuss necessary and sufficient conditions for the Kolmogorov and Marcinkiewicz-Zygmund type strong laws of large numbers and for the law of the iterated logarithm for renewal processes defined in two different ways.

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1. Introduction

Let $X; X_1, X_2, \dots$ be a sequence of independent identically distributed (i.i.d.) random variables, $S_n = X_1 + \dots + X_n \quad (n = 1, 2, \dots)$, and $S_0 = 0$. Throughout this paper we assume that

$$S_n \to +\infty \qquad \text{a.s.} \tag{1}$$

In this case the corresponding renewal counting process can be defined almost surely (a.s.) in two different ways:

$$N(t) = \sup\{n : S_n \le t\}, \qquad t \ge 0 ,$$

or

$$\nu(t) = \inf\{n : S_n > t\}, \qquad t \ge 0 .$$

It is clear that $\nu(t) \le N(t) + 1$. If $X \ge 0$ a.s. then $\nu(t) = N(t) + 1$ a.s.. Various limit theorems for $\{\nu(t)\}$ (or, equivalently, for $\{N(t)\}$) can be found in Gut (1988). Some converses of these limit theorems have been proved in Gut, Klesov and Steinebach (1997). Kesten and Maller (1999) have found criteria for the stability of $\{\nu(t)\}$. Aim of this paper is to

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provide some further stability criteria concerning $\{\nu(t)\}$ and $\{N(t)\}$ and to prove their equivalence to the corresponding counterparts for partial sums and maxima.

In general, the two definitions above can result in different asymptotics. If $EX = a > 0$, then, by the strong law of large numbers (SLLN), $S_n \sim na$ a.s., and $N(t) \sim \nu(t) \sim t/a$ a.s. For a series of other limit theorems concerning $\nu(t)$ confer Gut (1988).

Here we mainly investigate the almost sure asymptotics of $N(t)$ as $t \to \infty$. We find necessary and sufficient conditions for the Kolmogorov SLLN, the Marcinkiewicz-Zygmund SLLN and the law of the iterated logarithm (LIL) for $\{N(t)\}$. The following correspondence between $N(t)$ and $T_n = \inf_{k \geq n} S_k$ will be repeatedly used:

$$\{N(t) \leq n\} = \{T_n > t\}. \tag{2}$$

A similar correspondence of $\nu(t)$ and $M_n = \max_{1 \leq k \leq n} S_k$ is well-known (see Gut (1988)):

$$\{\nu(t) > n\} = \{M_n \leq t\}. \tag{3}$$

In view of (2) and (3), we first study the asymptotic behavior of $\{T_n\}$ and $\{M_n\}$. Necessary and sufficient conditions for the weak and strong stability of $\{T_n\}$ and $\{M_n\}$ are derived, which are of independent interest. These results are applied to the renewal processes $\{N(t)\}$ and $\{\nu(t)\}$.

The paper is organized as follows. In Section 2, we discuss the stability of maxima and infima of partial sums. Strong limit theorems for renewal processes are stated in Section 3. The proofs of our results are given in Section 4.

In the sequel all limits are taken either as $n \to \infty$ or $t \to \infty$ if otherwise is not pointed out. We denote by $\xrightarrow{P}$, $\limsup_P$ etc. the corresponding convergences in probability.

2. Stability of maxima of partial sums

Let $\{b_n\}$ be a sequence of positive real numbers. Put

$$M_n = \max_{1 \leq k \leq n} S_k, \quad T_n = \inf_{k \geq n} S_k. \tag{4}$$

In this section we discuss necessary and sufficient conditions for the convergences of $M_n/b_n \to 1$ and $T_n/b_n \to 1$ either in probability or almost surely. A priori, the sequences $\{T_n\}$ and $\{M_n\}$ look more stable than $\{S_n\}$. They are, for example, nondecreasing. Nevertheless, Theorems

1 and 2 below show that these three random sequences are simultanuously stable.

Theorem 1. *Assume that* (1) *holds. The following statements are equivalent:*

(1a) $M_n/b_n \xrightarrow{P} 1$ *for some positive sequence* $\{b_n\}$;

(1b) $T_n/b_n \xrightarrow{P} 1$ *for some positive sequence* $\{b_n\}$;

(1c) $S_n/b_n \xrightarrow{P} 1$ *for some positive sequence* $\{b_n\}$;

(1d) $\mu(x) = \mathbf{E}XI(|X| < x)$ *is positive for large* x *and slowly varying at* ∞ *such that*

$$\frac{xP(|X| \geq x)}{\mu(x)} \to 0 \qquad as \qquad x \to \infty\ .$$

If anyone of the assertions (1a)–(1c) *holds, then* $\{b_n\}$ *is a regularly varying sequence (with index* 1*), and the two other assertions hold with the same* $\{b_n\}$.

Rogozin (1976) has proved that ($1c$) and ($1d$) are equivalent. For the equivalence of ($1a$) and ($1c$) confer Kesten and Maller (1999). For the equivalence of ($1b$) and ($1c$) see Section 4.

Theorem 2. *Assume that* (1) *holds. The following statements are equivalent.*

(2a) $M_n/b_n \to 1$ a.s. *for some positive sequence* $\{b_n\}$;

(2b) $T_n/b_n \to 1$ a.s. *for some positive sequence* $\{b_n\}$;

(2c) $S_n/b_n \to 1$ a.s. *for some positive sequence* $\{b_n\}$;

(2d) $0 < \mathbf{E}X < \infty$.

The equivalence of (2c) and (2d) can be found in Maller (1978), and the equivalence of (2a) and (2c) has been proved by Kesten and Maller (1999). The equivalence (2b) and (2c) is proved in Section 4.

The results of Kesten and Maller (1999) have been proved without assumption (1). In Ulm (1999), the authors have independently presented the necessary and sufficient conditions for (1a) with a given sequence $b_n \nearrow \infty$ assuming $\liminf_P S_n/b_n > -\infty$, which makes our proofs much simpler (see Frolov, Martikainen, Steinebach (2000) for details).

3. Equivalences in strong limit theorems for renewal processes

Various strong limit theorems for renewal processes are well-known in the ordinary case, i.e., for $X \geq 0$ a.s. Sufficient conditions for the

Kolmogorov SLLN, Marcinkiewicz-Zygmund SLLN and the LIL can be found, for example, in Gut (1988). Necessity in these limit theorems has been derived by Gut, Klesov and Steinebach (1997). In Gut (1988), one can also find generalizations of the above mentioned limit theorems for the case when X can be negative. But only sufficient conditions are given there, and only for the case when the renewal process is defined as $\{\nu(t)\}$.

In this section main attention will be paid to deriving necessary and sufficient conditions in strong limit theorems when the renewal process is defined as $\{N(t)\}$. Moreover, we prove that sufficient conditions in the corresponding limit theorems for $\{\nu(t)\}$ are also necessary.

We start from the Kolmogorov SLLN.

Theorem 3. *Assume that* (1) *holds. The following statements are equivalent:*

(3a) $N(t)/t \to 1/a$ a.s. *for some positive* a*;*

(3b) $\nu(t)/t \to 1/a$ a.s. *for some positive* a*;*

(3c) $M_n/n \to a$ a.s. *for some positive* a*;*

(3d) $T_n/n \to a$ a.s. *for some positive* a*;*

(3e) $0 < \mathbf{E}X < \infty$.

Moreover, in (3a)–(3d) *we must have* $a = \mathbf{E}X$.

Indeed, if (3e) holds then, by the Kolmogorov SLLN, (3c) and (3d) hold. Moreover, by Heyde (1966), (3e) implies (3b) (cf. Gut (1988)). By Theorems 1 and 2, (3e) follows from (3c) or (3d). Kesten and Maller (1999) proved that (3b) implies (3e). In Section 4 it is shown that (3a) is equivalent to (3d). In fact, this follows from the correspondence in (2).

The next result can be derived from Theorem 3.

Theorem 4. *Assume that* (1) *holds. Then there exist a.s. finite limits* $\lim N(t)/t$ *and* $\lim \nu(t)/t$ *, and these limits are equal a.s. They are a.s. positive if* $\mathbf{E}|X| < \infty$ *, and are equal to* 0 *a.s., otherwise.*

Now we turn to the Marcinkiewicz-Zygmund SLLN.

Theorem 5. *Assume that* (1) *holds. Let* $1 < r < 2$. *The following statements are equivalent:*

(5a) $t^{-1/r}(N(t) - t/a) \to 0$ a.s. *for some positive* a*;*

(5b) $t^{-1/r}(\nu(t) - t/a) \to 0$ a.s. *for some positive* a*;*

(5c) $n^{-1/r}(T_n - na) \to 0$ a.s. *for some positive* a*;*

(5d) $n^{-1/r}(M_n - na) \to 0$ a.s. *for some positive* a*;*

(5e) $0 < \mathbf{E}X < \infty$, $\mathbf{E}|X|^r < \infty$.

Moreover, in (5a)–(5d) *we must have* $a = \mathbf{E}X$.

Theorem 6. *Assume that* (1) *holds. Let* $0 < r < 1$. *The following statements are equivalent:*

(6a) $t^{-r} N(t) \to \infty$ a.s.;

(6b) $t^{-r} \nu(t) \to \infty$ a.s.;

(6c) $n^{-1/r} T_n \to 0$ a.s.;

(6d) $n^{-1/r} M_n \to 0$ a.s.;

(6e) $\mathbf{E}|X|^r < \infty$.

Note that, under condition (1), (6e) is equivalent to $\mathbf{E}(X^+)^r < \infty$. In fact, the latter condition implies $M_n \le \sum_{k=1}^n X_k^+ = o(n^{1/r})$ a.s., so that (6d) holds.

Next we deal with the law of the iterated logarithm.

Theorem 7. *Assume that* (1) *holds, and* X *is non-degenerate. Then the following statements are equivalent:*

(7a) $\limsup \frac{N(t)-t/a}{\sqrt{2t \log\log t}} \in (0, +\infty)$ *a.s.* *for some positive* a;

(7b) $\limsup \frac{N(t)-t/a}{\sqrt{2t \log\log t}} \in (-\infty, +\infty)$ *a.s.* *for some positive* a;

(7c) $\liminf \frac{T_n - na}{\sqrt{2n \log\log n}} \in (-\infty, 0)$ *a.s.* *for some positive* a;

(7d) $\liminf \frac{T_n - na}{\sqrt{2n \log\log n}} \in (-\infty, +\infty)$ *a.s.* *for some positive* a;

(7e) $\mathbf{E}X = a$, $\mathbf{var}X < \infty$;

Theorem 8. *Assume that* (1) *holds. Then the following statements are equivalent:*

(8a) $\limsup \frac{N(t)-t/a}{\sqrt{2t \log\log t}} = \frac{\sigma}{a^{3/2}}$ *a.s. and* $\liminf \frac{N(t)-t/a}{\sqrt{2t \log\log t}} = -\frac{\sigma}{a^{3/2}}$ *a.s. for some positive* a, σ^2;

(8b) $\mathbf{E}X = a$ *and* $\mathbf{var}\, X = \sigma^2$ *for some positive* a, σ^2.

Finally we state corresponding results for $\{\nu(t)\}$ and $\{M_n\}$.

Theorem 9. *Assume that* (1) *holds, and* X *is non-degenerate. Then the following statements are equivalent:*

(9a) $\liminf \frac{\nu(t)-t/a}{\sqrt{2t \log\log t}} \in (-\infty, 0)$ *a.s.* *for some positive* a;

(9b) $\liminf \frac{\nu(t)-t/a}{\sqrt{2t \log\log t}} \in (-\infty, +\infty)$ *a.s.* *for some positive* a;

(9c) $\limsup \frac{M_n - na}{\sqrt{2n \log\log n}} \in (0, \infty)$ *a.s.* *for some positive* a;

(9d) $\limsup \frac{M_n - na}{\sqrt{2n \log\log n}} \in (-\infty, +\infty)$ *a.s.* *for some positive* a;

Moreover these statements are equivalent to (7a)–(7e).

Theorem 10. *Assume that* (1) *holds. Then the following statements are equivalent:*

(10a) $\limsup \frac{\nu(t)-t/a}{\sqrt{2t\log\log t}} = \frac{\sigma}{a^{3/2}}$ *a.s. and* $\liminf \frac{\nu(t)-t/a}{\sqrt{2t\log\log t}} = -\frac{\sigma}{a^{3/2}}$ *a.s. for some positive* a, σ^2;
(10b) $\mathbf{E}X = a$ *and* $\mathbf{var}X = \sigma^2$ *for some positive* a, σ^2.

The sufficiency of (10b) can be found in Gut (1988, p. 102).

4. Proofs

Proof of Theorem 1. We have $T_n = S_n + U_n$, where $U_n = \inf_{k\geq n}(S_k - S_n)$. Here U_n are identically distributed random variables, and $U_n/b_n \xrightarrow{P} 0$ since $b_n \to \infty$ in view of condition (1).

Hence $T_n/b_n \xrightarrow{P} 1$ iff $S_n/b_n \xrightarrow{P} 1$.

Proof of Theorem 2. We need to prove that (2b) $\iff$ (2c).

If (2c) holds then, by Rogozin (1976), $(1/b_n)\max_{1\leq k\leq n} b_k \to 1$, and (2c) holds with a nondecreasing sequence $\{b_n\}$. Hence we get (2b).

Now we prove that (2b) $\Longrightarrow$ (2c). We will use the following result.

Lemma 1. *Let* $\{U_n\}$ *and* $\{V_n\}$ *be sequences of random variables. Suppose that, for any* $n \geq 1$, *the random variables* $U_1, U_2, \ldots, U_n$ *do not depend on* V_n, *and* V_n *has a symmetric distribution. Then*

a) $U_n + V_n \to 0$ *a.s. implies* $U_n \to 0$ *a.s. and* $V_n \to 0$ *a.s.;*

b) $\limsup U_n \leq \limsup(U_n + V_n)$ *a.s.;* $\limsup V_n \leq \limsup(U_n + V_n)$ *a.s.*

Statement a) can be found in Martikainen (1985). Part b) is proved in the same way.

Applying Lemma 3 to $T_n^s = S_n^s + (\inf_{k\geq n}(S_k - S_n))^s = o(b_n)$ a.s. (here and in the sequel Z^s denotes the symmetrization of a random variable Z), we conclude that $S_n^s/b_n \to 0$ a.s., and

$$\sum_{n=1}^{\infty} \mathbf{P}(|X| > 2b_n) < \infty. \tag{5}$$

By Theorem 1, $S_n/b_n \xrightarrow{P} 1$.

We further need the following lemma which is an immediate consequence of Petrov (1972), Ch. 9, Theorem 17.

Lemma 2. *If $b_n \nearrow \infty$, $\sum_{k=n}^{\infty} 1/b_k^2 = O(n/b_n^2)$, and (5) holds, then*

$$\frac{S_n - mS_n}{b_n} \to 0 \quad \text{a.s.},$$

where mS_n denotes the median of S_n.

Note that $S_n/b_n \xrightarrow{P} 1$ implies $b_{2n}/b_n \to 2$ (cf. Rogozin (1976)). We have

$$\sum_{k=n}^{\infty} \frac{1}{b_k^2} = \sum_{j=0}^{\infty} \sum_{k=n2^j}^{n2^{j+1}-1} \frac{1}{b_k^2} \le \sum_{j=0}^{\infty} \frac{n2^j}{b_{n2^j}^2} \le C\frac{n}{b_n^2}.$$

Applying Lemma 2, we conclude that

$$\frac{S_n - mS_n}{b_n} \to 0 \quad \text{a.s.}$$

Since $S_n/b_n \xrightarrow{P} 1$ and $mS_n/b_n \to 1$, we have $S_n/b_n \to 1$ a.s.

Proof of Theorem 3. By the correspondence (2) we get

$$\mathbf{P}(N(t) \le (1+\varepsilon)t/a \text{ eventually } \forall\ \varepsilon > 0) =$$
$$= \mathbf{P}(T_n > na/(1+\varepsilon) \text{ eventually } \forall\ \varepsilon > 0) ,$$

and

$$\mathbf{P}(N(t) > (1-\varepsilon)t/a \text{ eventually } \forall\ \varepsilon > 0) =$$
$$= \mathbf{P}(T_n \le na/(1-\varepsilon) \text{ eventually } \forall\ \varepsilon > 0) .$$

So, (3a) is equivalent to (3d).

Proof of Theorem 4. Assume first that $\mathbf{E}|X| < \infty$. By (1) , we have $\mathbf{E}X > 0$, and Theorem 4 follows from Theorem 3.

Assume now that $\mathbf{E}|X| = \infty$. As in the previous proof we have

$$\mathbf{P}(N(t) \le t\varepsilon \text{ eventually } \forall\ \varepsilon > 0) = \mathbf{P}(S_n > n/\varepsilon \text{ eventually } \forall\ \varepsilon > 0) .$$

Hence $N(t)/t \to 0$ a.s. is equivalent to $S_n/n \to +\infty$ a.s.

Now if $\mathbf{E}X^- < \mathbf{E}X^+ = +\infty$, then $S_n/n \to +\infty$ a.s. If $\mathbf{E}X^- = \mathbf{E}X^+ = +\infty$, then by Kesten (1970) we have again $S_n/n \to +\infty$ a.s. (since $S_n \to +\infty$ a.s.)

So, if $\mathbf{E}|X| = \infty$ we retain the conclusion of Theorem 4.

Concerning $\nu(t)$, taking a result of Gut (1988, p. 83) into account, we only need to discuss the case $\mathbf{E}|X| = \infty$ in which the desired statement follows immediately from the inequality $\nu(t) \le N(t) + 1$.

Proof of Theorem 5. It follows from the Marcinkiewicz-Zygmund SLLN (cf., for example, Loève (1960)) that (5e) is equivalent to the following statement:

$\exists a > 0$ such that

$$\lim \frac{S_n - na}{n^{1/r}} = 0 \quad \text{a.s.} \tag{6}$$

The equivalences (5a) $\iff$ (5c) and (5b) $\iff$ (5d) can be proved in the same way as in the proof of Theorem 3. The further steps require the following lemma:

Lemma 3. *Let $\{a_n\}$, $\{b_n\}$ and $\{S_n\}$ be sequences of real numbers such that $a_n \nearrow \infty$, $b_n > 0$, $b_n \nearrow$, and let $\{M_n\}$ and $\{T_n\}$ be as in (4).*

If $(S_n - a_n)/b_n \to 0$, then $(M_n - a_n)/b_n \to 0$, and if additionally $b_n/a_n \searrow$, then $(T_n - a_n)/b_n \to 0$.

Moreover, for any $C > 0$, $\limsup(S_n - a_n)/b_n \leq C$ implies $\limsup(M_n - a_n)/b_n \leq C$.

Proof. Concerning $\{M_n\}$, we only need to prove the last (more refined) statement.

Assume that $C > 0$ and $\limsup(S_n - a_n)/b_n \leq C$. We have $S_n \leq M_n$ and, for some $k_n \leq n$,

$$M_n - a_n = S_{k_n} - a_n \leq S_{k_n} - a_{k_n} \ .$$

Therefore

$$\limsup \frac{M_n - a_n}{b_n} \leq \limsup \frac{S_{k_n} - a_{k_n}}{b_{k_n}} \frac{b_{k_n}}{b_n} \leq C.$$

The proof for $\{T_n\}$ is a little more complicated. We only prove the lower bound because $T_n \leq S_n$. Put $K_n = \{k : k \geq n, a_k \leq 2a_n\}$, $l_n = \sup\{k : k \in K_n\}$.

Since $b_n = O(a_n)$, we get for $k > l_n$,

$$S_k = a_k + o(b_k) \geq \frac{2}{3} a_k \geq \frac{4}{3} a_n > a_n + o(b_n) = S_n$$

for all large n. Hence it follows that $T_n = \inf_{k \in K_n} S_k$, and for some $k_n \in K_n$,

$$T_n = S_{k_n} = a_{k_n} + o(b_{k_n}) \geq a_n + o(b_{k_n}) = a_n + o(b_n) \ ,$$

since

$$b_{k_n} = \frac{b_{k_n}}{a_{k_n}} a_{k_n} \leq \frac{b_{k_n}}{a_n} a_{k_n} \leq 2b_n \ .$$

This completes the proof of Lemma 3.

Applying Lemma 3 we conclude that (mzl) $\Longrightarrow$ (5c) and (5d).

Next we show that (5c) $\Longrightarrow$ (5e). We have

$$\frac{T_n - na}{n^{1/r}} = U_n + V_n, \quad \text{where} \quad U_n = \frac{S_n - na}{n^{1/r}}, \quad V_n = \frac{\inf_{k\geq n}(S_k - S_n)}{n^{1/r}}.$$

By Lemma 1, $U_n^s \to 0$ a.s. This yields $\mathbf{E}|X|^r < \infty$.

Moreover, (5c) implies $N(t)/t \to 1/a$ a.s., which gives $\mathbf{E}X = a$.

Finally we prove that (5d) $\Longrightarrow$ (5e). From (5d) we get that $M_n/n \to a$ a.s., and $M_{2n}/2n \to a/2$ a.s. This yields

$$\frac{\max_{n\leq k\leq 2n} S_k}{2n} = \frac{S_n + \max_{n\leq k\leq 2n}(S_k - S_n)}{2n} \to a \quad \text{a.s.}$$

By Lemma 1, $S_n^s/n \to 0$ a.s. which gives $\mathbf{E}|X| < \infty$. (1) implies $\mathbf{E}X > 0$ and, by (5d), $\mathbf{E}X = a$.

Let now $k_n = [n - n^{1/r}]$. Then $k_n \nearrow \infty$ and $k_{n+1} - k_n \leq 1$. We have

$$\frac{M_{k_n} - k_n a}{n^{1/r}} \to 0 \quad \text{a.s.}$$

and

$$\frac{M_{k_n} - na}{n^{1/r}} \to -a \quad \text{a.s.}$$

From (5d) it follows that $M_{k_n} < M_n$ for large n , and therefore

$$\frac{\max_{k_n<k\leq n}(S_k - na)}{n^{1/r}} \to 0 \quad \text{a.s.}$$

Since

$$\max_{k_n<k\leq n}(S_k - na) = S_{k_n} - na + \max_{k_n<k\leq n}(S_k - S_{k_n}) \ ,$$

we have by Lemma 1,

$$\frac{S_{k_n}^s}{n^{1/r}} \to 0 \quad \text{a.s.,}$$

i.e.,

$$\frac{S_n^s}{n^{1/r}} \to 0 \quad \text{a.s.,}$$

which gives $\mathbf{E}|X|^r < \infty$.

Proof of Theorem 6. Evidently, (6e) $\Longrightarrow$ (6d) $\Longrightarrow$ (6c).

From (6c) we get

$$\frac{S_n + \inf_{k\geq n}(S_k - S_n)}{n^{1/r}} \to 0 \quad \text{a.s.}$$

By Lemma 1, $S_n^s/n^{1/r} \to 0$ a.s., and therefore (6e) holds.

The proof of (6c)$\Longleftrightarrow$(6a) and (6d)$\Longleftrightarrow$(6b) simply makes use of the correspondences (2) and (3).

Proof of Theorem 7. The following lemma will be used.

Lemma 4. *Let $\{a_n\}$, $\{b_n\}$ and $\{S_n\}$ be sequences of real numbers such that*
$a_n \nearrow \infty$, $b_n/a_n \searrow 0$, $S_n/a_n \to 1$, and let $\{T_n\}$ be as in (4). Then

$$\liminf \frac{S_n - a_n}{b_n} = \liminf \frac{T_n - a_n}{b_n}.$$

Note that Lemma 5 is not valid with lim inf's replaced by lim sup's.

Proof of Lemma 4. Since $T_n \leq S_n$, we only need to prove that, for any C,

$$\liminf \frac{S_n - a_n}{b_n} \geq C$$

implies

$$\liminf \frac{T_n - a_n}{b_n} \geq C.$$

Choose $\delta > 1$. Put $K_n = \{k : k \geq n, a_k \leq \delta a_n\}$, $l_n = \sup\{k : k \in K_n\}$. It is clear that, for $\varepsilon > 0$, n large, and $k > l_n$,

$$S_k \geq a_k + (C-\varepsilon)b_k \geq \delta^{-1/2}a_k \geq \delta^{1/2}a_n \geq \delta^{1/3}S_n,$$

which yields $S_k > S_n$. Then, for some $k_n \in K_n$,

$$T_n = \inf_{k\in K_n} S_k = S_{k_n} \geq a_{k_n} + (C-\varepsilon)b_{k_n} \geq a_n + (C-\varepsilon)b_{k_n} \geq a_n + C_1 b_n,$$

since

$$b_n \leq b_{k_n} = \frac{b_{k_n}}{a_{k_n}}a_{k_n} \leq \frac{b_n}{a_n}a_{k_n} \leq \delta b_n.$$

So, Lemma 4 is proved.

Assume now that (7e) holds. By the Hartman–Wintner LIL and Lemma 4, we have (7c). Clearly, (7c) $\Longrightarrow$ (7d). Furthermore, Lemma 4 and (7d) imply

$$-\infty < \liminf \frac{S_n - na}{\sqrt{n\log\log n}} < +\infty \quad \text{a.s. ,}$$

and by the converse to the LIL (cf. Martikainen (1980), Rosalsky (1980)), we get (7e).

The equivalences (7a)$\Longleftrightarrow$(7c) and (7b)$\Longleftrightarrow$(7d) can be proved as in the previous theorems.

Proof of Theorem 8. We need two lemmas.

Lemma 5. *Assume that $b(x)$ is a continuous increasing function with $b(0) = 0$, and $b(+\infty) = +\infty$. Then*

$$\mathbf{P}(N(t) \le b(t) \text{ eventually}) = \mathbf{P}(T_n \ge b^{-1}(n) \text{ eventually}) = \\ = \mathbf{P}(S_n \ge b^{-1}(n) \text{ eventually}).$$

Proof of Lemma 5. In fact, we prove that the above events coincide on the set $\{\omega : T_n(\omega) \to \infty\}$.

If $N(t) \le b(t) \quad \forall\ t \ge t_\omega$, then $N(T_n) \le b(T_n) \quad \forall\ n \ge n_\omega$. This implies
$n \le b(T_n) \quad \forall\ n \ge n_\omega$, which yields $T_n \ge b^{-1}(n) \quad \forall\ n \ge n_\omega$.

Evidently, $S_n \ge T_n$. Therefore, if $T_n \ge b^{-1}(n)$ then $S_n \ge b^{-1}(n)$.

Finally, if $S_n \ge b^{-1}(n) \quad \forall\ n \ge n_\omega$, then $S_{N(t)} \ge b^{-1}(N(t)) \quad \forall\ t \ge t_\omega$. The latter relation gives $t \ge b^{-1}(N(t)) \quad \forall\ t \ge t_\omega$, which implies $N(t) \le b(t) \quad \forall\ t \ge t_\omega$. This completes the proof of Lemma 5.

Lemma 6. *Assume that $b(x)$ is a continuous increasing function with $b(0) = 0$, and $b(+\infty) = +\infty$. Then*

$$\mathbf{P}(N(t) \ge b(t) \text{ eventually}) \le \mathbf{P}(T_n \le b^{-1}(n) \text{ eventually}) \le \\ \le \mathbf{P}(N(t) \ge b(t) - 1 \text{ eventually}).$$

Proof of Lemma 6. We follow the lines of proof of Lemma 5.

If $N(t) \ge b(t) \quad \forall\ t \ge t_\omega$, then $N(T_n - 0) \ge b(T_n - 0) \quad \forall\ n \ge n_\omega$. This yields
$b(T_n) \le n \quad \forall\ n \ge n_\omega$, which gives $T_n \le b^{-1}(n) \quad \forall\ n \ge n_\omega$.

If $T_n \le b^{-1}(n) \quad \forall\ n \ge n_\omega$, then $T_{N(t)+1} \le b^{-1}(N(t) + 1) \quad \forall\ t \ge t_\omega$. Hence it follows that $t \le b^{-1}(N(t) + 1) \quad \forall\ t \ge t_\omega$. The latter relation implies $N(t) \ge b(t) - 1 \quad \forall\ t \ge t_\omega$, which completes the proof of Lemma 6.

Now, put $\chi_t = \sqrt{2t \log\log t}$, $b^{-1}(n) = na + \varepsilon\sigma\chi_n$, so that $b_n = (n/a) - (\varepsilon\sigma/a^{3/2})\chi_n + o(1)$. By Lemma 6 and the LIL,

$$\liminf \frac{N(t) - t/a}{\chi_t} \ge -\frac{\sigma}{a^{3/2}} \qquad \text{a.s.}$$

By Lemma 5 and the LIL,

$$\limsup \frac{N(t) - t/a}{\chi_t} = \frac{\sigma}{a^{3/2}} \qquad \text{a.s.}$$

Since $S_n/n \to a$ a.s., $T_n/n \to a$ a.s. and $a > 0$, we have $\inf_{k\geq(1+\varepsilon)n} S_k > (1+\varepsilon/2)na \ \forall\ n \geq n_\omega$ and $T_n = \inf_{n\leq k\leq(1+\varepsilon)n} S_k = U_n$ $\forall\ n \geq n'_\omega$ and $\forall\ \varepsilon > 0$.

We will show that

$$\limsup \frac{U_n - na}{\chi_n} \geq \frac{\sigma}{\sqrt{1+\varepsilon}} \quad \text{a.s.,}$$

which implies

$$\limsup \frac{T_n - na}{\chi_n} \geq \frac{\sigma}{\sqrt{1+\varepsilon}} \quad \text{a.s.} \tag{7}$$

Since $V_n = \inf_{n\leq k\leq(1+\varepsilon)n}(S_k - ka) \leq U_n - na$, we only need to prove that

$$\limsup \frac{V_n}{\chi_n} \geq \frac{\sigma}{\sqrt{1+\varepsilon}} \quad \text{a.s.} \tag{8}$$

Let $\{Y_k\}$ be a sequence of i.i.d. random variables, $EY_k = 0$, $EY_k^2 = \sigma^2 \in (0,\infty)$. Put $R_0 = 0$, $R_n = \sum_{k=1}^n Y_k$, $W_n = \inf_{n/(1+\varepsilon)\leq k\leq n} R_k$. Let $f_n(k/n) = R_k/\chi_n$, $k = 0, 1, \ldots, n$, and $f_n(t)$ be linear for $t \in [k/n, (k+1)/n]$.

By Strassen's (1964) LIL, the set of a.s. limit points of $\{f_n\}$ coincides a.s. in $C[0,1]$ with $\overline{K}_\sigma$, where

$$K_\sigma = \{f : f \text{ is absolutely continuous, } f(0) = 0, \int_0^1 (f'(t))^2 dt \leq \sigma^2\}.$$

We define the mapping $T : C[0,1] \to C[0,1]$ by $(Tf)(t) = f(t)$ if $t \leq 1/(1+\varepsilon)$, and $(Tf)(t) = \inf_{1/(1+\varepsilon)\leq s\leq t} f(s)$ if $1/(1+\varepsilon) \leq t \leq 1$. Evidently, this mapping is continuous and $Tf_n(1) = W_n/\chi_n$. By invariance, the set of a.s. limit points of the sequence $\{W_n/\chi_n\}$ coincides a.s. with $D = \{Tf(1) : f \in K_\sigma\}$.

Put $f(t) = \sigma t$, $t \in [0,1]$. Then $f \in K_\sigma$ and $Tf(1) = \sigma/(1+\varepsilon)$. It holds that

$$\limsup \frac{W_n}{\chi_n} \geq \frac{\sigma}{1+\varepsilon} \quad \text{a.s.}$$

Checking the proof of Strassen's LIL, we conclude that, for any integer $m \geq 1$, the set of a.s. limit points of the sequence $\{f_{nm}\}$ again coincides a.s. in $C[0,1]$ with K_σ. Hence it follows that

$$\limsup \frac{W_{nm}}{\chi_{nm}} \geq \frac{\sigma}{1+\varepsilon} \quad \text{a.s. },$$

for $m = 1, 2, \ldots$ Thus, for rational $1+\varepsilon$, we have

$$\limsup \frac{V_n}{\chi_{[(1+\varepsilon)n]}} \geq \frac{\sigma}{1+\varepsilon} \quad \text{a.s. },$$

which is equivalent to (8). By (7) we get

$$\limsup \frac{T_n - na}{\chi_n} \geq \sigma \quad \text{a.s.}$$

An application of Lemma 6 with $b(t) = ta + (1-\varepsilon)\sigma\chi_t$ completes the proof of Theorem 8.

Proof of Theorem 9. We restrict our attention to the proof of implication (9d) $\Longrightarrow$ (7e). In Gut (1988), one can find the proof of (7e) $\Longrightarrow$ (9a), and evidently (9a) $\Longrightarrow$ (9b), (9c) $\Longrightarrow$ (9d), (9a)$\Longleftrightarrow$(9c), (9b)$\Longleftrightarrow$(9d).

Assume that (9b) holds. We have $\limsup M_n/n = a$ a.s., and therefore a is a limit point of S_n/n with probability 1. Hence $EX = a$. By Lemma 3 and $M_n \geq S_n$, we get

$$-\infty < \limsup \frac{S_n - na}{\chi_n} < +\infty \quad \text{a.s.}$$

So, by the converse to the LIL (cf. Martikainen (1980), Rosalsky (1980)), we get (7e).

Proof of Theorem 10. It is immediate from Theorem 9 in combination with Gut (1988, p. 102).

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