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PLANAR MANHATTAN LOCAL MINIMAL AND CRITICAL NETWORKS

Abstract. One-dimensional branching extremals of Lagrangian-type functionals are considered. Such extremals appear as solutions to the classical Steiner problem on a shortest network, i.e., a connected system of paths that has the smallest total length among all the networks spanning a given finite set of terminal points in the plane. In the present paper, the *Manhattan-length* functional is investigated, with Lagrangian equal to the *sum* of the absolute values of projections of the velocity vector onto the coordinate axes. Such functionals are useful in problems arising in Electronics, Robotics, chip design, etc. In this case, in contrast to the case of the Steiner problem, local minimality does not imply extremality (however, each extreme network is locally minimal). A ctriterion of extremality is presented, which shows that the extremality with respect to the Manhattan-length functional is a global topological property of networks.

Introduction

The present work is devoted to the investigation of branching extremals, i.e., extremal networks, of the Manhattan length functional. Recall that the **Manhattan length of a straight segment in** $\mathbb{R}^n$ is defined as the sum of the lengths of the segment projections to the Cartesian coordinate axis. The **Manhattan length of a curve** can be defined as the limit of the Manhattan lengths of polygonal lines inscribed into the curve. The **Manhattan length of a network** (i.e., of a connected set of curves–edges endowed with a graph structure) is the sum of its edges Manhattan lengths.

Traditionally the shortest networks are investigated. In the present work we investigate wider classes of networks: locally shortest (so-called **local minimal**) networks, and **critical** networks, i.e., critical points of the Manhattan length functional, see exact definitions below. Notice that in the case of Riemannian length functional, local minimal networks are critical points of the length functional, and, if splitting of vertices is permitted, then the reverse statement is also true, see [67]. In other words, the class of local minimal networks and the class of critical networks coincide for the case of Riemannian length functional. It turns out, that in the case of Manhattan length functional the class of local minimal networks is wider than the one of critical networks. The main aim of the present work is the description of the difference between these classes.

The first works on the shortest networks in the sense of Manhattan length appeared at 60th, see [34], due to the intensive development of Electronics and Robotics. The interest to the Manhattan length appeared due to the fact that, as a rule, conductors on printed circuits have a form of polygonal lines formed by horizontal and vertical segments, and therefore the Manhattan length of the conductors coincides with their Euclidean length. Similar situation takes place in Robotics. Rather the first systematical investigation of the shortest networks in the sense of Manhattan length (so-called **shortest rectilinear trees**) was made by Hanan [42] in 1966, who described some important general properties of such networks. In particular, Hanan showed that among the shortest rectilinear trees there always exists one which is a subset of the union of all vertical and horizontal straight lines passing through the boundary points (the union is called **Hanan lattice**). Notice that the edges of the shortest rectilinear tree can be chosen in many different ways without changing the length of the tree. But, starting from the work of Hanan [42] the edges of the shortest trees are traditionally chosen in the form of polygonal lines whose links are parallel to the Cartesian coordinate axis.

After 10 years Hwang [47] described the possible structures of the shortest rectilinear trees under the assumption that the given boundary set can be spanned by at least one **nondegenerate** shortest tree Γ_0. The latter means that the degrees of all the boundary vertices in Γ_0 equal to 1. In particular, the tree Γ_0 has no vertices of degree 2. Hwang proved that in this case the shortest tree has one of the following two possible structures, which are depicted in Figure 1.

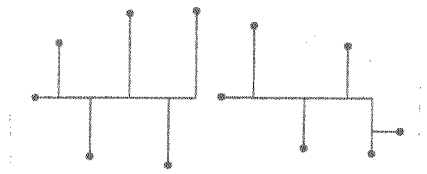

Fig. 1. Two possible types of nondegenerate shortest networks.

However, an efficient algorithm constructing a shortest rectilinear tree had not been found yet. The reason was explained in 1977 by Garey and Johnson [35], who proved that the problem of finding the shortest rectilinear tree is NP-hard, i.e., a polynomial algorithm solving the problem does not exist most likely. The fact makes the investigation of the restrictions to the structure of the shortest networks even more actual.

The idea to consider **local minimal** networks appeared first during the investigation of the shortest networks in the case of Euclidean length. Namely, after describing of the local structure of the shortest networks (to do this it suffices to solve the problem with at most one additional vertex) it is natural to try to describe all the networks satisfying the conditions obtained. These networks form a wider class and usually are referred as **local minimal** networks. Notice that the first real-life algorithms constructing the shortest Euclidean trees (they were based on the Melzak algorithm [Melz1]) enumerated all possible local minimal trees and selected the shortest one among them.

In real situations, sometimes it is essentially easier to find the shortest tree than to enumerate all local minimal networks. For example, if the boundary set is the vertex set of a regular n-gon, $n \geqslant 6$, then the shortest network is the n-gon itself without one of its sides (this result was obtained by Jarnik and Kössler [68] for $n \geqslant 13$, and by Du, Hwang, and Weng [26] for $6 \leqslant n \leqslant 12$). But the set of local minimal networks in that case is rather more complicated, see [60]. In the case of the shortest Manhattan networks with so called rectilinear-convex boundaries, a polynomial algorithm constructing such networks is known, see for example the work of Richards and Salowe [84].

Notice that the works listed above investigate some algorithmic aspects of the problem mainly. In particular, the difference between local minimal networks (i.e., in the context, the networks having the same local structure as the shortest networks have) and critical networks has not been observed. The geometry and topology of local minimal and critical networks have also not been investigated.

Our approach gives an opportunity to consider the problems of that type as natural generalizations of classical (one-dimensional) variational problem to the case of branching extremals[1]. The approach is based on a synthesis of ideas of differential geometry and convex geometry on the one

[1]Problems of that type were stated first by French mathematicians, among them are Gergonne, Clapeyron and Lame. Gauss also interested with problems of this type. In his letter to Schumacher [37] he mentioned the problem of building a shortest railway system between the for German cities: Harburg (close to Hamburg), Bremen, Hanover, and Braunschweig. General problem of finding the shortest network spanning

hand, and the ideas of discrete geometry and combinatorics on the other hand. In the case of Euclidean length, where the extremal can be considered as branching geodesics, we mention, beside the works of the authors, the works of A. Vdovina, I. Iskhakov, G. Karpunin, G. Lawlor, V. Manturov, F. Morgan, T. Pavlyukevich (Anikeeva), M. Pronin, I. Ptitsyna (Shklyanko), E. Selivanova, J. Hass, and others, who consider minimal networks from the same point of view, see a review in [ITbookGan]. Critical networks with respect to some other functionals of "elliptic type" were investigated by G. Lawlor and F. Morgan [74, 75].

From the geometrical point of view, interests to the networks extremal with respect to Euclidean and Manhattan length can be explained by the fact that these two functionals are, in some sense, the limiting cases of the functionals generated by Banach–Minkowski metrics. Namely, the Euclidean length is generated by the most symmetrical and strictly convex norm, and the Manhattan length is generated by the non-strictly convex norm with the maximal possible length of the circle. Just the failure of the strict convexity property leads to the fact that in the case of Manhattan length the classes of local minimal and critical networks are different, unlike the Euclidean case. Similar effects apparently take place also for the other Banach–Minkowski norms, see [53].

The debate on the necessity to investigate not only absolute minima but other critical points has a long history[2]. Notice that the critical

a given set of points of the plane was stated by Jarnik and Kössler [68] in 1934. Afterwards the problem became well-known as **Steiner Problem**, see the historical reviews in [10, 50, 58] or in [67].

Notice that the simplest case of Steiner problem when the boundary set consists of 3 points coincides with the following problem stated by Fermat: find the point in the plane such that the sum of the distances from that point to the vertices of a given triangle is the least. The principle difference between these two problems consists in the following: solving Steiner problem one does not know *a priori* neither the number of additional, i.e., non boundary vertices (these vertices are usually referred to as **Steiner points**), nor the way how the vertices are joined. Underline that just that undeterminacy makes the Steiner problem NP-hard.

[2]In 1746 Maupertuis published his famous Principle of Least Action: **if there occurs some change in Nature, the amount of action necessary for this change must be as small as possible**. Just after publication of the Principle, Maupertuis was criticized. On the one hand the question of priority was discussed, and on the other hand, it was understood fast that the Principle is not correct in such reading. For example, in 1749 and 1752 d'Arcy demonstrated how the Principle leads to false assertions using as a model the reflection of the light. D'Arcy showed that the "thrift" or "wastage" of the Nature (i.e., minimality or maximality of the necessary action) depends on the form of the mirror. The details of the dispute and some other

points different from absolute minimum or maximum appear naturally in multidimensional variational calculus too. A classical example is given by theory of minimal surfaces (minimal surface is a mathematical model of soap film or, more general, of an interface surface between two physical mediums in equilibrium). It is well-known that, generally speaking, the same one-dimensional contour can be spanned by several soap films having not only different geometrical form, but also having different topological structure, see, for example [17, 31, 32, 92, 44]. These films can correspond not only to absolute minima, but also to local minima and saddle points of the Area functional. Notice that the latter ones are non stable (see, for example [17], where the experiments of Poisson observing nonstable soap films are described). The questions of stability, i.e., the questions of finding out if a given surface is a local minimum or a saddle point are explored intensively in minimal surfaces theory, in particular, by the authors, see [51, 45, 46, 91].

On the other hand, the conversion to the investigation of all critical points leads to a new view of the nature of absolute minima and maxima, makes their global properties more clear. Just that idea works in Morse Theory.

In the present work the following results are obtained. Theorem 2.3 gives a criterion of that a local minimal network in the sense of Manhattan length is a critical network with respect to the Manhattan length functional. The main difficulty here consists of the necessity to control global deformations of local minimal networks attending all possible bifurcations of the vertices.

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interesting facts can be found in [44].

D'Arcy in his examples showed that the maxima of a functional can appear in Nature as well as minima of the same functional (by the way, Euler already understood this). But it is well-known that the critical points do not consist only of global maxima and global minima: there are local maxima and minima and saddle points also. As a standard example containing all possible types of critical points, one can take the geodesics on Riemannian manifolds.

§1. Preliminary results

In this section we recall some standard definitions necessary for what follows, see details in [58]. The main definitions of classical graph theory can be found, for example, in [80]. The necessary information concerning topology and differential geometry can be found in [78] and [71].

1.1. Graphs, networks. We shall consider graphs from topological point of view. The idea of topological approach to Graph Theory is to consider graphs as cell spaces. **A topological graph** G is a topological space obtained from a finite set $\{I_\alpha\}$ of line segments by gluing some of their ending points. Let $\pi\colon \sqcup_\alpha I_\alpha \to G$ be a canonical projection. The images of interiors of segments I_α under the mapping π are called **edges** of the graph G, and π-images of the ending points of the segments I_α are called **vertices**.

A homeomorphism of graphs taking vertices into vertices is called a **(topological) equivalence**. Graphs G_1 and G_2 are called **equivalent** if there exists an equivalence $\varphi\colon G_1 \to G_2$. In the sense of Graph Theory equivalent graphs are organized in the same way and they are not distinguished. In what follows all terminology from Graph Theory and Topological Spaces Theory will be applied to topological graphs without additional comments.

Let H be a subgraph in a topological graph G. The factor space G/H can be endowed evidently with the unique natural structure of a topological graph. The graph G/H is called the **factor graph** of the graph G over the subgraph H. A mapping φ from a graph G_1 to a graph G_2 is called a **projection**, if it coincides with the standard projection $\pi\colon G_1 \to G_1/H = G_2$ for some subgraph $H \subset G_1$. Note that the projection is continuous and closed by definition.

Suppose that a subset B of the vertices set of a graph G is marked. Such graph G is called a **graph with the boundary** $\partial G = B$. The vertices from ∂G are called **boundary vertices** or **fixed vertices**, and all the other vertices are called **interior** ones or **movable** ones. The edges of the graph incident to its boundary vertices are called **boundary edges**, and an edge which is not incident to any boundary vertex is called an **interior** one.

Let G be a graph with some boundary B. Let $\widehat{G}_B$ denote the subgraph in G generated by all movable vertices of the graph G. The subgraph $\widehat{G}_B$ is called the **movable subgraph of the graph** G (with respect to the boundary B). Note that the movable subgraph consists exactly of all movable edges of the graph G.

We defined above the equivalence of graphs. In the case of graphs with boundaries we always shall suppose that any equivalence generates a one-to-one mapping between the boundaries of such graphs. Also, if G is a graph with a boundary ∂G, then the **boundary of a factor-graph** obtained from G is defined as the image of ∂G under canonical projection.

Let G be an arbitrary graph with a boundary ∂G (the boundary can be empty) and $P \in G$ be some of its points. **An admissible neighbourhood** $U \subset G$ of a **point** P in the graph G is a closure of a connected neighbourhood of this point which does not contain any vertices of the graph G different from P, if P is a vertex, and which does not contain loops from G. Let us endow the neighbourhood U with a structure of graph by calling the vertices all the points from $\partial U \cup \{P\}$ and by calling the edges the interiors of the segments from U joining these points. The obtained star we denote by G_U and call the **local graph centered at** P. Define the **canonical boundary** ∂G_U **of a local graph** G_U by putting into it all the vertices from ∂U and also the vertex P if P is a boundary vertex of the graph G. In other words, $\partial G_U = \big(\partial G \cap U\big) \cup \big(G \cap \partial U\big)$.

Note that the number of edges of an arbitrary local graph G centered at a vertex v of the graph G equals the degree of this vertex in the graph G.

Let G be a graph with boundary ∂G. Cut the graph G over all its boundary vertices of degree more than 1 and denote the obtained connected components by G_i. For each component G_i we define the boundary ∂G_i as the set of all the vertices from G_i of degree 1 which obtained from the boundary vertices of the initial graph G. Each component G_i with the boundary ∂G_i is called **an nondegenerate component of the graph** G.

Definition. *Let G be an arbitrary topological graph and ∂G be its boundary.* **A parametric network of topology** G *in the topological space X is a continuous mapping Γ from G to X. In this case the topological graph G is called* **a parametrizing graph of the parametric network** Γ *or its* **topology**.

All terminology from graph theory, theory of topological spaces and their mappings can be extended to parametric networks. For example, the restriction of the mapping Γ onto its vertices, edges, boundary, a connected subgraph of the parameterizing graph, local graph, etc. are called the **vertices**, the **edges**, the **boundary**, a **subnetwork**, a **local network**, etc. of the parametric network Γ. Similarly, two parametric networks $\Gamma_i\colon G_i \to X$, $i = 1$, 2, are called **equivalent** if there exists an equivalence of graphs $\varphi\colon G_1 \to G_2$ such that $\Gamma_2 \circ \varphi = \Gamma_1$.

Further, if $M \subset X$ is the image of a boundary mapping $\partial\Gamma$, then we say that the **parametric network Γ spans the set** M (by the mapping $\partial\Gamma$).

Remark. Above we have represented any topological graph as a union of segments factored by an equivalence identifying some ending points of the segments. In the same way each parametric network can be represented as a union of continuous curves in a topological space such that some ending points of the curves are identified.

Now, let $X = W$ be a smooth manifold. A parametric network $\Gamma\colon G \to W$ is called (**piecewise**) **smooth**, if the restriction of the mapping Γ onto the closure of any edge of the graph G is of this type. Notice that the notion of a smooth parametric network is a natural generalization of a smooth curve. Piecewise smooth parametric networks without multiple edges and loops we often call **immersed** for brevity. An immersed parametric network is called **embedded** if the mapping Γ is one-to-one with its image. Note that due to compactness of a graph G, any embedded parametric network gives a homeomorphism with its image. Thus, embedded parametric network is a topological embedding of a connected graph into a manifold such that the restriction of this embedding onto any closed edge is a piecewise regular curve.

In what follows, we shall always assume that equivalent networks have the same smoothness.

Let $\Gamma\colon G \to X$ be an arbitrary parametric network and $I = [a, b]$ be a segment.

Definition. *A continuous mapping $\Psi\colon G \times I \to X$ such that $\Psi(g, a) = \Gamma(g)$ for all $g \in G$ is called* **a deformation of the parametric network** Γ. *If the initial network Γ is (piecewise) smooth (immersed, embedded), then we shall assume that each parametric network $\Psi(\cdot, t) = \Gamma_t$ is of this type and also for the closure $\overline{e}$ of any edge e of the graph G the restriction of the mapping Ψ onto $\overline{e} \times I$ is either smooth or piecewise smooth. The family of velocity vectors of the curves $\Gamma_t(g)$ at $t = 0$ over all point $g \in G$ is called* **the deformation field of** Γ_t.

Below we shall define the notion of trace which enables to model splitting of networks vertices and, hence, the reconstructions of networks topologies under deformations.

Define a new equivalence ρ on the class of all parametric networks in a topological space X as follows. We say that a parametric network Γ_1 can be **projected onto a parametric network** Γ_2 if there exists a projection $\pi: G_1 \to G_2$ such that $\Gamma_2 \circ \pi = \Gamma_1$. The projection π induces a mapping $\pi: \Gamma^1 \to \Gamma^2$ from one of the networks to the other one which will be called **projection**[3]. Two parametric networks $\Gamma_i: G_i \to X$ are called **ρ-adjacent** if one of them can be projected onto the other one. Notice that the relation of ρ-adjacency is reflective and symmetric. However, it is not transitive. We extend this relation upto an equivalence relation as follows. Two parametric networks Γ and Γ' are called **ρ-equivalent** if there exists a finite sequence $\{\Gamma = \Gamma_1, \Gamma_2, \dots, \Gamma_n = \Gamma'\}$ of parametric networks such that any two networks Γ_i and Γ_{i+1} in this sequence are ρ-adjacent. The classes of ρ-equivalence are called **traces** or simply **networks**. If a parametric network Γ belongs to a trace Υ, then we write $\Upsilon = [\Gamma]$.

A network–trace Υ is said to **span** a subset $M \subset X$ if M is the image of the boundary mapping of some and, thus, of any parametric network from Υ.

A canonical representative of a trace Υ is a parametric network $\Gamma \in \Upsilon$ such that any parametric network Γ' from Υ can be projected onto Γ. It is not difficult to show that up to topological equivalence any trace has a unique canonical representative, see [66]. **A local trace** or a **a local network of a trace-network** Υ is any trace of the form $[\Gamma_l]$, where Γ_l is an arbitrary local network of a canonical representative Γ of the trace Υ.

(Local) **deformation of a trace** Υ is any one-parametric family of networks $\Upsilon_t = [\Gamma_t]$, where Γ_t is a deformation of some parametric network Γ from Υ.

Remark. Many properties of networks–traces are completely determined by the corresponding properties of their canonical representatives. Indeed, the notion of trace is useful only when we investigate deformations (namely during a deformation the topology of a network can be changed, i.e., only during a deformation one can get networks–traces with nonequivalent parametric graphs of their canonical representatives). In what follows we will often identify a network–trace with its canonical representative if we speak about "static" properties of the network–trace. In other words,

[3]When we speak about a mapping from one network $\Gamma_1: G_1 \to X$ to another one $\Gamma_2: G_2 \to X$ we mean the corresponding mapping of the sets $\Big\{(g, \Gamma_i(g))\Big\}$.

we shall speak about networks without specifying which kind of networks, parametric ones or traces, we mean.

In particular, the notions of **embedded** and **immersed networks** have sense. This means that the canonical representatives of these networks are of the corresponding type.

The notation Υ for traces we won't use below.

In what follows we shall work mainly with networks in the plane. A finite collection of curves immersed into the plane is called **an immersed planar graph**. If all these immersions are embeddings and the images of any two such embeddings do not intersect each other, then the immersed planar graph is called simply **planar graph**.

Let G be an arbitrary planar tree. Using G we construct an immersed planar graph K_G coinciding with its unique cycle as follows. A path γ in G joining two vertices of degree 1 and such that all the edges from G not belonging to γ but incident to the vertices of γ lie from one side of γ is called **a Steiner path**. Consider all Steiner paths in G. We distinguish the edges and the vertices of different Steiner paths also if they coincide in G. Now, glue all these Steiner paths by their common (in G) beginning and ending vertices. It is easy to see that we obtain an immersed planar graph coinciding with its unique cycle. This graph is called **the contour of the planar tree** G and is denoted by K_G.

1.2. Local minimal and critical networks. The definitions presented in this part can be given for much more general objects. For simplicity reasons **we restrict ourselves with case of piecewise smooth networks in linear spaces with norms**.

Let X be a finite dimensional linear space with some norm $\|\cdot\|$ and $\gamma\colon [a,b] \to X$ be a continuous curve. The curve γ is called **measurable** if there exists a limit $\ell(\gamma)$ of lengths of the polygonal lines inscribed into the curve. Such number $\ell(\gamma)$ is called **the length of the curve** γ. Notice that if the curve γ is piecewise smooth, then it is measurable and

$$\ell(\gamma) = \int_a^b \|\dot\gamma(t)\|\, dt.$$

A network Γ is called **piecewise smooth** if all its edges are piecewise smooth curves.

Let Γ be an arbitrary piecewise smooth network in X. Then **the length $\ell(\Gamma)$ of the network** Γ is the sum of the lengths of its edges. A network Γ spanning a set $M \subset X$ is called **the shortest network** if its length does not exceed the length of any other piecewise smooth network spanning M.

Definition. *A network Γ is called* **local minimal** *if each its point is contained inside a shortest local network (with respect to the canonical boundary).*

Definition. *A network Γ is called* **critical** *or* **extreme** *if for any deformation Γ_t, $t \in [0,1]$, where $\Gamma_{t=0} = \Gamma$, the following condition holds*

$$\frac{d}{dt}\Big|_{t=0+} \ell(\Gamma_t) \geqslant 0.$$

One can show that in the case of the standard Euclidean norm in n-dimensional vector space the sets of local minimal and critical networks coincide. However, it turns out that there are norms for which that is not true. An example of such a norm is Manhattan norm or ℓ_1-norm.

Suppose that some Cartesian coordinates $(x^1, \dots, x^n)$ are fixed in $\mathbb{R}^n$. Recall that the **Manhattan norm** of a vector $y = (y^1, \dots, y^n)$ is the following number:

$$\|y\|_1 = \sum_{i=1}^{n} |y^i|.$$

The corresponding space with such a norm we shall call the **n-dimensional Manhattan space**. Below we shall show that the class of local minimal networks in a Manhattan space is much more wide than the class of critical networks.

1.3. Planar linear trees. In investigation of local minimal and critical networks the notion of linear tree and some important properties of such trees obtained in [62] will be useful. This is due to the following fact.

Assertion 1.1. *Let A and B be arbitrary points of a linear space X with a norm. Then any straight segment $[A, B]$ is the shortest curve joining these points.*

Remark. A shortest curve is not unique in general. For example, for Manhattan space the shortest path joining a pair of points is an arbitrary curve whose coordinate functions are monotonic.

A linear network is an embedded network in $\mathbb{R}^n$ whose edges are straight segments. For any embedded planar linear tree Γ, i.e., for an embedded planar tree whose edges are nondegenerate straight segments, we define its twisting number as follows. Let a and b be arbitrary edges from Γ. Consider the unique oriented path $\gamma(a,b)$ in Γ starting at a and ending at b. The path $\gamma(a,b)$ is an oriented polygonal line in the plane whose consecutive links $a = e_1, \dots, e_n = b$ can be considered as vectors. **The twisting number of the ordered pair** (a,b) **of edges of the tree** Γ is the sum of oriented angles between the consecutive links (e_i, e_{i+1}), $i = 1, \dots, n-1$, of the polygonal line $\gamma(a,b)$, multiplied by $3/\pi$. Recall that the oriented angle between ordered pair (v_1, v_2) of nonopposite vectors equals zero if v_1 is codirected with v_2, and equals the value of the least angle between v_1 and v_2 taken with the sign of the oriented frame (v_1, v_2) in the opposite case.

Definition. The twisting number $\mathrm{tw}\Gamma$ **of an embedded linear tree** Γ *is the maximum of the twisting numbers taken over all ordered pairs of edges from* Γ.

Now, we define the notion of the geometric boundary of an embedded planar linear tree Γ.

Definition. *A vertex P of an embedded planar linear tree Γ is called a* **boundary vertex** *if there exists a straight line ℓ passing through P such that one of the open half-planes bounded by ℓ contains all the edges from Γ incident P. The set of all boundary vertices of the tree Γ is called* **the geometric boundary** *of the tree Γ and is denoted by $\partial_g\Gamma$.*

We recall the definition of the partition of an arbitrary nonempty finite subset M of the plane into convexity levels. Put into **the first convexity level** M^1 **of the set** M all the points from M lying on the boundary of the convex hull of M. Through nonempty set M^1 out of M. If the obtained set is not empty, then we transform it in the same way. Namely, all points from $M \setminus M^1$ lying on the boundary of the convex hull of this set we put into **the second convexity level** M^2 **of the set** M. Continue this process until all points from M will be placed in the convexity levels. The set M^i is called **the** i**-th convexity level of the set** M. The number of convexity levels of a set M is denoted by $\varkappa(M)$.

Proposition 1.1. *Let Γ be an arbitrary embedded planar linear tree and $M = \partial_g\Gamma$ be its geometric boundary. Then*

$$\mathrm{tw}\Gamma \leqslant 12(\varkappa(M) - 1) + 6.$$

This estimation is exact. Namely, for any integer $k \geqslant 1$ there exists a planar linear tree Γ such that $\mathrm{tw}\Gamma = 12(k-1) + 6$ and $\varkappa(\partial_g\Gamma) = k$.

§2. Manhattan local minimal and critical networks

In this section we shall describe the local structure of local minimal and critical networks in a Manhattan space and give a necessary and sufficient condition of that planar Manhattan local minimal network is critical. It turns out that the criticality conditions are nonlocal. For simplicity reasons we restrict ourselves with the case of embedded networks.

2.1. General Properties. A curve γ in Manhattan space $\mathbb{R}^n$ is called **monotonic** if each its coordinate function (in canonical coordinates) is monotonic.

To start with, we describe critical and local minimal curves.

Assertion 2.1. *A curve γ joining a fixed pair of points is critical if and only if it is monotonic. Moreover, any critical curve is a shortest one.*

A curve γ joining a fixed pair of points is local minimal if and only if there exists a finite open covering of its parametric segment such that the restriction of the curve γ onto the closure of any of these covering elements is a monotonic curve.

Now, let us describe local minimal networks.

Let Γ be a network in Manhattan space all of whose edges are local minimal curves and let V be an arbitrary its vertex. By the **monotonic neighbourhood of the vertex** V we call the local network consisting of monotonic segments of the edges incident to V such that each of these segments contains V and is the maximal segment with such properties. A monotonic neighbourhood of the vertex V with V removed is called **the punctured monotonic neighbourhood of** V.

Theorem 2.1. *A network Γ with a boundary is local minimal in Manhattan space if and only if the following conditions hold.*

- *Any edge of the network Γ is a local minimal curve, see Assertion 2.1.*
- *Let V be an arbitrary vertex of the network Γ. Then any open half-space bounded by a hyperplane passing through V and parallel to a coordinate hyperplane contains completely at most 1 edge of the punctured monotonic neighbourhood of the vertex V.*
- *Any vertex of degree 1 of the network Γ is a boundary one.*

Corollary 2.1. *The degree of a vertex of a local minimal network in n-dimensional Manhattan space does not exceed $2n$.*

The following result holds.

Theorem 2.2. *Any critical network in Manhattan space is local minimal. In particular, the local structure of critical networks is described by Theorem 2.1.*

Remark. It is not true that any local minimal Manhattan network is critical. The simplest example can be constructed if one consider any nonmonotonic polygonal line whose edges parallel to coordinate axis.

More delicate effect one can see in the following example. The edges of the network Γ depicted in Figure 2 are shortest curves, however, this network is not critical. In Figure 2 a deformation which linearly decreases the length is shown.

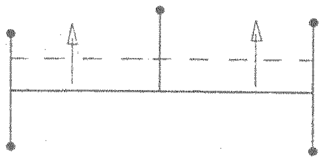

Fig. 2. Local minimal noncritical network in the plane.. fig:lc-n-cr

Now, let us describe some properties of critical networks in Manhattan space $\mathbb{R}^n$ which immediately follow from Assertion 2.1. Let Γ be an arbitrary network. A maximal path in Γ (it can be cyclic) all of whose interior vertices have degree 2 in Γ and which are not boundary vertices of Γ is called **a thread**. Notice that any edge of Γ is a thread by definition.

Corollary 2. *Any thread of a critical network on Manhattan space is a monotonic curve.*

2.2. Critical network and linear networks. Let Γ be an arbitrary embedded network in $\mathbb{R}^n$. Using it, construct a linear network $\overline{\Gamma}$ by changing the threads of Γ with straight segments joining the same vertices. The network $\overline{\Gamma}$ is called **the linearization of the network** Γ.

The following important result is evident.

Assertion 2.2. *A network Γ is a critical one in Manhattan space $\mathbb{R}^n$ if and only if all its threads are monotonic and its linearization is a critical network.*

Corollary 2.3. *Any (immersed) critical network in Manhattan space $\mathbb{R}^n$ can be obtained from some critical linear network without movable vertices of degree 2 by replacing of edges of the latter one with monotonic threads joining the same points.*

Thus, it suffices to describe only linear critical networks without movable vertices of degree 2.

2.3. Critical Manhattan networks in the plane. Now, we describe critical networks in Manhattan plane $\mathbb{R}^2$ with Cartesian coordinates (x, y). Let Γ be a network with monotonic edges. An edge of the network Γ is called **vertical** (**horizontal**) if it is a straight segment parallel to X-axis (Y-axis). All other edges we call **free**. Connected components constructed of horizontal (vertical) edges are called **horizontal** (**vertical**) **fragments**. Maximal fragments are called **sections**. Notice that every fragment is a path joining a pair of vertices of the network Γ. These vertices are called **the ending vertices** of this fragment. A network obtained from a fragment γ by adding all edges from Γ incident to the vertices of γ we denote by $T(\gamma)$ and call **the extension of the fragment** γ.

Let γ be a horizontal fragment. A nonhorizontal edge e incident to a vertex v from γ is called **onesided** if it is the unique nonhorizontal edge incident to v. A onesided edge incident to an interior vertex v of a fragment γ is called **static** if v is a boundary vertex of the tree Γ. A onesided edge incident to an ending vertex of a fragment γ is also called **static**. All remained onesided edges we call **floating**.

Further, a vertex v of a fragment γ such that v is not incident to a onesided edge is called **static** if in the tree Γ it is a boundary vertex of degree 1 or 2. Note that a vertex of γ is static if and only if it is not incident to any nonhorizontal edge of the network Γ and belongs to the boundary of Γ.

Partition the set of all nonhorizontal edges incident to a fragment γ into two classes C_1 and C_2 depending on which half-plane bounded by a straight line passing through γ these edges lie in. By t_i^s and t_i^n we denote the numbers of static and floating edges in the class C_i, $i = 1, 2$. The number $\operatorname{ind}_i = 2t_i^n + t_i^s$ is called **the index of the class** C_i and the number $\operatorname{ind}(\gamma) = |\operatorname{ind}_1 - \operatorname{ind}_2|$ is called **the index of the fragment** γ. To be definite, we assume that the class C_1 consists of edges placed in upper half-plane with respect to γ.

Further, by e^s we denote the total number of onesided edges, i.e., $e^s = t_1^s + t_2^s$ and by v^s the total number of static vertices of the fragment

γ. The number $\operatorname{stat}(\gamma) = 2v^s + e^s$ is called **the static degree of the fragment** γ.

The corresponding notions for a vertical fragment can be defined similarly.

Theorem 2.3. *A network Γ in Manhattan plane $\mathbb{R}^2$ is critical if and only if all its threads are monotonic curves, the linearization Γ_l of Γ is local minimal, and the index of any fragment γ of Γ_l does not exceed the static degree of this fragment:*

$$\operatorname{ind}(\gamma) \leqslant \operatorname{stat}(\gamma).$$

Proof. By Assertion 2.2, we can assume that the network Γ is linear local minimal one without movable vertices of degree 2. In particular, the network Γ coincides with its linearization Γ_l. Besides that, it is sufficient to consider only those deformations of the network Γ which preserves the edges monotonicity. This implies that it suffices to consider only deformations in the class of linear networks.

To calculate the first variation of a straight segment length with respect to Manhattan norm the following function will be useful

$$f(x, y) = \begin{cases} y & \text{for } x > 0, \\ -y & \text{for } x < 0, \\ |y| & \text{for } x = 0. \end{cases}$$

Define three vector-valued functions on $\mathbb{R}^n$ as follows

$$\begin{aligned} \operatorname{sign}(X) &= \bigl(\operatorname{sign}(X_1), \dots, \operatorname{sign}(X_n)\bigr), \\ \operatorname{null}(X) &= \bigl(1 - \operatorname{sign}^2(X_1), \dots, 1 - \operatorname{sign}^2(X_n)\bigr), \\ \operatorname{mod}(X) &= \bigl(|X_1|, \dots, |X_n|\bigr). \end{aligned}$$

where $X = (X_1, \dots, X_n)$. Note that a component of the vector $\operatorname{null}(X)$ equals zero if and only if the corresponding component of the vector X does not vanish. In the opposite case this component of the vector $\operatorname{null}(X)$ equals 1. Notice also that the functions $\operatorname{null}(X)$ and $\operatorname{mod}(X)$ are even, but the function $\operatorname{sign}(X)$ is odd.

In what follows it will be more convenient to consider the functions $\operatorname{null}(X)$ and $\operatorname{mod}(X)$ as the ones defined on nonordered pairs of vectors, namely

$$\operatorname{null}\bigl(\{A, B\}\bigr) = \operatorname{null}(A - B), \qquad \operatorname{mod}\bigl(\{A, B\}\bigr) = \operatorname{mod}(A - B).$$

Lemma 2.1 (The first variation of segment length). *Let $[A,B]$ be an arbitrary segment in Manhattan space $\mathbb{R}^n$ (this segment can be degenerated). Consider some its linear deformation such that the point $A=(A_1,\dots,A_n)$ moves along the curve $A(t)=A+t\xi+o(t)$ and the point $B=(B_1,\dots,B_n)$ moves along the curve $B(t)=B+t\eta+o(t)$, where $t\in[0,1]$, and $o(t)$ stands for an infinitesimal of order t, $t\to 0$. Then the function $\ell(t)$ of the Manhattan length of $[A(t),B(t)]$ is differentiable at $t=0$ and*

$$\frac{d}{dt}\Big|_{t=0}\ell(t)=\sum_i f(B_i-A_i,\eta_i-\xi_i),$$

where $\xi=(\xi_1,\dots,\xi_n)$ and $\eta=(\eta_1,\dots,\eta_n)$. In other words,

$$\frac{d}{dt}\Big|_{t=0}\ell(t)=\langle \mathrm{sign}(A-B),\xi\rangle+\langle \mathrm{sign}(B-A),\eta\rangle+\big\langle \mathrm{null}(\{A,B\}),\mathrm{mod}(\{\xi,\eta\})\big\rangle.$$

Lemma 2.1 gives us an opportunity to write down the following formula of the first variation for a linear network.

Lemma 2.2 (The first variation of network length). *Let Γ be a linear network in $\mathbb{R}^n$ and $\Gamma_t=[\Phi_t]$, $t\in[0,1]$, be its local deformation in the class of linear networks, where Φ_t is the corresponding deformation of some parametric network $\Phi=\Phi_0$. By ξ we denote the deformation field of Φ_t. Let V be an arbitrary movable vertex of the network Φ. By S_V we denote the sum of vectors $\mathrm{sign}(V-V')$ over all nondegenerate edges VV' of the network Φ. Then*

$$\frac{d}{dt}\Big|_{t=0}\ell(\Gamma_t)=\frac{d}{dt}\Big|_{t=0}\ell(\Phi_t)=$$
$$=\sum_{V\ \text{is movable}}\langle S_V,\xi_V\rangle+\sum_{e\ \text{is an edge}}\big\langle \mathrm{null}(\partial e),\mathrm{mod}(\xi|_{\partial e})\big\rangle.$$

The first term from the formula of Lemma 2.2 is called **the linear part of the first variation formula** and the second term is called **the nonlinear part**.

Return to the proof of Theorem 2.3. Suppose first that Γ is a critical network and γ is its arbitrary fragment. Consider a linear deformation Γ_t of the network Γ which moves the fragment γ in a direction perpendicular to γ in such a way that γ remains parallel to its initial position and each vertex which moves under the deformation Γ_t does that with unit speed. To be definite, we assume that γ is a horizontal fragment which moves

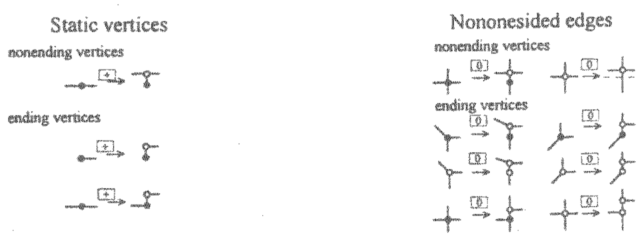

Fig. 3. Splitting the vertices under deformation Γ_t. White vertices are non-boundary ones; black vertices are boundary ones; "-", "+", or "0" means negative, positive, or zero contribution in that case. fig:vert49mm

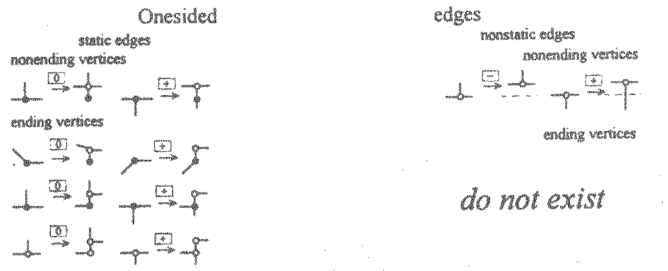

Fig. 4. Splitting the vertices under deformation Γ_t. White vertices are non-boundary ones; black vertices are boundary ones; "-", "+", or "0" means negative, positive, or zero contribution in that case. fig:vert160mm

upward. In the neighbourhoods of vertices we define the deformation Γ_t as shown in Figures 3 and 4.

As above, partition nonhorizontal edges into two classes C_i by putting into the class C_1 the edges from upper half-plane. Recall that by t_i^s and t_i^n we have denoted the numbers of static and, respectively, floating onesided edges from the ith class, by v^s the total number of static vertices in γ, and by e^s the sum $t_1^s + t_2^s$. The first variation formula, see Lemma 2.2, and the fact that the network Γ is critical imply

$$\frac{d}{dt}\Big|_{t=0} \ell(\Gamma_t) = t_2^n - t_1^n + v^s + t_2^s \geqslant 0.$$

In the same way, the deformation moving γ downward gives the following inequality:

$$t_1^n - t_2^n + v^s + t_1^s \geqslant 0.$$

Thus,

$$-v^s - t_1^s \leqslant t_1^n - t_2^n \leqslant v^s + t_2^s.$$

Adding $(t_1^s - t_2^s)/2$ to the latter inequality, we obtain

$$-v^s - e^s/2 \leqslant \frac{2t_1^n + t_1^s - 2t_2^n - t_2^s}{2} \leqslant v^s + e^s/2.$$

It remains to notice that $2t_i^n + t_i^s = \mathrm{ind}_i$ by definition, thus, since $\mathrm{stat}(\gamma) = 2v^s + e^s$, we obtain the inequality sought for.

Note that our calculations leads to the following result.

Lemma 2.3. *The inequality* $\mathrm{ind}(\gamma) \leqslant \mathrm{stat}(\gamma)$ *is equivalent to the following system of inequalities:*

$$\begin{cases} t_1^n \leqslant t_2^n + v^s + t_2^s \\ t_2^n \leqslant t_1^n + v^s + t_1^s. \end{cases}$$

Now, prove the converse statement of the theorem. Let Γ be a local minimal network without movable vertices of degree 2 such that for any its fragment γ' the inequality $\mathrm{ind}(\gamma') \leqslant \mathrm{stat}(\gamma')$ holds. Consider an arbitrary deformation $\Gamma_t = [\Phi_t]$ of the network–trace Γ through the class of linear networks remaining fixed the boundary of the network. This means that we fix a parameterization $\Phi = \Phi_0$ of the trace Γ (recall that such a parameterization can contain some degenerate edges). Show that the first variation of this deformation is nonnegative.

Without loss of generality we assume that the deformation Γ_t possesses the following properties.

- Any vertex of the network moves uniformly along a straight line.
- Any vertex is splitted on at most a tree and any such tree contains at most one fixed vertex with respect to the deformation Φ_t. In particular, the networks Γ and Φ have the same numbers of boundary vertices.
- The network Φ (and, thus, all the networks Φ_t) does not have movable vertices of degree 1 and 2 (notice that the deformation depicted in Figures 2 and 3 does not satisfy the mentioned property, however, it can be evidently corrected without changing the lengths of the networks in consideration in such a way that all threads become straight segments).

We transfer all terminology introduced above to the parametric network Φ. Let $\pi\colon \Phi \to \Gamma$ be a projection of the parametric network Φ onto the canonical representative Γ. An edge of Φ such that its image under the projection π is a horizontal (vertical, free) edge is called **horizontal** (respectively, **vertical**, **free**). Notice that horizontal, vertical and free edges of the network Φ are all its nondegenerate edges.

Let γ' be an arbitrary fragment of the network Γ. **The fragment γ of the parametric network Φ** corresponding to the fragment γ' is the preimage of the path γ' under the projection π. **A section of the network Φ** is any its fragment corresponding to a section.

Let γ be a fragment of the network Φ corresponding to a fragment γ'. **The extension $T(\gamma)$ of the fragment** γ is a subnetwork in Φ obtained from γ by adding all the edges of Φ incident to it. Notice that all added edges are nondegenerate. Recall that the nonhorizontal edges of the network Γ corresponding to the edges added are partitioned into the classes C_1 and C_2. This partition induces a partition of the corresponding edges of the network Φ into two classes which will be denoted by C_1 and C_2 as well.

Write down the first variation formula for the deformation Φ_t. Lemma 2.2 implies that the obtained expression can be represented as the sum of two expressions $\sum^v$ and $\sum^h$ such that the first one depends only on vertical components of the deformation field, but the second one depends only on horizontal components. We shall show that each of these terms is nonnegative.

Consider the expression $\sum^v$. Lemma 2.2 shows that this expression consists of two parts: linear $\sum^v_l$ and nonlinear $\sum^v_n$. And also, the part $\sum^v_n$ is a sum of absolute values, thus, it is nonnegative. We shall show that the linear part $\sum^v_l$ can be compensated by the nonlinear one $\sum^v_n$.

Lemma 2.4. *Nonzero contribution into $\sum^v_l$ is given only by deformation vectors of those vertices of the network Φ which lie in horizontal sections.*

Proof. Note first that the nonzero contribution into linear part $\sum^v_l$ is given by the deformation vectors of only those vertices of the network Φ which move during the deformation and to which vertical or free edges are incident. Moreover, if two such edges are incident to such a vertex, then the contribution of the vertex vanishes. Thus, the deformation vector at a vertex of the network Φ gives a nonzero contribution into $\sum^v_l$ only if this vertex moves and among the edges incident to it there exists just one nondegenerate nonhorizontal edge.

Let V be such a vertex and $V' = \pi(V)$ be its projection into Γ. If V' is incident to a horizontal edge (this is certainly true if the degree of the vertex V' is more than 2), then the vertex V, evidently, belongs to a horizontal section. Now suppose that the degree of the vertex V' does not exceed 2. If this degree equals 1, then our assumptions on the deformation Φ_t of the trace Γ imply that the both vertices V and V' are boundary ones, this contradicts to the choice of V. If this degree equals 2, then the vertex V' is a boundary one and the vertex V has degree 3 and it is incident to two nondegenerate edges. By the choice of the vertex V, one of these edges is horizontal, q.e.d.

Thus, $\sum_l^v$ can be decomposed into the sum of terms $\sum_l^v(\gamma)$ each of which corresponds to vertices of some horizontal section γ of the network Φ. We shall show that every such term can be compensated independently on the other ones.

Let γ be a horizontal section of the network Φ and $T(\gamma)$ be its extension. By definition of a section, all horizontal edges of the tree $T(\gamma)$ belong to γ. Besides, all degenerated edges from $T(\gamma)$ lie, evidently, in the preimage of the vertex set of the corresponding section γ' of the network Γ. Therefore, the extensions of any two different horizontal sections of the network Φ intersect each other neither by horizontal, nor by degenerate edges. On the other hand, Lemma 2.2 implies that the value $\sum_n^v$ equals to the sum of terms over all horizontal and degenerate edges, and each of these terms depends only on the value of the deformation field on the boundary of the corresponding edge. Thus, $\sum_n^v$ is more or equal the sum of terms $\sum_n^v(\gamma)$ over all horizontal sections, where $\sum_n^v(\gamma)$ denotes the contribution of the edges from $T(\gamma)$ into nonlinear part. Our aim is to prove that

$$\Sigma_l^v(\gamma) + \Sigma_n^v(\gamma) \geqslant 0. \qquad (*)$$

Denote the left hand side of the latter inequality by $\sum^v(\gamma)$.

Prove the inequality $(*)$. Notice first that, by definition, the expression $\sum^v(\gamma)$ depends only on vertical components of the vectors of the deformation field. Thus, to prove the inequality $(*)$ we may assume that the deformation field is vertical.

For what follows we need to correct the network Γ and the deformation Φ_t preserving the inequalities from the condition of the theorem and the both linear and nonlinear parts $\sum_l^v$ and $\sum_n^v$ (the same can be done for horizontal components).

Lemma 2.5. *Under the above assumptions on the deformation Φ_t, sup-*

pose that one of the following possibilities takes place.

1. *The deformation field vanishes at a nonboundary vertex V of the network Φ.*[4]
2. *The deformation field vanishes at an interior point V of an edge e of the network Φ.*

Reconstruct the networks Φ and Γ by adding the point V and, respectively, the point $\pi(V)$ to the boundary sets of these networks. The obtained networks we denote by $\overline{\Phi}$ and $\overline{\Gamma}$. Then the network $\overline{\Phi}$ can be projected onto the network $\overline{\Gamma}$, the network $\overline{\Gamma}$ is a canonical representative of the corresponding trace, and $\overline{\Gamma}$ satisfies the conditions of the theorem. The deformation Φ_t generates a deformation $\overline{\Phi}_t$ of the parametric network $\overline{\Phi}$ and defines a local deformation of the trace $\overline{\Gamma}$. The linear and nonlinear parts of the vertical and horizontal components of the first variation formulas for the deformations Φ_t and $\overline{\Phi}_t$ coincide.

Lemma 2.5 gives an opportunity to assume without loss of generality that the deformation Φ_t remains fixed only the boundary vertices of the network Φ and all nonzero deformation vectors are codirected with Y-axis, i.e., they directed upwards.

Let γ' be a section of the network Γ. Cut γ' over all its noninterior vertices incident to those edges from C_1 which are not floating. The closures of the obtained components we denote by γ'_i.

Consider an arbitrary component γ'_i. Partition the vertices from γ'_i onto two classes U_1 and U_2 by putting into the first one all the vertices incident to the edges from C_1. Orient γ'_i in one of two possible ways that gives an order on the set of vertices from γ'_i.

Lemma 2.6. *The class U_1 cannot contain more than two consecutive vertices.*

Proof. Suppose otherwise. Consider the smallest fragment δ containing three consecutive vertices from U_1. For this fragment we have:

$$t_1^n = 1, \qquad t_1^s \leqslant 2, \qquad t_2^n = t_2^s = v^s = 0, \qquad e^s \leqslant 2,$$

thus,

$$\operatorname{ind}(\delta) = 2(t_1^n - t_2^n) + t_1^s - t_2^s = 2 + t_1^s,$$

and

$$\operatorname{stat}(\delta) = 2v^s + e^s = t_1^s,$$

which contradicts to assertions of the theorem. Lemma is proved.

Lemma 2.7. *Between each two pairs H_1 and H_2 of adjacent vertices from U_1 there is a pair of adjacent vertices from U_2.*

Proof. Without loss of generality we assume that between H_1 and H_2 there is no any other pairs of adjacent vertices from U_1. Suppose otherwise, i.e., between H_1 and H_2 there is no a pair of adjacent vertices from U_2. Consider the smallest fragment δ containing H_1 and H_2. Denote by u_i the number of vertices in $U_i \cap \delta$. Lemma 2.6 implies that $u_2 = u_1 - 3$. On the other hand,

$$u_2 = v^s + t_2^n + t_2^s - t_1^s + 2, \qquad u_1 = t_1^n + 2,$$

thus, $t_1^n = v^s + t_2^n + t_2^s - t_1^s + 3$. By Lemma 2.3 the following inequality holds: $t_1^n \leqslant t_2^n + v^s + t_2^s$, therefore, $t_1^s \geqslant 3$. But $t_1^s \leqslant 2$ because static edges from C_1 can not be attached to the ending vertices of the fragment γ_i'. This contradiction completes the proof.

Lemmas 2.6 and 2.7 imply the following result.

Lemma 2.8. *Each nonending vertex x from the class U_1 can be paired with a vertex from U_2 adjacent with x in such a way that the obtained pairs do not intersect each other.*

Proof. Suppose first that there exists at least one pair $H = \{h_1, h_2\}$ of adjacent vertices from the class U_1. If the both vertices are ending ones, then the class U_1 consists just of two ending vertices and lemma is trivial. Let one of the vertices in consideration, say, h_2 is not an ending one. By $b_1 \neq h_1$ we denote a vertex from g_i' adjacent with h_2. By Lemma 2.6, we have $b_1 \in U_2$. So, we take $\{h_2, b_1\}$ as the first pair.

Now, let us move from the pair H in the direction toward b_1 and continue constructing the partition into pair upto coming to an ending vertex of γ_i'. If the both vertices from H are nonending ones, then we move from H into the both possible directions. For each next nonending vertex h from U_1 we proceed as follows.

If the vertex x preceding to h belongs to U_2 but it is not contained in a pair still , then we take the pair $\{x, h\}$ as the next one and continue the procedure. Otherwise, by Lemmas 2.6 and 2.7, the vertex y consequent to h belongs to U_2 and we take the pair $\{h, y\}$ as the next one and continue the procedure.

Let us show that the described procedure is correct and we always come to an ending vertex of γ_i' as a result. The fragment γ_i' is partitioned by the pairs of adjacent vertices from U_1 into several smaller fragments

γ'_{ij}. Each of these fragments is either bounded from the both its sides by pairs of adjacent vertices from U_1, or some pair of this type is placed at one end of the fragment γ'_{ij} and the other end of the fragment coincides with an ending vertex of γ'_i.

Suppose first that γ'_{ij} is bounded by two pairs H_1 and H_2 of adjacent vertices from U_1. By Lemma 2.7, the fragment γ'_{ij} contains a pair of adjacent vertices from U_2 and all the vertices from U_1 lying inside γ'_{ij} are alternated with the vertices from U_2. Thus, it is easy to see that our procedure starts from the first pair H_i, constructs the desired pairs for all the vertices from U_1 lying inside γ'_{ij}, and after that it comes to the last vertex from γ'_{ij}. If the fragment γ'_{ij} is bounded by exactly one pair H_1 of adjacent vertices from U_1, then, by definition, during the motion along γ'_{ij} from this pair we meet a vertex from U_2 after each nonending vertex from U_1. Thus, our procedure starts with H_1, constructs all desired pairs for all the vertices from U_1 lying inside γ'_{ij}, and after that comes to the ending vertex from γ'_{ij}.

Now consider the remained possibility: suppose that the fragment γ'_i has no adjacent vertices from U_1. Then each vertex from U_1 lying inside γ'_i has an adjacent vertex from U_2 at its both sides. Now the desired pairs can be constructed obviously. Lemma is proved.

By $U_2(\gamma')$ we denote the union of the sets U_2 over all fragments γ'_i. Lemma 2.8 implies the following result.

Lemma 2.9. *Each floating vertex from the section γ' can be paired with an adjacent vertex from U_2 in such a way that the obtained pairs do not intersect each other.*

Proof. The desired set of pairs can be obtained as the union of all the pairs constructed for the fragments γ'_i with the help of Lemma 2.8. Since the pairs corresponding to one fragment γ'_i do not intersect each other by construction, it remains to check that the pairs corresponding to different adjacent fragments do not intersect each other too. Really, any two adjacent fragments γ'_i and γ'_j intersect each other by a vertex which belongs to U_1 with respect to the both fragments, thus, since this vertex is an ending one for these fragments, it does not belong to any pair. Lemma is proved.

Note that the edges from the class C_1 are not incident with the vertices from $U_2(\gamma')$.

Let us return to investigation of the deformation Φ_t. Recall that the both floating vertices and the other nonboundary vertices incident to the edges from C_1 give the positive contribution into the linear vertical part $\sum_l^v(\gamma)$ of the first variation. To get the desired estimation for $\sum^v(\gamma)$ we include every such a vertex into some path Ψ in the network Φ.

Consider first an arbitrary pair $\{h,x\}$ from Lemma 2.9, where h is a floating vertex and x is vertex from U_2. By e_h we denote the unique edge from C_1 incident to h.

Using the pair $\{h,x\}$ we construct a path Ψ in the network Φ as follows. There are two possibilities: either the vertex x is a boundary one of the network Γ, or it is movable vertex of degree 3 of this network. In the first case we define Ψ to be the unique path in the extension $T(\gamma)$ of the section γ of the network Φ joining the unique boundary vertex X of the network Φ projecting into x with the unique nondegenerate edge E_h of the network Φ projecting into e_h. In the second case, by e_x we denote the unique edge from C_2 incident to x and define Ψ as the unique path in the extension $T(\gamma)$ of the section γ of the network Φ joining the edge E_h and the unique nondegenerate edge E_x of the network Φ projecting into e_x.

Let E be a floating edge from C_1 incident with the section γ. Using E we construct a path Ψ in the network Φ as follows. By e we denote the image of projection into Γ of the edge E and let x be a vertex of the section γ' of the network Γ incident to e. There are two possibilities: either the vertex x is a boundary one of the network Γ, or it is a movable vertex of this network. In the first case we define the path Ψ to be the unique path in the extension $T(\gamma)$ of the section γ of the network Φ joining the edge E with the unique boundary vertex X of the network Φ projecting into x. In the second case, by e_x we denote the unique edge from C_2 incident to x and define Ψ as the unique path in the extension $T(\gamma)$ of the section γ of the network Φ joining the edge E and the unique nondegenerate edge E_x of the network Φ projecting into e_x.

By construction, the following Lemma holds.

Lemma 2.10. *Let γ be an arbitrary section of the network Φ. The paths Ψ constructed above for the section γ do not intersect each other and contain all the vertices giving a negative contribution into $\sum_l^v(\gamma)$.*

Let $V_1,\dots,V_k$ be consecutive nonending vertices of the path Ψ enumerated from the unique edge E_1 of Ψ belonging to C_1. Notice that each vertex V_j is not a boundary one for the network Φ and all the edges joining the consecutive vertices V_j are degenerate. By ξ_j we denote the projection of the deformation field vector at V_j onto Y-axis. There are two possibilities: either the ending edge $E_2 \neq E_1$ of the path Ψ is degenerated, or not. In the latter case we have $E_2 \in C_2$ by construction.

In the first case the total contribution of the vectors ξ_j into $\sum^v(\gamma)$ has the form

$$-\xi_1 + |\xi_1 - \xi_2| + \cdots + |\xi_{k-1} - \xi_k| + |\xi_k| \geqslant -\xi_1 + |\xi_1| \geqslant 0.$$

In the second case the total contribution of the vectors ξ_j into $\sum^v(\gamma)$ has the form

$$-\xi_1 + |\xi_1 - \xi_2| + \cdots + |\xi_{k-1} - \xi_k| + \xi_k \geqslant -\xi_1 + \xi_k + |\xi_1 - \xi_k| \geqslant 0.$$

Thus, the total contribution of the vectors ξ_j into $\sum^v(\gamma)$ is nonnegative always. Now Lemma 2.10 implies that $\sum^v(\gamma)$ is nonnegative. Theorem 2.3 is completely proved.

Corollary 2.4. *Let Γ be an embedded nondegenerate local minimal Manhattan network, i.e., Γ coincides with its nondegenerate component. Then for any fragment γ the following conditions hold.*

- *The numbers t_i^s and v^s do not exceed 2 and they are completely determined by types of the ending vertices of the fragment.*
- *Moreover, $v^s + t_i^s \leqslant 2$ for $i = 1, 2$.*

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