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ON BACKWARD UNIQUENESS FOR PARABOLIC EQUATIONS

Abstract. We prove backward uniqueness result for the heat operator with variable lower order terms which implies full regularity of $L_{3,\infty}$-solutions of the the three-dimensional Navier–Stokes equations.

Dedicated to Olga Alexandrovna Ladyzhenskaya

In this note we address a backward uniqueness problem for the heat operator with variable lower order terms. The problem was raised in connection with regularity theory of the Navier-Stokes equations [11], and, independently, in connection with control theory for parabolic equations [10].

We will work with the backward operator $\partial_t + \Delta$ rather than the more usual $\partial_t - \Delta$, since this will save us writing some minus signs in many formulae. We fix $R > 0$ and $T > 0$ and consider the domain $Q_{R,T} = (R^n \setminus \overline{B}_R) \times (0,T) \subset R^{n+1}$. We consider vector-valued functions $u = (u_1, \dots, u_m)$, $u = u(x_1, \dots, x_n, t)$ defined in $Q_{R,T}$ and satisfying

$$|u_t + \Delta u| \leqslant C(|\nabla u| + |u|) \quad \text{a. e. in} \quad Q_{R,T} \tag{1}$$

for some $C > 0$. Natural regularity assumptions under which (1) can be considered are for example that u_t and $\nabla^2 u$ be square integrable over bounded subdomains of $Q_{R,T}$. We will also assume the following bound on the growth of $|u(x,t)|$ as $|x| \to \infty$,

$$|u(x,t)| \leqslant C \exp(C|x|^2). \tag{2}$$

Our main result is as follows.

Theorem 1. *In the above notation, assume that u satisfies* (1) *and* (2) *for some $C > 0$. If, in addition, $u(x,0) = 0$ in $R^n \setminus \overline{B}_R$, then $u = 0$ in $Q_{R,T}$.*

One interesting consequence of Theorem 1 is that it gives an answer to a well-known problem in the regularity theory of the Navier-Stokes equations. This is explained in [11] in detail, and we briefly recall it here. We consider the classical Cauchy problem for the incompressible Navier–Stokes equations in $R^3 \times (0, \infty)$

$$v_t + v\nabla v + \nabla p - \Delta v = 0 \tag{3}$$

$$\operatorname{div} v = 0 \tag{4}$$

$$v(x, 0) = v_0(x). \tag{5}$$

We assume that v_0 is a smooth divergence-free vector field with suitable decay at ∞. It is known that the problem (3)–(5) has at least one Leray–Hopf weak solution. (See, for example, [9].) As shown in [11], Theorem 1 implies the following result.

Theorem 2. *In the notation introduced above, assume that a Leray–Hopf weak solution v of (3)–(5) is bounded in the space $L^\infty(0, T; L^3(R^3))$. Then v is smooth in $R^3 \times [0, T]$.*

Remark. Strictly speaking, a literal application of Theorem 2. 3 in [11] gives smoothness of v only in $R^3 \times [0, T)$. However, it is easy to see from the local-in-time well-posedness of the Navier–Stokes equations in L^3 ([8]), the weak lower-semicontinuity of the L^3-norm, and Theorem 2.3 in [11] that the stronger conclusion in Theorem 2 can be made.

Apart from Theorem 1, the main step in the proof of Theorem 2 is a blow-up argument. The $L^\infty(0, T; L^3(R^3))$-norm is preserved under the natural scaling of the equation $u(x, t) \to \lambda u(\lambda x, \lambda^2 t)$. Assuming that a singularity develops while $\int_{R^3} |v(x, t)|^3 \, dx$ stays bounded, we can use rescaling at the singularity together with "small data" regularity results to produce a non-trivial weak solution $w \in L^\infty(-\infty, \infty; L^3(R^3))$ which vanishes for $t \geqslant 0$. Moreover, w is regular in domains of the form $(R^3 \setminus \overline{B}_\rho) \times (-t_1, \infty)$, where $t_1 > 0$ and $\rho = \rho(t_1)$. Theorem 1 is then applied to $\omega = \operatorname{curl} w$. Details can be found in [11].

Theorem 1 is also of interest in control theory, see [10]. Namely, one can consider the following control problem.

$$v_t - \Delta v = A(x, t)\nabla v + B(x, t)v \quad \text{in} \quad Q_{R,T} \tag{6}$$

$$v = g(x, t) \quad \text{at} \quad \partial B_R \times (0, T) \tag{7}$$

$$v(x, 0) = v_0(x). \tag{8}$$

We can think of $g(x, t)$, $A(x, t)$, $B(x, t)$ as "controls," and for a given v_0 we can try to find bounded controls so that $v(x, T) = 0$. Theorem 1 says that, unless $v_0 = 0$, this is not possible if we want the growth condition (2) to be satisfied.

We now sketch the main steps in the proof of Theorem 1. A detailed proof will be given in a forthcoming paper. The proof uses two different Carleman ineqaulities. The first inequality, based on ideas developed in [2, 4, 3] and [5], is as follows. We fix $T > 0$, a small $\epsilon > 0$ so that $\epsilon \exp(T^\epsilon) < 1/4$, and set $\sigma(t) = t \exp(-t^\epsilon)$. Then for all functions u compactly supported in $Q_T = R^n \times (0,T)$ and all sufficiently large $a > 0$ one has

$$\int_{Q_T} \left(au^2/t^{2-\epsilon} + |\nabla u|^2/t^{1-\epsilon}\right) \sigma(t)^{-2a} e^{-|x|^2/4t}\, dx\, dt \leqslant C \int_{Q_T} (u_t + \Delta u)^2 \sigma(t)^{-2a} e^{-|x|^2/4t}\, dx\, dt. \tag{9}$$

The reason for using $\sigma(t)$ above rather than just $\sigma(t) = t$ is that, in suitable variables, it has "more convexity." This makes it possible to handle the term with ∇u. A similar trick in the context of elliptic operators can be found already in [1] and [7].

Inequality (9) can be used to prove that, under the assumptions of Theorem 1, one has

$$|u(x,t)| \leqslant C \exp(-|x|^2/Ct) \tag{10}$$

in Q_{R_1,T_1} for some $C > 0$ and a suitable $T_1 > 0$ and $R_1 > 0$.

The next step of the proof uses the following Carleman inequality: We fix $T > 0$, $R_0 > 0$ and let $\varphi_\beta(x,t) = \beta(T-t)(|x| - R_0) + |x|^2$. Then for any u compactly supported in $Q_{R,T}$ and all sufficiently large $\beta > 0$ one has

$$\int_{Q_{R,T}} \left((\beta^2(T-t) + \beta|x|)u^2 + |\nabla u|^2\right) e^{2\varphi_\beta}\, dx\, dt \leqslant C \int_{Q_{R,T}} (u_t + \Delta u)^2 e^{2\varphi_\beta}\, dx\, dt. \tag{11}$$

Using (11) and (10), one can prove Theorem 1 by a slight modification of usual unique continuation arguments. One can also use the following variant of (11) for this step. We let $\psi(x,t) = \frac{T-t}{T}\frac{|x|^{2\alpha}}{t^\alpha}$ with $\alpha \in (1/2, 1)$

fixed. Let us also fix a sufficiently large $R > 0$. Then for all v compactly supported in $Q_{R,T}$ and all sufficiently large $a > 0$ we have

$$\int\limits_{Q_{R,T}} (au^2/t^2 + |\nabla u|^2/t)e^{2a\psi} \leqslant C \int\limits_{Q_{R,T}} (u_t + \Delta u)^2 e^{2a\psi}. \tag{12}$$

Both (11) and (12) seem to be new, although they seem to fit quite well into the standard L^2-Carleman theory. The proofs are based on a suitable integration by parts, as is the proof of (9). The calculations can be organized either by using identities developed in [3], or by following the usual calculation with commutators, as found for example in [6, 12], or [7].

References

1. S. Alinhac, M. S. Baouendi, *Uniqueness for the characteristic Cauchy problem and strong unique continuation for higher order partial differential inequalities.* — Amer. J. Math.} **102** (1980), 179–217.
2. L. Escauriaza, *Carleman inequalities and the heat operator.* — Duke Math. J., **104**, No. 1 (2000), 113–126.
3. L. Escauriaza, F. J. Fernández, *Unique continuation for parabolic operators.* — (to appear).
4. L. Escauriaza, L. Vega, *Carleman inequalities and the heat operator. II.* – Indiana Univ. Math. J. **50**, No. 3 (2001), 1149–1169.
5. F. J. Fernández Unique continuation for parabolic operators. II, (to appear).
6. L. Hörmander, *Linear Partial Differential Operators.* Springer Verlag (1963).
7. L. Hörmander, *Uniqueness theorems for second order elliptic differential equations.* Comm. in PDE, **8**(1) (1983), 21–64.
8. T. Kato, *Strong L^p-solutions of the Navier-Stokes equations in R^m with applications to weak solutions.* — Math. Zeit. **187** (1984), 471–480.
9. O. A. Ladyzhenskaya, *Mathematical problems of the dynamics of viscous incompressible fluids.* Gorden and Breach (1969).
10. S. Micu, E. Zuazua, *On the lack of null-controllability of the heat equation on the half space.* — Portugaliae Mathematica, **58**, 1 (2001), 1–24.
11. G. Seregin, V. Šverák, *Navier–Stokes and backward uniqueness for the heat equation.* — IMA preprint No. 1852 (April 2002), downloadable from http://www.ima.umn.edu.
12. F. Trèves, *Linear Partial Differential Equations.* Gordon and Breach (1970).

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