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THE REIDEMEISTER NUMBER OF ANY AUTOMORPHISM OF A GROMOV HYPERBOLIC GROUP IS INFINITE

ABSTRACT. It is shown that the number of twisted conjugacy classes is infinite for any automorphism of a nonelementary, Gromov hyperbolic group. An analog of the Selberg theory for twisted conjugacy classes is suggested.

To Aleksandr Aleksandrovich Ivanov
on his 80th birthday

§0. INTRODUCTION

Let G be a finitely generated group and $\phi : G \to G$ an endomorphism. Two elements $\alpha, \alpha' \in G$ are *ϕ-conjugate* if there exists $\gamma \in G$ such that

$$\alpha' = \gamma\alpha\phi(\gamma)^{-1}.$$

We write $\{x\}_\phi$ for the ϕ-conjugacy class of the element $x \in G$. The number of ϕ-conjugacy classes is the *Reidemeister number* of an endomorphism ϕ, denoted by $R(\phi)$. If ϕ is the identity map, then the ϕ-conjugacy classes are the usual conjugacy classes in G.

We note that $R(\phi)$ is infinite if G is free Abelian and the action of ϕ on G has 1 as an eigenvalue [6].

In [5], we conjectured that the Reidemeister number is infinite as long as the endomorphism ϕ is injective and G has exponential growth.

In this paper, we prove our conjecture for any automorphism of any nonelementary (i.e., not virtually cyclic), Gromov hyperbolic group. We also prove some generalizations of this result.

The main result of the paper has a topological counterpart. Let X be a connected, compact polyhedron and $f : X \to X$ a continuous map. Let $p : \tilde{X} \to X$ be a universal covering of X and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lifting

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of f, i.e., $p \circ \tilde{f} = f \circ p$. Two liftings $\tilde{f}$ and $\tilde{f}'$ are *conjugate* if there is a element γ in the deck transformation group $\Gamma \cong \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. The subset $p(\mathrm{Fix}(\tilde{f})) \subset \mathrm{Fix}(f)$ is the *fixed-point class of f determined* by the lifting class $[\tilde{f}]$. Two fixed points x_0 and x_1 of f belong to the same fixed point class iff there is a path c from x_0 to x_1 such that $c \cong f \circ c$ (homotopy relative endpoints). This fact can be considered as an equivalent definition of a nonempty fixed point class. Every map f has only finitely many nonempty fixed point classes, each a compact subset of X.

A fixed point class is *essential* if its index is nonzero. The number of lifting classes of f (= the number of fixed point classes, empty or not) is the *Reidemeister number* of f, denoted by $R(f)$. This is a positive integer or infinity. The number $N(f)$ of essential fixed-point classes is the *Nielsen number* of f.The Nielsen number is always finite.

Theorem 3 implies that the topological Reidemeister number $R(f)$ is infinite for a homeomorphism f of a compact polyhedron X with nonelementary, Gromov hyperbolic fundamental group $\pi_1(X, x_0)$.

In the case of pseudo-Anosov homeomorphisms of surfaces, we can develop an analog of the Selberg theory, and we obtain an asymptotic expansion for the number of twisted conjugacy classes or for the number of Nielsen fixed-point classes whose norm is at most x.

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§1. Twisted conjugacy classes and the Reidemeister number of a group endomorphism

Lemma 1. *If G is a group and ϕ is an endomorphism of G, then any element $x \in G$ is always ϕ-conjugate to its image $\phi(x)$.*

Proof. If $\gamma = x^{-1}$, then one immediately has $\gamma x = \phi(x)\phi(\gamma)$. The existence of a γ satisfying this equation implies that x and $\phi(x)$ are ϕ-conjugate. □

The *mapping torus* $M(\phi)$ of a group endomorphism $\phi : G \to G$ is obtained from G by adding a new generator z and the relations $z^{-1}gz = \phi(g)$, $g \in G$. It means that $M(\phi)$ is a semi-direct product of G and $\mathbb{Z}$.

Lemma 2. *Two elements x, y of G are ϕ-conjugate if and only if xz and yz are conjugate in the usual sense in $M(\phi)$. Therefore, $R(\phi)$ is the number of the usual conjugacy classes in the coset $G \cdot z$ of G in $M(\phi)$.*

Proof. If x and y are ϕ-conjugate, then there is a $\gamma \in G$ such that $\gamma x = y\phi(\gamma)$. This implies $\gamma x = yz\gamma z^{-1}$, whence $\gamma(xz) = (yz)\gamma$. So, xz and yz are conjugate in the usual sense in $M(\phi)$. Conversely, suppose xz and yz are conjugate in $M(\phi)$. Then there is a $\gamma z^n \in M(\phi)$ with $\gamma z^n xz = yz\gamma z^n$. From the relation $zxz^{-1} = \phi(x)$, we obtain $\gamma\phi^n(x)z^{n+1} = y\phi(\gamma)z^{n+1}$, whence $\gamma\phi^n(x) = y\phi(\gamma)$. This shows that $\phi^n(x)$ and y are ϕ-conjugate. However, by Lemma 1, x and $\phi^n(x)$ are ϕ-conjugate, so x and y must be ϕ-conjugate. $\square$

Lemma 3 (T. Delzant)**.** *Let J be a nonelementary, Gromov hyperbolic group. Let K be a normal subgroup with Abelian quotient. Then every coset C of J* mod *K contains infinitely many conjugacy classes.*

Proof (cf. [8])**.** (1) Fix $u \in C$. Suppose for a moment that we can find $c, d \in K$ generating a free group of rank 2, such that $uc^{\infty} \neq c^{-\infty}$ and $ud^{\infty} \neq d^{-\infty}$ (recall that we denote $g^{-\infty} = \lim_{n \to +\infty} g^{-n}$ for g of infinite order). Consider $x_k = c^k uc^k$ and $y_k = d^k ud^k$. For k large, the above inequalities imply that these two elements have infinite order, and do not generate a virtually cyclic group because ${x_k}^{+\infty}$ and ${x_k}^{-\infty}$(respectively ${y_k}^{+\infty}$ and ${y_k}^{-\infty}$) are close to $c^{+\infty}$ and $c^{-\infty}$ (respectively, $d^{+\infty}$ and $d^{-\infty}$). Fix k, and consider the elements $z_n = {x_k}^{n+1}{y_k}^{-n}$. They belong to the coset C, because J/K is Abelian, and their stable norm goes to infinity with n. Therefore, C contains infinitely many conjugacy classes.

(2) Now let us construct c, d as above. Choose $a, b \in K$ generating a free group of rank 2. First we explain how to get c. There is a problem only if $ua^{\infty} = a^{-\infty}$ and $ub^{\infty} = b^{-\infty}$. In that case, there exist integers p, q with $ua^p u^{-1} = a^{-p}$ and $ub^q u^{-1} = b^{-q}$. We take $c = a^p b^q$, noting that $ucu^{-1} = a^{-p}b^{-q} \neq c^{-1} = b^{-q}a^{-p}$.

Once we have c, we choose $c^* \in K$ with $\langle c, c^* \rangle$ free of rank 2, and we obtain d by applying the preceding argument using c^* and cc^* instead of a and b. The group $\langle c, d \rangle$ is free of rank 2 because d is a positive word in c^* and cc^*. $\square$

1.1. Automorphisms of Gromov hyperbolic groups. Now let ϕ be an automorphism of the Gromov hyperbolic group G and let $\|\cdot\|$ denote the word metric with respect to some finite generating set for G. The automorphism ϕ is *hyperbolic* if there is an integer m and a number

$\lambda > 1$ such that for all $g \in G$ we have

$$\max(\|\phi^m(g)\|, \|\phi^{-m}(g)\|) \geq \lambda \|g\|.$$

For example, a pseudo-Anosov homeomorphism of a closed surface of genus larger than one induces a hyperbolic automorphism on the level of the fundamental group. Also, an automorphism of a finitely generated free group with no nontrivial periodic conjugacy classes is hyperbolic.

Lemma 4 [2]**.** *The mapping torus $M(\phi)$ of a hyperbolic automorphism ϕ is a Gromov hyperbolic group.*

Theorem 1. *The Reidemeister number $R(\phi)$ is infinite if G is Gromov hyperbolic and nonelementary and ϕ is a hyperbolic automorphism.*

Proof. The proof immediately folows from Lemmas 2–4. □

Let us consider an outer automorphism $\Phi \in \mathrm{Out}\, G$ corresponding to automorphism of $\phi \in \mathrm{Aut}\, G$ and viewed as a collection of ordinary automorphisms $\alpha \in \mathrm{Aut}\, G$. We define $\alpha, \beta \in \Phi$ to be *isogredient* if $\beta = i_h \cdot \alpha \cdot {i_h}^{-1}$ for some $h \in G$, where $i_h(g) = hgh^{-1}$.

Lemma 5 [8]**.** *The set $S(\Phi)$ of isogredience classes is infinite if group G is Gromov hyperbolic, nonelementary, and Φ has finite order in the group* $\mathrm{Out}\, G$.

Proof. Let J be the subgroup of $\mathrm{Aut}\, G$ consisting of all automorphisms whose image in $\mathrm{Out}\, G$ is a power of Φ. The exact sequence

$$1 \to K \to J \to \langle \Phi \rangle \to 1,$$

with $K = G/\mathrm{Center}$ and $\langle \Phi \rangle$ finite, shows that J is hyperbolic, nonelementary. The set of automorphisms $\alpha \in \Phi$ is a coset of J mod K. If $\alpha, \beta \in \Phi$ are isogredient, they are conjugate in J. The proof is therefore concluded by applying Lemma 3. □

Lemma 6 [3]**.** *The set $S(\Phi)$ of isogredience classes is infinite if G is a free group F_n and $\Phi \in \mathrm{Out}\,(F_n)$ fixes a nontrivial conjugacy class.*

Theorem 2 [8]**.** *For any $\Phi \in \mathrm{Out}\, G$, with G Gromov hyperbolic and nonelementary, the set $S(\Phi)$ of isogredience classes is infinite.*

Proof. We describe main steps of the proof. By Lemma 5, we may assume that Φ has infinite order. By Paulin's theorem [11], Φ preserves some R-tree T with nontrivial minimal small action of G (recall that an action of G is *small* if all arc stabilisers are virtually cyclic; the action of

G on T is always irreducible (no global fixed points, no invariant lines, no invariant ends)). This means that there is an R-tree T equipped with an isometric action of G whose length function satisfies $l \cdot \Phi = \lambda l$ for some $\lambda \geq 1$.

Step 1. Suppose $\lambda = 1$. Then $S(\Phi)$ is infinite.

Step 2. Suppose $\lambda > 1$. Assume that the arc stabilisers are finite, and there exists $N_0 \in N$ such that, for every $Q \in T$, the action of Stab Q on $\pi_0(T - Q)$ has at most N_0 orbits. Then $S(\Phi)$ is infinite.

Step 3. If $\lambda > 1$, then T has finite arc stabilisers. If $\lambda > 1$, then from work of Bestwina–Feighn [2] it follows that there exists $N_0 \in N$ such that, for every $Q \in T$, the action of Stab Q on $\pi_0(T - (Q))$ has at most N_0 orbits. $\Box$

Theorem 3. *The Reidemeister number $R(\phi)$ is infinite if G is Gromov hyperbolic and nonelementary, and ϕ is any automorphism of G.*

Proof. By definition, the automorphisms $\beta = i_m \cdot \alpha$ and $\gamma = i_n \cdot \alpha$ are isogredient if and only if there exists $g \in G$ with $\gamma = i_g \cdot \beta \cdot {i_g}^{-1}$, or equivalently $n = gm\alpha(g^{-1})c$ with c in the center of G. So, the set $S(\Phi)$ of isogredience classes of automorphisms representing Φ may be identified with the set of twisted conjugacy classes of $G \mod$ its center. If ϕ is of finite order in Aut G, then the theorem immediately follows from Lemma 5. If an automorphism ϕ has infinite order, then the theorem follows from Theorem 2. $\Box$

1.2. The co-Hopf property. A group G is *co-Hopf* if every monomorphism of G into itself is an isomorphism. It is fairly immediate to see that a freely decomposable group is not co-Hopf.

Lemma 7 [14]**.** *Let G be a nonelementary, torsion-free, Gromov hyperbolic group. Then G is co-Hopf if and only if G is freely indecomposable.*

Theorem 4. *The Reidemeister number $R(\phi)$ is infinite if G is Gromov hyperbolic, nonelementary, torsion free, and freely indecomposable and ϕ is any monomorphism of G into itself.*

Proof. This follows from Lemma 7 and Theorem 3. $\Box$

1.3. Reduction to injective endomorphisms. Let G be a group and $\phi : G \to G$ an endomorphism. An element $x \in G$ is *nilpotent* if there is an $n \in \mathbb{N}$ such that $\phi^n(x) = \mathrm{id}$. Let N be the set of all nilpotent elements of G.

Lemma 8. *The set N is a normal subgroup of G. We have $\phi(N) \subset N$ and $\phi^{-1}(N) = N$. Thus, ϕ induces an endomorphism $[\phi/N](xN) := \phi(x)N$. The endomorphism $[\phi/N] : G/N \to G/N$ is injective, and we have $R(\phi) = R([\phi/N])$.*

Proof. (i) Let $x \in N, g \in G$. Then for some $n \in \mathbb{N}$ we have $\phi^n(x) = \mathrm{id}$. Therefore, $\phi^n(gxg^{-1}) = \phi^n(gg^{-1}) = \mathrm{id}$. This shows that $gxg^{-1} \in N$, so N is a normal subgroup of G.

(ii) Let $x \in N$ and choose n such that $\phi^n(x) = \mathrm{id}$. Then $\phi^{n-1}(\phi(x)) = \mathrm{id}$, so $\phi(x) \in N$. Therefore, $\phi(N) \subset N$.

(iii) If $\phi(x) \in N$ then there is n such that $\phi^n(\phi(x)) = \mathrm{id}$. Therefore $\phi^{n+1}(x) = \mathrm{id}$, so $x \in N$. This shows that $\phi^{-1}(N) \subset N$. The converse inclusion follows from (ii).

(iv) We write $\mathcal{R}(\phi)$ for the set of ϕ-conjugacy classes of elements of G. Now we show that the map $x \mapsto xN$ induces a bijection $\mathcal{R}(\phi) \to \mathcal{R}([\phi/N])$. Suppose $x, y \in G$ are ϕ-conjugate. Then there is $g \in G$ with $gx = y\phi(g)$. Projecting to the quotient group G/N, we have $gnxN = yN\phi(g)N$, so

$$gNxN = yN[\phi/N](gN).$$

This means that xN and yN are $[\phi/N]$-conjugate in G/N.

Conversely, suppose that xN and yN are $[\phi/N]$-conjugate in G/N. Then there is $gN \in G/N$ such that $gNxN = yN[\phi/N](gN)$. In other words, $gx\phi(g)^{-1}y^{-1}) = \mathrm{id}$. Therefore,

$$\phi^n(g)\phi^n(x) = \phi^n(y)\phi^n(\phi(g)).$$

This shows that $\phi^n(x)$ and $\phi^n(y)$ are ϕ-conjugate. However, by Lemma 1, x and $\phi^n(x)$ are ϕ-conjugate as are y and $\phi^n(y)$. Therefore, x and y are ϕ-conjugate.

(v) We have shown that x and y are ϕ-conjugate iff xN and yN are ϕ/N-conjugate. It follows that $x \mapsto xN$ induces a bijection $\mathcal{R}(\phi) \to \mathcal{R}([\phi/N])$. Therefore, $R(\phi) = R([\phi/N])$. □

Theorem 5. *The Reidemeister number $R(\phi)$ is infinite if the group G/N is Gromov hyperbolic, nonelementary, torsion-free, freely indecomposable and ϕ is any endomorphism of G into itself.*

Proof. This follows from Lemma 8 and Theorems 3 and 4.

§2. The Reidemeister number of a continuous map

Now let X be a connected, compact polyhedron and $f : X \to X$ a continuous map. Let a specific lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ be chosen as a reference.

Then every lifting of f is uniquely written as $\gamma \circ \tilde{f}$, with $\gamma \in \Gamma$. So the elements of Γ serve as coordinates of liftings with respect to the reference $\tilde{f}$. Now, for every $\gamma \in \Gamma$, the composition $\tilde{f} \circ \gamma$ is a lifting of f; so there is a unique $\gamma' \in \Gamma$ such that $\gamma' \circ \tilde{f} = \tilde{f} \circ \gamma$. The correspondence $\gamma \mapsto \gamma'$ is determined by the reference $\tilde{f}$, and is obviously a homomorphism. □

Definition 1. *The endomorphism $\tilde{f}_* : \Gamma \to \Gamma$ determined by the lifting $\tilde{f}$ of f is defined by*

$$\tilde{f}_*(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma.$$

It is well known that $\Gamma \cong \pi_1(X)$. We identify $\pi = \pi_1(X, x_0)$ and Γ in the following way. Pick base points $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0) \subset \tilde{X}$ once and for all. Now, the points of $\tilde{X}$ are in a 1-1 correspondence with homotopy classes of paths in X which start at x_0: for $\tilde{x} \in \tilde{X}$, take any path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}$ and project it onto X; conversely, for a path c starting at x_0, lift it to a path in $\tilde{X}$ which starts at $\tilde{x}_0$, and then take its endpoint. In this way, we identify a point of $\tilde{X}$ with a path class $\langle c \rangle$ in X starting at x_0. Under this identification, $\tilde{x}_0 = \langle e \rangle$ is the unit element in $\pi_1(X, x_0)$. The action of the loop class $\alpha = \langle a \rangle \in \pi_1(X, x_0)$ on $\tilde{X}$ is then given by

$$\alpha = \langle a \rangle : \langle c \rangle \mapsto \alpha \cdot c = \langle a \cdot c \rangle.$$

Now we have the following relationship between $\tilde{f}_* : \pi \to \pi$ and

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, f(x_0)).$$

Lemma 9. *Suppose $\tilde{f}(\tilde{x}_0) = \langle w \rangle$. Then the following diagram commutes:*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(X, f(x_0)) \\ & \tilde{f}_* \searrow & \downarrow w_* \\ & & \pi_1(X, x_0) \end{array}$$

where w_* is the isomorphism induced by the path w.

In other words, for every $\alpha = \langle a \rangle \in \pi_1(X, x_0)$, we have

$$\tilde{f}_*(\langle a \rangle) = \langle w(f \circ a)w^{-1} \rangle. \quad \square$$

Remark 1. In particular, if $x_0 \in p(\mathrm{Fix}(\tilde{f}))$ and $\tilde{x}_0 \in \mathrm{Fix}(\tilde{f})$, then $\tilde{f}_* = f_*$.

Lemma 10. *Lifting classes of f (and hence fixed-point classes, empty or not) are in a 1-1 correspondence with $\tilde{f}_*$-conjugacy classes in π, the lifting class $[\gamma \circ \tilde{f}]$ corresponding to the $\tilde{f}_*$-conjugacy class of γ. We therefore have $R(f) = R(\tilde{f}_*)$.*

We say that the fixed point class $p(\mathrm{Fix}(\gamma \circ \tilde{f}))$, which is labeled with the lifting class $[\gamma \circ \tilde{f}]$, *corresponds* to the $\tilde{f}_*$-conjugacy class of γ. Thus $\tilde{f}_*$-conjugacy classes in π serve as coordinates for fixed point classes of f, once a reference lifting $\tilde{f}$ is chosen.

Theorem 6. *The Reidemeister number $R(f)$ is infinite for any homeomorphism f of a compact polyhedron X with nonelementary, Gromov hyperbolic fundamental group $\pi_1(X, x_0)$.*

Proof. This immediately follows from Theorem 3 and Lemma 10. □

Theorem 7. *The Reidemeister number $R(f)$ is infinite if fundamental group $\pi_1(X, x_0)$ is Gromov hyperbolic, nonelementary,torsion free, freely indecomposable and the map f induces a monomorphism $\tilde{f}_*$ of $\pi_1(X, x_0)$ into itself.*

Proof. This follows from Lemma 10 and Theorem 4. □

Now let f be any continious map of X. Let N be the set of all nilpotent elements of $\pi_1(X, x_0)$ under $\tilde{f}_*$.

Theorem 8. *For any map f, the Reidemeister number $R(f)$ is infinite if the group $\pi_1(X, x_0)/N$ is Gromov hyperbolic, nonelementary torsion free, and freely indecomposable.*

Proof. This follows from Lemma 10 and Theorem 5. □

§3. Pseudo-Anosov homeomorphisms and an asymptotic expansion

Now we assume X to be a compact surface with negative Euler characteristic and $f : X \to X$ a *pseudo-Anosov* homeomorphism, i.e., there is a number $\lambda > 1$ and a pair of transverse measured foliations (F^s, μ^s) and (F^u, μ^u) such that

$$f(F^s, \mu^s) = (F^s, \frac{1}{\lambda}\mu^s) \text{ and } f(F^u, \mu^u) = (F^u, \lambda\mu^u).$$

The *mapping torus* T_f of $f : X \to X$ is obtained from $X \times [0, 1]$ by identifying $(x, 1)$ with $(f(x), 0)$ for all $x \in X$. Take the base point x_0 of

X as the base point of T_f. According to the van Kampen theorem, the fundamental group $\pi_1(T_f, x_0)$ is obtained from $\pi_1(X, x_0)$ by adding a new generator z and adding the relations $z^{-1}gz = \tilde{f}_*(g)$ for all $g \in \pi_1(X, x_0)$, where z is the generator of $\pi_1(S^1, x_0)$. This means that $\pi_1(T_f, x_0)$ is a semi-direct product of $\pi_1(X, x_0)$ with $\mathbb{Z}$.

From Lemma 2 it follows that two elements x, y of $\pi_1(X, x_0)$ are $\tilde{f}_*$-conjugate if and only if xz and yz are conjugate in the usual sense in $\pi_1(T_f, x_0)$. Therefore $R(f) = R(\tilde{f}_*)$ is the number of the usual conjugacy classes in the coset $\pi_1(X, x_0) \cdot z$ of $\pi_1(X, x_0)$ in $\pi_1(T_f, x_0)$. There is a canonical projection

$$\tau : T_f \to \mathbb{R}/\mathbb{Z}, \quad (x, s) \mapsto s.$$

This induces a map $\pi_1(\tau) : G = \pi_1(T_f, x_0) \to \mathbb{Z}$.

We see that the Reidemeister number $R(f)$ is equal to the number of homotopy classes of closed paths γ in T_f whose projections onto $\mathbb{R}/\mathbb{Z}$ are homotopic to the path

$$\begin{array}{cccc} \sigma : & [0,1] & \to & \mathbb{R}/\mathbb{Z} \\ & s & \mapsto & s. \end{array}$$

Corresponding to this, there is a group-theoretical interpretation of $R(f)$ as the number of usual conjugacy classes of elements $\gamma \in \pi_1(T_f)$ satisfying $\pi_1(\tau)(\gamma) = z$.

Lemma 11 [15]. *The interior of the mapping torus* $\mathrm{Int}(T_f)$ *admits a hyperbolic structure of finite volume if and only if f is isotopic to a pseudo-Anosov homeomorphism.*

So, if the surface X is closed and f is isotopic to a pseudo-Anosov homeomorphism, the mapping torus T_f can be realized as a hyperbolic 3-manifold, $H^3/\pi_1(T_f, x_0)$, where H^3 is the Poincare upper half-space $\{(x, y, z) : z > 0, (x, y) \in \mathbb{R}^2\}$ with metric $ds^2 = (dx^2 + dy^2 + dz^2)/z^2$.

The closed geodesics on a hyperbolic manifold are in one-to-one correspondence with the free homotopy classes of loops. These classes of loops are in one-to-one correspondence with the conjugacy classes of loxodromic elements in the fundamental group of the hyperbolic manifold.

This correspondence allowed Epstein (see [4], p. 127) to study the asymptotics of such functions as $p_n(x) = \#\{$primitive closed geodesics of length less than x represented by an element of the form $gz^n\}$ using the Selberg trace formula. A primitive closed geodesic is one which is not an iterate of another closed geodesic.

Phillips and Sarnak [12] generalized results of Epstein in the following way. Let M be a compact Riemannian manifold of negative curvature. Let $\psi : G \to \Lambda$ be a surjective homomorphism of the fundamental group G of M onto an Abelian group Λ. Let r be the rank of Λ and let m be the order of the torsion of Λ; i.e., Λ is isomorphic to $\mathbb{Z}^r \times F$, F being finite of order m. For $\beta \in \Lambda$, let $p_\beta(x)$ be the number of the primitive closed geodesics γ on M of length at most x for which $\psi(\gamma) = \beta$.

Theorem 9 [12]**.** *We have the asymptotic expansion*

$$p_\beta(x) \sim \frac{e^{(n-1)x}}{mx^{r/2+1}} \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots \right)$$

as $x \to \infty$. Here, C_0 is independent of β; C_0 is the determinant of a certain period matrix of harmonic 1*-forms on M.*

The proof of this result makes routine use of the Selberg trace formula.

If $\Lambda = H_1(M, Z)$, then Theorem 9 gives an asymptotic of the number of primitive closed geodesics of length at most x lying in a fixed homology class. In the more general case of variable negative curvature, similar asymptotic was obtained by Pollicott and Sharp [13]. They used dynamical approach based on the geodesic flow. We only need an asymptotic for $p_1(x)$. Note that closed geodesics represented by an element of the form gz are automatically primitive, because they wrap exactly once around the mapping torus (once around the generator z). Taking $\Lambda = \mathbb{Z}$, a surjective homomorphism $\pi_1(\tau) : G = \pi_1(T_f, x_0) \to \mathbb{Z}$, and $\beta = z$-generator of the group $\mathbb{Z}$ in Theorem 9, we have the following asymptotic expansion

$$p_1(x) = \frac{e^{hx}}{x^{3/2}} \left(\sum_{n=0}^{N} \frac{C_n}{x^{n/2}} + o\left(\frac{1}{x^{N/2}}\right) \right), \tag{1}$$

for any $N > 0$, where $h = \dim T_f - 1 = 2$ is the topological entropy of the geodesic flow on the unit-tangent bundle ST_f, the constant $C_0 > 0$, and the constants C_n vanish if n is odd. So in particular, using $N = 1$, we have

$$p_1(x) \sim C_0 \frac{e^{hx}}{x^{3/2}}, \quad x \to \infty. \tag{2}$$

Since $C_0 > 0$, we see that $p_1(x)$ goes to infinity as x does.

Now, using the one-to-one correspondences in Lemmas 2 and 10, we define the norm of a fixed point class, or of the corresponding lifting class, or of the corresponding twisted conjugacy class $\{g\}_{\tilde{f}_*}$ in the fundamental group $\pi_1(X, x_0)$, as the length of the primitive closed geodesic γ on T_f represented by an element of the form gz. So, for example, the norm function l^* on the set of twisted conjugacy classes equals $l^* = l \circ B$, where l is the length function on geodesics ($l(\gamma)$ is the length of the primitive closed geodesic γ) and B is a bijection between the set of twisted conjugacy classes $\{g\}_{\tilde{f}_*}$ in the fundamental group $\pi = \pi_1(X, x_0)$ and the set of closed geodesics represented by elements of the form gz in the fundamental group $G := \pi_1(T_f, x_0)$. We introduce the following counting functions:

FPC(x)= # {fixed point classes of f of norm less than x};

L(x) = # {lifting classes of f of norm less than x};

Tw(x) = #{twisted conjugacy classes for $\tilde{f}_*$ in the fundamental group of surface of norm less than x}.

We have the following asymptotic expansion.

Theorem 10. *Let X be a closed surface of negative Euler characteristic and let $f : X \to X$ be a pseudo-Anosov homeomorphism. Then*

$$\mathrm{FPC}(x) = L(x) = Tw(x) = \frac{e^{2x}}{x^{3/2}}\left(\sum_{n=0}^{N}\frac{C_n}{x^{n/2}} + o\left(\frac{1}{x^{N/2}}\right)\right),$$

where the constant $C_0 > 0$ depends on the volume of the hyperbolic 3-manifold T_f, and the constants C_n vanish if n is odd.

Proof. This follows from Lemmas 2 and 10 and asymptotic expansion (1). □

Corollary 1. *For pseudo-Anosov homeomorphisms of closed surfaces, the Reidemeister number is infinite.* □

As we see, Theorem 10 gives an analytical proof of Lemma 3 in the case of a pseudo-Anosov automorphism of the surface group.

Theorem 10 indicates also that we can develop an analog of the Selberg theory for twisted conjugacy classes of a group automorphism. We have the following question in this direction:

Question 1. *Is there an asymptotic of the counting function for twisted conjugacy classes of a hyperbolic automorphism of a Gromov hyperbolic group?*

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