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HARMONIC SERIES SUMMATION LEMMA AND VASYUNIN'S FORMULAS

1. Introduction

Introducing a rather simple map of the Nyman-Beurling approach, we arrive at Vasyunin style cotangent formulas for the relevant inner products. This is accomplished by means of a summation formula for certain series derived from arithmetical progression subseries of the harmonic series. The main attractiveness of the resulting cotangent sums for the inner products is the fact that the coefficients of the cotangents are only ± 1, which gives perhaps some hope to lighten up the calculation of the Gramians.

2. Summing a modification of the harmonic series

We find many uses for the following summation lemma.

Lemma 2.1. *Let $\epsilon : \mathbb{Z} \to \mathbb{R}$ be an odd function with period $2b$, $b \in \mathbb{N}$, then*

$$\sum_{n=1}^{\infty} \frac{\epsilon(n)}{n} = \frac{\pi}{2b} \sum_{r=1}^{b-1} \epsilon(r) \cot \frac{\pi r}{2b}. \tag{2.1}$$

Proof. Note that

$$\sum_{n=1}^{2b} \epsilon(n) = 0, \tag{2.2}$$

hence the series in (2.1) converges, albeit conditionally. Now by Abel's theorem we can write

$$\begin{aligned} S := \sum_{n=1}^{\infty} \frac{\epsilon(n)}{n} &= \lim_{t \uparrow 1} \sum_{n=1}^{\infty} \frac{\epsilon(n)}{n} t^n \\ &= \lim_{t \uparrow 1} \sum_{n=1}^{\infty} \epsilon(n) \int_0^t x^{n-1} dx = \int_0^1 \sum_{n=1}^{\infty} \epsilon(n) x^{n-1} dx. \end{aligned} \tag{2.3}$$

Next we rewrite the above integrand as

$$\begin{aligned} I(x) := \sum_{n=1}^{\infty} \epsilon(n) x^{n-1} &= \sum_{j=0}^{\infty} \sum_{r=1}^{2b-1} \epsilon(2bj+r) x^{2bj+r-1} \\ &= \sum_{j=0}^{\infty} x^{2bj} \sum_{r=1}^{2b-1} \epsilon(r) x^{r-1} = \frac{P(x)}{1-x^{2b}}, \end{aligned} \tag{2.4}$$

where we have set

$$P(x) := \sum_{r=1}^{2b-1} \epsilon(r) x^{r-1}. \tag{2.5}$$

Note that $P(1) = 0$. The next obvious task is to decompose the last fraction in (2.4) in partial fractions. This function is of order $|x|^{-2}$ at infinity, having at most simple poles at the $2b$-th roots of unity, so it is equal to the sum of its principal parts at each such root ω, which are equal to

$$\lim_{x \to \omega} \frac{P(x)}{1-x^{2b}} = -\frac{\omega}{2b} P(\omega).$$

Now adding up for $\omega = e^{i\frac{\pi p}{b}}$, $p = 1, 2, \ldots, 2b-1$, since $\omega = 1$ is not a pole, and noting that $\epsilon(b-n) = -\epsilon(b+n)$, so $\epsilon(b) = 0$, we obtain

$$\begin{aligned} I(x) &= -\frac{1}{2b} \sum_{\omega} \frac{\omega P(\omega)}{x-\omega} = -\frac{1}{2b} \sum_{\omega} \frac{1}{x-\omega} \sum_{r=1}^{2b-1} \epsilon(r) \omega^{r} \\ &= -\frac{i}{b} \sum_{p=1}^{2b-1} \frac{1}{x - e^{i\frac{\pi p}{b}}} \sum_{r=1}^{b-1} \epsilon(r) \sin \frac{\pi p r}{2b} = -\frac{i}{b} \sum_{r=1}^{b-1} \epsilon(r) \sum_{p=1}^{2b-1} \frac{\sin \frac{\pi p r}{2b}}{x - e^{i\frac{\pi p}{b}}}. \end{aligned}$$

Now we want to plug this expression for $I(x)$ back into (2.3). Since the integral must be real, we compute

$$\Im \int_0^1 \frac{dx}{x - e^{i\frac{\pi p}{b}}} = \frac{\pi}{2}\left(1 - \frac{p}{b}\right),$$

which then leads elementarily, and tediously as well, to

$$S = \frac{\pi}{2b} \sum_{r=1}^{b-1} \epsilon(r) \sum_{p=1}^{2b-1} \left(1 - \frac{p}{b}\right) \sin \frac{\pi p r}{b} = \frac{\pi}{2b} \sum_{r=1}^{b-1} \epsilon(r) \cot \frac{\pi r}{2b}.$$

3. An application to the inner products in our mapped Beurling approach

3.1. Nyman's theorem and its mappings. We let $\mathcal{H} := L_2((0,\infty), dx)$. As usual we denote the inner product of $f, g \in \mathcal{H}$ by

$$(f,g) := \int_0^\infty f(t)\overline{g(t)}dt.$$

For any $a > 0$ let K_a be the operator on $\mathcal{H}$ given by $K_a f(x) := f(ax)$. We recall the following simple facts:

$$K_a K_b = K_{ab}, \quad K_a^* = \frac{1}{a} K_{\frac{1}{a}}, \quad \text{and} \quad \|K_a\| = a^{-1/2}.$$

With $\rho(x) := x - [x]$, i.e., the fractional part of the real number x, we denote $\rho_1(x) := \rho(1/x)$, and $\rho_a := K_a \rho_1$ ($\mathbb{R} \ni a > 0$). Unfortunately this notation deviates from that in other papers. A minor extension of the Nyman–Beurling criterion [4, 3] states that a necessary and sufficient condition for the Riemann hypothesis is (see [2])

$$\chi_{(0,1)} \in \overline{\operatorname{span}}_{\mathcal{H}}\{\rho_a | a \geq 1\}. \tag{3.1}$$

The so-called *natural conjecture* is the statement (as yet unproven) that the Riemann hypothesis is equivalent to

$$\chi_{(0,1)} \in \overline{\operatorname{span}}_{\mathcal{H}}\{\rho_n | n \in \mathbb{N}\}. \tag{3.2}$$

We aim now to "map" all these statements. Note that $Q_2 := I - K_2$ is a bounded, invertible operator of $\mathcal{H}$ onto itself since $\|K_2\| = \frac{1}{\sqrt{2}} < 1$. Now we bring in the unitary operator U introduced in [1], defined on $\mathcal{H}$ by the requirements that it commute with all K_a and that $U\rho_1(x) := 1/x\rho(x)$. We know also, see [1], that

$$U\chi_{(0,1)}(x) = \frac{\sin 2\pi x}{\pi x}.$$

Now we define

$$T := K_{\frac{1}{2}} Q_2 U. \tag{3.3}$$

Clearly T is a bounded invertible operator on $\mathcal{H}$ commuting with all K_λ. It is then a trivial computation to see that

$$T\rho_1(x) = \frac{1}{x}\chi_E(x), \tag{3.4}$$

where

$$E := [1,2) \cup [3,4) \cup (5,6] \cup \dots . \tag{3.5}$$

and, just as trivially,

$$\xi(x) := T\chi_{(0,1]}(x) = \frac{1}{\pi x} \sin^3 \pi x. \tag{3.6}$$

We denote

$$\phi_a := T\rho_a \quad (a > 0).$$

Then we can immediately translate the closure statements (3.1) and (3.2) as follows:

$$\xi \in \operatorname{span}_{\mathcal{H}}\{\phi_a | a \geq 1\}, \tag{3.7}$$

$$\xi \in \operatorname{span}_{\mathcal{H}}\{\phi_n | n \in \mathbb{N}\}. \tag{3.8}$$

The first statement above is equivalent to the Riemann hypothesis; the second one is equivalent to the natural conjecture. It is natural then to desire a thorough knowledge of the family of inner products

$$c(a,b) := (\phi_a, \phi_b) = \frac{1}{ab} \int_0^\infty \chi_E(at)\chi_E(bt)\frac{dt}{t^2} \quad (0 < a, b \in \mathbb{R}). \tag{3.9}$$

We see trivially that

$$c(\lambda a, \lambda b) = \frac{1}{\lambda} c(a,b). \tag{3.10}$$

It is easy to see that

$$\chi_E = \sum_{h=1}^\infty (-1)^{h+1} \chi_{[h,\infty)}, \tag{3.11}$$

with L_p convergence for $1 \leq p < \infty$, therefore

$$\begin{aligned} c(a,b) &= \frac{1}{ab} \int_0^\infty \chi_E(at)\chi_E(bt)\frac{dt}{t^2} \\ &= \lim_{m,n\to\infty} \sum_{h=1}^m \sum_{k=1}^n (-1)^{h+k} \frac{1}{ab} \int_0^\infty \chi_{[h,\infty)}(at)\chi_{[k,\infty)}(bt)\frac{dt}{t^2} \\ &= \lim_{m,n\to\infty} \sum_{h=1}^m \sum_{k=1}^n (-1)^{h+k} \frac{1}{ak} \wedge \frac{1}{bh}, \end{aligned} \tag{3.12}$$

a limit that is valid when m, n go independently to infinity in any way desired, including in particular the iterated series case.

3.2. The application to summing $c(a,b)$. With the summation Lemma 2.1 at hand and formula (3.12), we can now find a closed finite expression for $c(a,b)$, when both $a,b\in\mathbb{N}$. Even if a,b are not positive integers, a great deal of the calculation below is valid, which should be of some importance later. In view of the homogeneity property (3.10) there is no loss of generality in assuming that $\gcd(a,b)=1$. On account of (3.12) we may write

$$c(a,b)=\lim_{N\to\infty}\sum_{h=1}^{2bN}\sum_{k=1}^{2aN}(-1)^{h+k}\frac{1}{ah}\wedge\frac{1}{bk}.$$

Now we partition the above sum in three nondisjoint subsums corresponding to the conditions $ah\le bk$, $bk\le ah$, and $ah=bk$. By coprimality the sum over the last subset is

$$\frac{1}{ab}\sum_{j=1}^{2N}\frac{(-1)^{(a+b)j}}{j}.$$

The first one on the other hand easily computes to

$$\frac{1}{2a}\sum_{h=1}^{2bN}\frac{(-1)^{h+1}}{h}+\frac{1}{2a}\sum_{h=1}^{2bN}\frac{(-1)^{h+\left[\frac{ah}{b}\right]}}{h},$$

while obviously the second subsum has a corresponding symmetric expression.

Remark 3.1. Note here that should a,b not be integers and a/b be an irrational number, then the diagonal term would disappear, while the others would remain as written above. However it is not easy, or impossible, to find a close expression in this case when $N\to\infty$.

Adding the first two terms together and subtracting the third term, i.e., the diagonal term, we obtain

$$c(a,b)=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\log 2+\frac{1}{2a}\sum_{\substack{h=1\\ h\not\equiv 0 \,(\mathrm{mod}\, b)}}^{\infty}\frac{(-1)^{h+\left[\frac{ah}{b}\right]}}{h}$$
$$+\frac{1}{2b}\sum_{\substack{k=1\\ k\not\equiv 0 \,(\mathrm{mod}\, a)}}^{\infty}\frac{(-1)^{k+\left[\frac{bk}{a}\right]}}{k}. \tag{3.13}$$

It is easy to see that the two series above satisfy the conditions set forth in our Lemma 2.1, thus leading to the desired expression

$$c(a,b) = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\log 2 + \frac{\pi}{4ab}\sum_{k=1}^{b-1}(-1)^{k+\left[\frac{ak}{b}\right]}\cot\frac{\pi k}{2b}$$
$$+ \frac{\pi}{4ab}\sum_{h=1}^{a-1}(-1)^{h+\left[\frac{bh}{a}\right]}\cot\frac{\pi h}{2a}, \tag{3.14}$$

a formula quite like that in Vasyunin's paper [5], but which can be considered simpler as it only involves a sum of cotangents multiplied by plus or minus one.

References

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