Записки научных
семинаров ПОМИ
Том 279, 2001 г.

S. M. Finashin

AN INTEGRAL FORMULA FOR THE COMPLEX INTERSECTION NUMBER OF REAL CYCLES IN A REAL ALGEBRAIC VARIETY WITH TOPOLOGICALLY RATIONAL SINGULARITIES

Abstract. A formula of type indicated in the title is presented and discussed.

§1. The complex intersection form and its local version

Considering a real algebraic variety, I denote by $\mathbb{C}X$ its complex-point set and by $\mathbb{R}X$ its real part, which is the fixed-point set of the complex conjugation $\mathrm{conj}\colon \mathbb{C}X \to \mathbb{C}X$. A similar notation is used also for conj-invariant subsets of real varieties $\mathbb{C}X$.

For a connected component Γ of a nonsingular variety $\mathbb{R}X$, it is well known that the self-intersection number $\langle\Gamma,\Gamma\rangle_{\mathbb{C}X}$ is equal to $(-1)^{d/2}\chi(\Gamma)$ if the dimension $d=\dim(\mathbb{R}X)$ is even, and to 0 if odd. This is because the multiplication by $i=\sqrt{-1}$ gives an isomorphism between the tangent and the normal bundle of $\mathbb{R}X$ in $\mathbb{C}X$. If $\mathbb{C}X$ is a singular variety being a rational homology manifold, at least in a neighborhood of $\mathbb{R}X$, then the intersection numbers $\langle\Gamma,\Delta\rangle_{\mathbb{C}X}$ for d-cycles Γ, Δ in $\mathbb{R}X$ are still well defined. It turns out that these numbers can be calculated using the formula (cf. [3])

$$\langle\Gamma,\Delta\rangle_{\mathbb{C}X} = \int\limits_{\Gamma\cap\Delta} \lambda_x(\Gamma,\Delta)\,d\chi(x). \tag{1}$$

Here, $\lambda_x(\Gamma,\Delta)\in\mathbb{Q}$ is the local intersection number of Γ and Δ at a point $x\in\Gamma\cap\Delta$, see below. The number $\lambda_x(\Gamma,\Delta)$ does not change as x varies within a connected component of a stratum S of a semi-algebraic Whitney stratification of $\mathbb{R}X$, and we can write $\lambda_x(\Gamma,\Delta)=\lambda_S(\Gamma,\Delta)$. The integral with respect to the Euler characteristic is understood in the

Partly supported by the RFBR (grant 98-01-00832).

sense of [6], i.e., it is a finite sum $\sum_{S\subset\Gamma\cap\Delta}\lambda_S(\Gamma,\Delta)\chi_c(S)$, where χ_c is the Euler characteristic with compact support (which is additive, cf. [4], so that $\chi_c(S)=\chi(\mathrm{Cl}(S))-\chi(\partial S)$), and the sum is taken with respect to the connected components of the strata $S\subset\Gamma\cap\Delta$.

The idea of the proof of (1) is to shift Δ along a suitable vector field and count the local intersection numbers of Γ with the shifted Δ.

§2. Topologically rational singularities

The singularities of a complex algebraic variety X at which X is a $\mathbb{Q}$-homology manifold will be called *topologically rational*. In other words, for the link L_x of such a singular point x (called briefly a *TR-point*),

$$H_*(L_x;\mathbb{Q}) = H_*(S^{2d-1};\mathbb{Q}),$$

where $d=\dim_{\mathbb{C}}X$. If a complex algebraic variety has only such singular points (i.e., is a $\mathbb{Q}$-homology manifold), then I call it a *TR-variety*.

By a *component* of $\mathbb{R}X$ I mean the closure of a connected component of $\mathbb{R}X\smallsetminus\mathrm{Sing}_{\mathrm{top}}(\mathbb{R}X)$. Here, $\mathrm{Sing}_{\mathrm{top}}(\mathbb{R}X)$ denotes the *topological singularities* of $\mathbb{R}X$, i.e., the set of the points $x\in\mathbb{R}X$ whose link is not homeomorphic to S^{d-1}. It is not difficult to prove that

Given a real TR-variety $\mathbb{R}X$, the subset $\mathrm{Sing}_{\mathrm{top}}(\mathbb{R}X)$ has codimension $\geqslant 2$ in $\mathbb{R}X$.

This implies that the components of $\mathbb{R}X$ for a TR-variety $\mathbb{C}X$ represent $(\mathbb{Z}/2)$-cycles generating $H_d(\mathbb{R}X;\mathbb{Z}/2)$. The orientable (outside $\mathrm{Sing}_{\mathrm{top}}(\mathbb{R}X)$) components of $\mathbb{R}X$ can similarly be regarded as the generators of $H_d(\mathbb{R}X)$, after we fix an orientation of the components.

§3. The form λ_x

Given a point $x\in X$ of an algebraic variety $X\subset\mathbb{C}\mathrm{P}^m$, let us consider its neighborhood $U_x=X\cap B_\varepsilon$, where B_ε is a compact ε-ball around x in $\mathbb{C}\mathrm{P}^m$ of sufficiently small radius $\varepsilon>0$ (say, in the Fubini–Studi metric), bounded by the *link* M_x of x, $M_x=X\cap S_\varepsilon$, where $S_\varepsilon=\partial B_\varepsilon$. By a *stratified outward-directed* vector field ξ around x, I mean a vector field defined along U_x, which is tangent to (the strata of) X and is directed outward the spheres S_δ for $0<\delta<\varepsilon$. Existence of such ξ is well known for any Whitney stratified pseudo-manifold. Indeed, one can fix some smooth triangulation of X containing x as a vertex, and take a *radial* vector field, i.e., a vector field defined inside the star of x and directed along line segments originated from x towards the simplicial link of x.

Any continuous vector field $\xi(z)$ defined along a subset $A \subset S^{2m-1} \subset \mathbb{C}^m$ and directed outward the sphere determines a continuous vector field $\xi_\tau(z)$ tangent to S^{2m-1} such that $\xi_\tau(z) = e^{ia(z)}\xi(z)$, where $0 < a(z) < \pi$. If $A = M_x \subset S_\varepsilon$ and ξ is an outward-directed stratified vector field, then ξ_τ is a nowhere-vanishing vector field tangent to M_x.

Suppose now that $\mathbb{C}X$ is a real algebraic variety, $x \in \mathbb{R}X$, and consider a neighborhood $\mathbb{C}U_x = \mathbb{C}X \cap B_\varepsilon$, the complex link $\mathbb{C}M_x = \mathbb{C}X \cap S_\varepsilon$ and their real parts

$$\mathbb{R}U_x = \mathbb{C}U_x \cap \mathbb{R}X, \quad \mathbb{R}M_x = \mathbb{C}M_x \cap \mathbb{R}X.$$

We may choose ξ to be tangent to $\mathbb{R}X$ at the points of $\mathbb{R}U$. Then $\xi_\tau(z) = i\xi(z)$ at $z \in \mathbb{R}M_x$, so ξ_τ is stratified normal to $\mathbb{R}M_x$. I call ξ_τ the *canonical framing* of the real link $\mathbb{R}M_x$ in the complex link $\mathbb{C}M_x$.

The normality guarantees that a small shift of $\mathbb{R}M_x$ in $\mathbb{C}M_x$ along ξ_τ gives a disjoint subset.

Assume that $x \in \mathbb{R}X$ is a TR-point, γ and δ are $(d-1)$-cycles representing homology classes $[\gamma]$, $[\delta] \in H_{d-1}(\mathbb{R}M_x)$, and δ_{ξ_τ} is the cycle in $\mathbb{C}M_x$ obtained by a small shift of δ in the direction of ξ_τ. Since δ_{ξ_τ} lies outside $\mathbb{R}M_x$, the linking number $\mathrm{lk}(\gamma, \delta_{\xi_\tau})$ of γ and δ_{ξ_τ} in $\mathbb{C}M_x$ is well defined and we can define a bilinear form

$$\lambda_x \colon H_{d-1}(\mathbb{R}M_x) \times H_{d-1}(\mathbb{R}M_x) \to \mathbb{Q}, \quad \lambda_x([\gamma],[\delta]) = \mathrm{lk}(\gamma, \delta_{\xi_\tau}).$$

It is conceptually useful to regard λ_x as the local complex intersection form

$$\lambda_x([\gamma],[\delta]) = \langle \mathrm{Con}(\gamma), \mathrm{Con}(\delta_{\xi_\tau})\rangle_{\mathbb{C}U_x}, \tag{2}$$

where $\mathbb{C}U_x = \mathbb{C}X \cap \mathbb{C}B_\varepsilon$, $\mathbb{C}B_\varepsilon$ being a ball bounded by $\mathbb{C}S_\varepsilon$ in $\mathbb{C}\mathrm{P}^m$, and $\mathrm{Con}(\gamma)$, $\mathrm{Con}(\delta_{\xi_\tau})$ denote the cones over the cycles γ, δ_{ξ_τ} in the cone-like neighborhood $\mathbb{C}U_x \cong \mathrm{Con}(\mathbb{C}M_x)$. The cone $\mathrm{Con}(\delta_{\xi_\tau})$ can be regarded as the result of a shift of $\mathrm{Con}(\delta)$ along the vector field ξ_τ.

Note that the form λ_x is independent of several auxiliary choices that we made (because any pair ξ_0, ξ_1 of stratified outward-directed vector fields along $\mathbb{R}M_x$ is connected by a linear homotopy defined pointwise as $\xi_t = t\xi_1 + (1-t)\xi_0$).

Recall that sufficiently small spheres $\mathbb{R}S_\varepsilon$ around $x \in \mathbb{R}X$ are transverse to the strata of $\mathbb{R}X$. This implies that any component $\Gamma \subset \mathbb{R}X$ cuts on $\mathbb{R}S_\varepsilon$ a cycle $\gamma = \Gamma \cap \mathbb{R}S_\varepsilon$, which is oriented if Γ is oriented (outside $\mathrm{Sing}_{\mathrm{top}}(\mathbb{R}X)$). Given a pair of oriented components Γ, Δ of $\mathbb{R}X$ for a real TR-variety $\mathbb{C}X$, and $x \in \Gamma \cap \Delta$, we put

$$\lambda_x(\Gamma, \Delta) = \lambda_x([\gamma],[\delta]),$$

where γ, δ are the cycles on $\mathbb{R}M_x$ cut by Γ and Δ.

It turns out that λ_x *is symmetric independently of the parity of the dimension* d, unlike the usual intersection form. For other properties of λ_x and its calculation in certain cases, see [3].

§4. Motivation: generalization of the Arnold inequalities

The Arnold inequalities [1] for a nonsingular real algebraic surface $\mathbb{C}X$ estimate the numbers l_+, l_-, and l_0 of the connected components of $\mathbb{R}X$ with positive, negative, and zero Euler characteristic. This estimate basically says that $l_\pm$ is not greater then the $\mp$-inertia number of the conj_*-invariant subspace of $H_2(\mathbb{C}X;\mathbb{R})$, just because the self-intersection is minus the Euler characteristic of a component. Similar estimates for $l_\pm + l_0$ involve an additional summand, the rank of $\mathrm{Ker}(H_2(\mathbb{R}X) \to H_2(\mathbb{C}X))$. The above inertia numbers were calculated in [1] for double coverings $\mathbb{C}X \to \mathbb{C}\mathrm{P}^2$ branched along nonsingular curves $\mathbb{C}A \subset \mathbb{C}\mathrm{P}^2$ of even degree $m = 2k$, which gave an enforcement of the Petrovskii inequalities.

The Arnold inequalities were extended by Viro [7] and Kharlamov [5] to nodal curves and nodal surfaces. In [2], I suggested a further extension, to curves and surfaces with topologically rational singularities, i.e., those for which $\mathbb{C}X$ is a rational homology manifold. The main difficulty in extending the Arnold inequalities is to write down an explicit formula for the intersection numbers $\langle\Gamma,\Delta\rangle_{\mathbb{C}X}$ of the cycles Γ, Δ in $\mathbb{R}X$, while the estimates themselves are similar to those in the nonsingular case. It turns out that $\langle\Gamma,\Delta\rangle_{\mathbb{C}X}$ can be treated as a weighted Euler characteristic of $\Gamma\cap\Delta$, in which nonsingular points contribute all -1, and the contribution of a singularity at $x \in \mathbb{R}X$ is measured by the quadratic form λ_x. A few methods to calculate these forms are suggested in [2] and [3].

Formula (1) is a further generalization, in the case of nonisolated singularities of higher dimensions. I refer to [3] for the discussion of the generalized Arnold inequalities and some of their applications and confine myself here just with the following example.

§5. Example

Let us consider the union $\mathbb{C}A = \mathbb{C}A_1 \cup \cdots \cup \mathbb{C}A_m \subset \mathbb{C}\mathrm{P}^d$ of real projective nonsingular hypersurfaces $\mathbb{C}A_i$. Assume that the hypersurfaces are in general position in the differential-topological sense, i.e., their mutual intersections (for any set of A_i) are transverse. Assume furthermore that the degree of $\mathbb{C}A$ is even and consider the double covering $p\colon X \to \mathbb{C}\mathrm{P}^d$ branched along $\mathbb{C}A$.

The singularities of X arising over the mutual intersections of $\mathbb{C}A_i$'s are *corner singularities.* They are quotient singularities and thus topologically rational. The quadratic forms λ_x for the corner singularities can easily be calculated (see [3]). It turns out that λ_x is described by a matrix with entries $\pm 2^{1-k}$ if $x \in \mathbb{R}X$ belongs to precisely k hypersurfaces $\mathbb{C}A_i$.

This makes computation of $\langle \Gamma, \Delta \rangle_X$ almost as simple as in the case of nodal curves [7]. For example, let us assume that $\mathbb{C}A_i$ are all hyperplanes. Then $\mathbb{R}\mathrm{P}^d \smallsetminus \mathbb{R}A$ splits into several open convex polyhedra. Let us denote by $W_1, \dots, W_s$ their closures and put $\Gamma_i = p^{-1}(W_i)$. Then our calculation shows that

$$\langle \Gamma_i, \Gamma_j \rangle_X = \pm 2^{1-d} f_{\Gamma_i \cap \Gamma_j}(-2),$$

where $f_\Delta(t) = \sum_{k=0}^{\dim(\Delta)} f_k t^k$ is the f*-polynomial* of Δ, and the coefficients f_k are the numbers of k-faces in Δ.

It is well known that $f_\Delta(-2)$ is equal to the signature $\sigma(T_\Delta)$ of the toric variety T_Δ associated to a convex polyhedron Δ. So, $\langle \Gamma_i, \Gamma_j \rangle_X$ is equal, up to coefficient $\pm 2^{1-d}$, to the signature $\sigma(T_\Delta)$, $\Delta = \Gamma_i \cap \Gamma_j$. (Strictly speaking, T_Δ is well defined only for a polyhedron Δ with rational vertices. However, its signature is determined by the combinatorial type of Δ and thus it makes sense to consider any polyhedron.)

In the case of a hyperplane arrangement $\mathbb{C}A$, the Arnold inequalities turn out to be equalities, which give precise values of the inertia numbers and the nullity of the complex intersection form, see [3] for more details.

References

1. V. I. Arnold, *On the arrangement of ovals of real plane algebraic curves, involutions on four-dimensional smooth manifolds, and the arithmetic of integer-valued quadratic forms.* — Funk. Anal. Prilozh. 5, No. 3 (1971) 1–9.
2. S. Finashin, *A generalization of the Arnold–Viro inequalities for real singular algebraic curves.* — Contemp. Math. (ed. C. Delzell, J. Madden). 2000, pp. 31–37.
3. S. Finashin, *Complex intersection of real cycles in real algebraic varieties and generalized Arnold–Viro inequalities.* Preprint, math.AG/9902022.
4. M. Goresky, R. MacPherson, *Local contribution to the Lefschetz fixed point formula.* — Invent. Math. (1993), 1–33.
5. V. M. Kharlamov, *Topology of Real Algebraic Varieties.* Nauka, M. (1986).
6. O. Ya. Viro, *Some integral calculus based on the Euler characteristic.* Springer Lecture Notes in Math. 1988, pp. 127–138.
7. О. Я. Виро, *Обобщение неравенств Петровского и Арнольда на кривые с особенностями.* — Успехи мат. наук **33, No. 3** (1978), 145–146.

С.-Петербургский электротехнический
университет (ЛЭТИ)
e-mail: serge@edu.metu.tr

Поступило 21 декабря 2000 г.