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ESTIMATES OF SOLUTIONS OF THE STOKES EQUATIONS IN S. L. SOBOLEV SPACES WITH A MIXED NORM

Abstract. We obtain some estimates of solutions of evolution Stokes and Navier-Stokes
equations in a bounded n-dimensional domain and we analyse the structure of the solution in the case of half-space using explicit formulas.

Dedicated to O. A. Ladyzhenskaya on the occasion of her jubilee

1. Introduction

Let Ω be a bounded domain in R^n, $n \geq 2$, with C^2-smooth boundary S. We consider initial boundary-value problem

$$\vec{v}_t - \nu\Delta\vec{v} + \nabla p = \vec{f}(x,t), \qquad \nabla\cdot\vec{v} = g(x,t), \qquad x \in \Omega,\ t \in (0,T),$$
$$\vec{v}(x,0) = \vec{v}_0(x), \qquad \vec{v}(x,t)|_{x\in S} = 0 \tag{1.1}$$

in anisotropic S. L. Sobolev spaces $W^{2,1}_{q,r}(Q^T)$, $Q^T = \Omega\times(0,T)$, with a mixed norm. We denote by $W^l_q(\Omega)$ with an integral positive l and with $q > 1$ a standard Sobolev space of scalar or vector-valued functions $u(x)$, $x \in \Omega$, with the norm

$$\|u\|_{W^l_q(\Omega)} = \Big(\sum_{0\leq|j|\leq l}\int_\Omega |D^j u(x)|^q dx\Big)^{1/q}.$$

By $W^{-1}_q(\Omega)$ we mean the space dual with respect to $W^1_{q'}(\Omega)$, $q' = q/(q-1)$, with a standard norm

$$\|u\|_{W^{-l}_q(\Omega)} = \sup_{\varphi\in W^1_{q'}(\Omega)} \frac{|\int_\Omega u(x)\varphi(x)dx|}{\|\varphi\|_{W^1_{q'}(\Omega)}}$$

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(in the case of scalar $u(x)$). Finally, we define the space $W^{2,1}_{q,r}(Q^T)$ as the set of functions $u(x,t)$, $x \in \Omega$, $t \in [0,T]$, possessing generalized derivatives $u_t(x,t)$, $D^j_x u(x,t)$, $|j| = 0,1,2$, such that the norm

$$\|u\|_{W^{2,1}_{q,r}(Q^T)} = \Big(\|u_t\|^r_{L_{q,r}(Q^T)} + \sum_{0\le|j|\le 2} \|D^j_x u\|^r_{L_{q,r}(Q^T)}\Big)^{1/r}, \tag{1.2}$$

where

$$\|u\|_{L_{q,r}(Q^T)} = \Big(\int_0^T \|u(\cdot,t)\|^r_{L_q(\Omega)} dt\Big)^{1/r}, \quad q,\ r > 1,$$

is finite.

It is known (see [1]) that the space of traces of functions from $W^{2,1}_{q,r}(Q^T)$ on the hyperplane $t = t_0 \in [0,T]$ coinsides with the O. V. Besov space $B^{2-2/r}_{q,r}(\Omega)$. If $\Omega = R^n$ or $\Omega = R^n_+ = \{x_n > 0\}$, then the norm in the space $B^l_{q,r}(\Omega)$ with arbitrary $l > 0$ can be defined by

$$\|u\|_{B^l_{q,r}(\Omega)} = \Big(\|u\|^r_{W^{\bar l}_q(\Omega)} + \|u\|^r_{\dot B^l_{q,r}(\Omega)}\Big)^{1/r}, \tag{1.3}$$

where $\bar l$ is a maximal integral number which is less than l,

$$\|u\|_{\dot B^l_{q,r}(\Omega)} = \Big(\sum_{|j|=\bar l} \|D^j u\|^r_{\dot B^{l-\bar l}_{q,r}(\Omega)}\Big)^{1/r},$$

$$\|v\|_{\dot B^\lambda_{q,r}(\Omega)} = \Big(\sum_{i=1}^n \int_0^\infty \|\Delta_i^{[\lambda]+1}(h)u\|^r_{L_q(\Omega)} \frac{dh}{h^{1+r\lambda}}\Big)^{1/r}, \tag{1.4}$$

and $\Delta^k_i(h)u(x)$ is a finite difference of the function $u(x)$ of the order k with respect to x_i. In particular,

$$\Delta^1_i(h)u(x) \equiv \Delta_i(h)u(x) = u(x + e_i h) - u(x), \quad e_i = (\delta_{ik})_{k=1,\dots,n},$$

$$\Delta^2_i(h)u(x) = u(x + 2e_i h) - 2u(x + e_i h) + u(x).$$

If Ω is a domain or a smooth manifold, then the norm in $B^l_{q,r}(\Omega)$ can be defined in a standard manner by means of partition of unity and local maps.

There are many other equivalent norms in $B^l_{q,r}(\Omega)$ (see [2, 3]).

If $l > 1/q$, then every function from $B^l_{q,r}(\Omega)$ has a trace on the boundary S belonging to $B^{l-1/q}_{q,r}(S)$ and

$$\|u\|_{B^{l-1/q}_{q,r}(S)} \leq c\|u\|_{B^l_{q,r}(\Omega)}.$$

(see [2]).

We prove the following theorem concerning the solvability of problem (1.1).

Theorem 1.1. *Assume that the functions $\vec{f}(x,t)$, $\vec{v}_0(x)$, and $g(x,t)$ possess the following properties.*

(i) *$\vec{f} \in L_{q,r}(Q^T)$, $g \in L_r(0,T;W^1_q(\Omega))$, $g_t \in L_r(0,T;W^{-1}_q(\Omega))$, and $\vec{v}_0 \in B^{2-2/r}_{q,r}(\Omega)$, moreover, $\vec{v}_0(x)|_S = 0$ if $2-2/r > 1/q$ and $\vec{v}_0$ has a finite norm*

$$\|\vec{v}^{(0)}_0\|_{B^{2-2/r}_{q,r}(R^n)},$$

where $\vec{v}^{(0)}_0(x) = \vec{v}_0(x)$ for $x \in \Omega$, $\vec{v}^{(0)}_0 = 0$ for $x \in R^n \setminus \Omega$ if $2-2/r = 1/q$;

(ii) *there holds the compatibility condition $\nabla \cdot \vec{v}_0 = g(x,0)$ understood in the following sense:*

$$\lim_{t\to 0}\int\limits_\Omega \nabla\varphi(x)\cdot(\vec{V}(x,t) - \vec{v}_0(x))dx = 0 \tag{1.5}$$

for arbitrary smooth $\varphi(x)$ and arbitrary $\vec{V} \in W^{2,1}_{q,r}(Q^T)$ such that $\nabla \cdot \vec{V}(x,t) = g(x,t)$.

Then problem (1.1) has a unique solution $\vec{v} \in W^{2,1}_{q,r}(Q^T)$ and $\nabla p \in L_{q,r}(Q^T)$, and the solution satisfies the inequality

$$\begin{aligned}\|\vec{v}\|_{W^{2,1}_{q,r}(Q^T)} + \|\nabla p\|_{L_{q,r}(Q^T)} \leq c\Big(\|\vec{f}\|_{L_{q,r}(Q^T)}\\ +\|\vec{v}_0\|^{(2-2/r)}_\Omega + \|g\|_{L_r(0,T;W^1_q(\Omega))} + \|g_t\|_{L_r(0,T;W^{-1}_q(\Omega))}\Big)\end{aligned} \tag{1.6}$$

where

$$\begin{aligned}\|\vec{v}_0\|^{(2-2/r)}_\Omega &= \|\vec{v}_0\|_{B^{2-2/r}_{q,r}(\Omega)} \qquad \textit{if}\quad 2-2/r \neq 1/q,\\ \|\vec{v}_0\|^{(2-2/r)}_\Omega &= \|\vec{v}^{(0)}_0\|_{B^{2-2/r}_{q,r}(R^n)} \qquad \textit{if}\quad 2-2/r = 1/q.\end{aligned}$$

For $g = 0$, the estimate (1.6) was proved by different methods in [4] and in [5], where the case of exterior domain Ω was considered (therefore the norm $\|\vec{v}_0\|_\Omega^{(2-2/r)}$ was defined in a different way). The case $g \neq 0$ was also studied in [6].

G. A. Seregin has obtained a local version of the estimate (1.6) and used it in the proof of local regularity near a flat part of the boundary of solution to the Navier–Stokes equations, under different assumptions concerning the solutions (see [7–9]). In particular, he has established the Hölder continuity of the solutions belonging to the class $L_{q,\infty}(Q_\rho)$, $q > 3 = n$, in a certain "parabolic cylinder" Q_ρ ajacent to the boundary [7].

We write a local estimate in a slightly different form.

Let $\omega_\rho(x_0) = \Omega \cap \{|x - x_0| < \rho\}$ where $\rho > 0$, $x_0 \in \bar{\Omega}$. By $Q_\rho(x_0, t_0) \equiv Q_\rho$, $t_0 \in (0, T]$, we mean "a parabolic cylinder"

$$Q_\rho = \omega_\rho(x_0) \times (t_0 - \rho^2, t_0).$$

Theorem 1.2. *Assume that $\vec{v} \in W^{2,1}_{q,r}(Q_\rho)$, $\nabla p \in L_{q,r}(Q_\rho)$ satisfy in Q_ρ the equations*

$$\vec{v}_t - \nu \Delta \vec{v} + \nabla p = \vec{f}(x,t), \quad \nabla \cdot \vec{v} = 0 \tag{1.7}$$

with $\vec{f} \in L_{q,r}(Q_\rho)$ and that $t_0 - \rho^2 > 0$, and

$$\vec{v}(x,t)|_{x \in s_\rho(x_0)} = 0, \tag{1.8}$$

where $s_\rho(x_0) = \partial\omega_\rho(x_0) \cap S$. Then

$$\begin{aligned} &\|\vec{v}\|_{W^{2,1}_{q,r}(Q_{\theta\rho})} + \|\nabla p\|_{L_{q,r}(Q_{\theta\rho})} \\ &\leq c\Big(\|\vec{f}\|_{L_{q,r}(Q_\rho)} + \rho^{-\mu_1}\|\vec{v}\|_{L_{q_1,r_1}(Q_\rho)} + \rho^{-\mu_2}\|p\|_{L_{q_2,r}(Q_\rho)}\Big) \end{aligned} \tag{1.9}$$

where $\theta \in (0,1)$, q_1, $q_2 \in [1, q]$, $r_1 \in [1, r]$, $\mu_1 = 2 - n/q - 2/r + n/q_1 + 2/r_1$, $\mu_2 = 1 - n/q + n/q_2$.

If $\vec{v} \in W^{2,1}_{q_0,r}(Q_\rho)$, p, $\nabla p \in L_{q_0,r}(Q_\rho)$ satisfy (1.7) and (1.8) with $\vec{f} \in L_{q,r}(Q_\rho)$, $q > q_0$, then $\vec{v} \in W^{2,1}_{q_0,r}(Q_\rho)$, $\nabla p \in L_{q,r}(Q_{\rho'})$ in arbitrary smaller cylinder $Q_{\rho'} : \rho' \in (0, \rho)$.

We could assume that $t_0 - \rho^2 < 0$; in this case the term c$\|\Phi \vec{v}_0\|_{\omega_\rho}^{(2-2/r)}$ appears in the right hand side of (1.9). Here $\Phi(x)$ is a smooth cut-off function, $\Phi(x) = 1$ in $\omega_{\theta\rho}(x_0)$, $\Phi(x) = 0$ for $|x - x_0| > \rho$.

Concerning solutions of the Navier–Stokes equations, we prove the following proposition.

Theorem 1.3. *Let $\vec{v} \in W^{2,1}_{q_0,r_0}(Q_\rho)$, $\nabla p \in L_{q_0,r_0}(Q_\rho)$, where*

$$\frac{n}{q_0} + \frac{2}{r_0} = 1 + n \tag{1.10}$$

and $r_0 < 2$ is sufficiently close to 2, satisfy in Q_ρ the Navier–Stokes equations

$$\vec{v}_t - \nu\Delta\vec{v} + \nabla p = \vec{f} - (\vec{v}\cdot\nabla)\vec{v}, \qquad \nabla\cdot\vec{v} = 0$$

and the boundary conditions

$$\vec{v}(x,t)|_{x\in s(x_0)} = 0,$$

where $s(x_0) = \partial\omega(x_0)\cap S$. Assume that $\nabla\vec{v} \in L_{2,2}(Q_\rho)$, $\vec{f}(x,t)$ is bounded in Q_ρ and that $t_0 - \rho^2 > 0$. If $\vec{v} \in L_{\sigma,\tau}(Q_\rho(x_0,t_0))$ with

$$\frac{n}{\sigma} + \frac{2}{\tau} = 1 - \epsilon, \qquad \epsilon > 0, \tag{1.11}$$

then $\vec{v} \in W^{2,1}_{M,r_0}(Q_{\rho'})$ with $M \gg 1$ in arbitrary smaller cylinder $Q_{\rho'}$: $\rho' < \rho$.

Theorem 1.4. *The statement of Theorem* 3.1 *remains true if σ and τ satisfy the conditions*

$$\frac{n}{\sigma} + \frac{2}{\tau} = 1, \qquad \sigma > n. \tag{1.12}$$

It follows from the imbedding theorems for the spaces $W^{2,1}_{M,r_0}(Q_{\rho'})$ that $\vec{v}(x,t)$ is Hölder continuous with respect to x and t in $Q_{\rho'}$ and that $\nabla\vec{v} \in L_{\infty,R}(Q_{\rho'})$ with R satisfying the relation

$$\frac{1}{R} = \frac{1}{r_0} - \frac{1}{2} + \frac{n}{2M}$$

($R \gg 1$ if $M \gg 1$ and r_0 is sufficiently close to 2). If the cylinder Q_ρ is a strictly interior subdomain of Q^{t_0}, and $\vec{f}$ is sufficiently regular, then we can differentiate the Navier–Stokes equations with respect to x_k and obtain the same result for all the derivatives $D^j_x\vec{v}(x,t)$, $|j| = 1, 2, \ldots$. This leads to well known result of J. Serrin [10]: if conditions (1.11) or (1.12) are satisfied in a strictly interior "parabolic cylinder," then all the derivatives $D^j_x\vec{v}(x,t)$ are bounded in a smaller cylinder. However, in the case $x_0 \in S$ this argument applies only to tangential derivatives of $\vec{v}$, and in general it seems difficult to expect the boundedness even of the first derivatives $\vec{v}_{x_k}(x,t)$ in $Q_{\rho'}$. We illustrate this (without a rigorous proof) in the case of half-space, analysing explicit formula for the solution of the Stokes problem.

In the case $\Omega = \mathbb{R}^n_+$, the Hölder continuity near the boundary of the solutions of the Navier–Stokes equations satisfying the conditions (1.12) is established in [11].

Finally, we consider initial boundary-value problem

$$\begin{aligned} &\vec{v}_t - \nu\Delta\vec{v} + \nabla p = \vec{f}(x,t), \qquad \nabla\cdot\vec{v} = 0, \qquad x \in R^n_+, \\ &\quad t \in (0,T), \qquad \vec{v}(x,0) = 0, \qquad \vec{v}(x,t)|_{x_n=0} = 0 \end{aligned} \tag{1.13}$$

and prove the following theorem.

Theorem 1.5. *Assume that* $\vec{f} \in L_{q_0,r_0}(\Pi^{t_0}_+)$ *and that* $\vec{f} \in L_{q,r}(Q_\rho)$, *where* $q > q_0, r > r_0$, $Q_\rho = \omega_\rho(x_0) \times (t_0 - \rho^2)$, $\omega_\rho(x_0) = \{x \in R^n_+ : |x - x_0| < \rho\}$, *and* $x_0 \in \bar{R}^n_+$. *Then the solution of problem* (1.13) *is representable in the form*

$$(\vec{v}, p) = (\vec{v}_1, p_1) + (\vec{v}_2, p_2) + (\vec{v}_3, p_3), \tag{1.14}$$

where $\nabla p_1 \in L_{q,r}(\Pi^{t_0}_+) \cap L_{q_0,r_0}(\Pi^{t_0}_+)$,

$$\|\nabla p_1\|_{L_{q,r}(\Pi^{t_0}_+)} \le \|\vec{f}\|_{L_{q,r}(Q_\rho)}, \tag{1.15}$$

$$\|p_1\|_{L_{q,r}(Q_{\rho_1})} \le c\|\vec{f}\|_{L_{q,r}(Q_\rho)}, \quad \rho_1 < \rho, \tag{1.16}$$

$p_2(x,t)$ *is a harmonic function in* Q_{ρ_1},

$$\|D^j\nabla p_2\|_{L_{\infty,r_0}(Q_{\rho_1})} \le c\|\vec{f}\|_{L_{q_0,r_0}(\pi(\rho,\rho_1))}, \qquad |j| = 0,1,\dots \tag{1.17}$$

where $\pi(\rho,\rho_1) = (R^n_+ \setminus \omega_{\rho'}(x_0)) \times (t_0 - {\rho'}^2, t_0)$, *and* $\rho' = (\rho+\rho_1)/2$,

$$\|\nabla p_2\|_{L_{q_0,r_0}(Q_{\rho_1})} \le c\|\vec{f}\|_{L_{q_0,r_0}(\pi(\rho,\rho_1))}, \tag{1.18}$$

$$\|p_2\|_{L_{\infty,r_0}(Q_{\rho_1})} \le c\|\vec{f}\|_{L_{q_0,r_0}(\pi(\rho,\rho_1))} \ \ \textit{if} \ \ q < n, \tag{1.19}$$

$$\|p_2\|_{L_{\infty,r_0}(Q_{\rho_1})} \le c\Big(\|\vec{f}\|_{L_{q_0,r_0}(\pi(\rho,\rho_1))} + \|p_2\|_{L_{n-\epsilon,r}(Q_{\rho_1})}\Big), \quad \forall\epsilon > 0 \tag{1.20}$$

if $q > n$ *and* $\vec{f} \in L_{n-\epsilon,r}(\pi_{\rho,\rho_1})$. *Finally,* $p_3(x,t)$ *is a harmonic infinitely differentiable function in* Q_{ρ_1} *with respect to* x *and* t, *and*

$$\sup_{Q_{\rho_1}} |D^j_x D^h_t p_3(x,t)| \le c\|\vec{f}\|_{L_{q_0,r_0}(\Pi^{t_0-{\rho'}^2})}, \quad |j|, \quad h = 0,1,\dots. \tag{1.21}$$

The functions $\vec{v}_i(x,t)$ satisfy the inequalities

$$\|\vec{v}_1\|_{W^{2,1}_{q,r}(Q_{\rho_1})} \leq c\|\vec{f}\|_{L_{q,r}(Q_\rho)} \tag{1.22}$$

and

$$\|\vec{v}_2\|_{W^{2,1}_{M,r}(Q_{\rho_1})} \leq c\|\vec{f}\|_{L_{q_0,r_0}(\pi(\rho,\rho_1))}, \tag{1.23}$$

where M is an arbitrarily large number, and

$$\sup_{Q_{\rho_1}} |D^j_x D^h_t \vec{v}_3(x,t)| \leq c\|\vec{f}\|_{L_{q_0,r_0}(\Pi^{t_0-\rho'^2})}, \quad |j|, h = 0, 1, \dots. \tag{1.24}$$

2. Auxiliary propositions

First of all, we recall some imbedding and trace theorems for the spaces $W^{2,1}_{q,r}(Q^T)$.

Proposition 2.1. *Arbitrary function $u \in W^{2,1}_{q,r}(Q^T)$ belongs to $L_{p_1,s_1}(Q^T)$, $\nabla u \in L_{p_2,s_2}(Q^T)$, and $\nabla u \in L_{p_3,s_3}(\Sigma^T)$, $\Sigma^T = S \times (0,T)$, where $p_i \geq q$, $s_i \geq r$ satisfy the conditions*

$$1 - \frac{n}{q} - \frac{2}{r} + \frac{n}{p_2} + \frac{2}{s_2} \equiv \kappa_2 \geq 0,$$
$$1 - \frac{n}{q} - \frac{2}{r} + \frac{n-1}{p_3} + \frac{2}{s_3} \equiv \kappa_3 \geq 0,$$

and there hold the estimates

$$\|u\|_{L_{p_1,s_1}(Q^T)} \leq \eta_1^{\kappa_1}\|u\|_{W^{2,1}_{q,r}(Q^T)} + c_1\eta_1^{\kappa_1-2}\|u\|_{L_{q,r}(Q^T)}, \tag{2.1}$$

$$\|\nabla u\|_{L_{p_2,s_2}(Q^T)} \leq \eta_2^{\kappa_2}\|u\|_{W^{2,1}_{q,r}(Q^T)} + c_2\eta_2^{\kappa_2-1}\|u\|_{L_{q,r}(Q^T)}, \tag{2.2}$$

$$\|\nabla u\|_{L_{p_3,s_3}(\Sigma^T)} \leq \eta_3^{\kappa_3}\|u\|_{W^{2,1}_{q,r}(Q^T)} + c_3\eta_3^{\kappa_3-1}\|u\|_{L_{q,r}(Q^T)} \tag{2.3}$$

with arbitrary $\eta_i \leq \eta_0$. If $\kappa_i > 0$, then both p_i and s_i may be infinite. If $u \in W^{2,1}_{q.r}(Q^T)$, then $u|_{t=t_0} \in B^{2-2/r}_{q,r}(\Omega)$ and

$$\|u\|_{B^{2-2/r}_{q,r}(\Omega)} \leq c\|u\|_{W^{2,1}_{q,r}(Q^T)}.$$

For arbitrary $\varphi \in B^{2-2/r}_{q,r}(\Omega)$ there exists $u \in W^{2,1}_{q,r}(Q^T)$ such that $u(x,t_0) = \varphi(x)$ and

$$\|u\|_{W^{2,1}_{q,r}(Q^T)} \leq c\|\varphi\|_{B^{2-2/r}_{q,r}(\Omega)}.$$

The proof of this proposition can be found in [1, 2].

We need the following "inverse trace theorem."

Proposition 2.2. *For arbitrary* $v_0 \in B^{2-2/r}_{q,r}(\Omega)$ *satisfying the hypotheses of Theorem 1.1* ($v_0|_S = 0$, *if* $2 - 2/r > 1/q$, $\|v_0\|^{(2-2/r)}_{\Omega} < \infty$, *if* $2 - 2/r = 1/q$), *there exists* $V_0 \in W^{2,1}_{q,r}(Q^T)$ *such that* $V_0(x, 0) = v_0(x)$, $V_0(x, t)|_{x\in S} = 0$ *and*

$$\|V_0\|_{W^{2,1}_{q,r}(Q^T)} \le c\|v_0\|^{(2-2/r)}_{\Omega}. \tag{2.4}$$

Proof. Let $\{\psi_k(x)\}$, $k = 1, \dots, k_1 + k_2$ be a smooth "partition of unity" in $\Omega : 1 = \sum_{k=1}^{k_1+k_2} \psi_k(x)$, with supp $\psi_k \subset \omega_k$, where ω_k are strictly interior subdomains for $k = 1, \dots, k_1$ and adjacent to the boundary for $k = k_1 + 1, \dots, k_2$. We assume that there exists a transformation $y^{(k)} \equiv y = T_k x$ which maps ω_k, $k = k_1 + 1, \dots, k_2$, into a subdomain $\tilde{\omega}_k$ of the half-space R^n_+ contained in a semi-ball $|y| \le d$, $y_n \ge 0$. Let $\tilde{\psi}_k(y) = \psi_k(T_k^{-1} y)$ and $\tilde{v}_{0k}(y) = v_0(T_k^{-1} y)$. We extend $v_k(x) = v_0(x)\psi_k(x)$, $k = 1, \dots, k_1$ by zero into $R^n \setminus \omega_k$ and $v_k(y) = \tilde{v}_{0k}(y)\tilde{\psi}_k(y)$ into $R^n_+ \setminus \tilde{\omega}_k$ and consider the problem of construction of the functions V_k possessing all the required properties in the space and in the half-space. We set

$$V_k(x, t) = \int\limits_{R^n} \Gamma(x - z, t) v_k(z) dz, \quad k = 1, \dots, k_1,$$

$$\begin{aligned} V_k(y, t) &= \int\limits_{R^n_+} \Big(\Gamma(y - z, t) - \Gamma(y - z^*, t)\Big) v_k(z) dz \\ &= \int\limits_{R^n} \Gamma(y - z, t) v_k^-(z) dz, \quad k = k_1 + 1, \dots, k_2, \end{aligned}$$

where $z^* = (z_1, \dots, z_{n-1}, -z_n)$,

$$\Gamma(z, t) = \frac{1}{(4\pi\nu t)^{n/2}} \exp\left(-\frac{|z|^2}{4\nu t}\right)$$

is a fundamental solution of the heat equation $u_t - \nu\Delta u = 0$ and v_k^- is an odd (with respect to y_n) extension of v_k into $R^n_- \equiv \{x_n < 0\}$. It was shown in [5] that

$$\|V_{kt}\|^r_{L_{q,r}(\Pi^T)} + \sum_{|j|=2} \|D^j V_k\|^r_{L_{q,r}(\Pi^T)} \le c\|v_k\|^r_{\dot{B}^{2-2/r}_{q,r}(R^n)},$$

$$\|V_k\|^r_{L_{q,r}(\Pi^T)} \le c\Big(\|v_k\|^r_{L_q(R^n)} + \|V_{kt}\|^r_{L_{q,r}(\Pi^T)}\Big), \quad k = 1, \dots, k_1,$$

$$\|V_{kt}\|^r_{L_{q,r}(\Pi^T_+)} + \sum_{|j|=2} \|D^j V_k\|^r_{L_{q,r}(\Pi^T_+)} \le c\|v^-_k\|^r_{\dot B^{2-2/r}_{q,r}(R^n)},$$

$$\|V_k\|^r_{L_{q,r}(\Pi^T_+)} \le c\Big(\|v_k\|^r_{L_q(R^n)} + \|V_{kt}\|^r_{L_{q,r}(\Pi^T_+)}\Big), \qquad k = k_1+1, , \dots , k_2,$$

where $\Pi^T = R^n \times (0,T)$, $\Pi^T_+ = R^n_+ \times (0,T)$. The norm of v^-_k can be estimated in the following way: if $2-2/r < 1+1/q$, then we can use an evident inequality

$$\|v^-_k\|^r_{\dot B^{2-2/r}_{q,r}(R^n)} \le c\|v^{(0)}_k\|^r_{\dot B^{2-2/r}_{q,r}(R^n_+)}$$

and if $2-2/r > 1$, then we have

$$\|v^-_k\|^r_{\dot B^{2-2/r}_{q,r}(R^n)} = \Big(\sum_{i=1}^{n-1} \|v^-_{kx_i}\|^r_{\dot B^{1-2/r}_{q,r}(R^n)} + \|(v_{kx_n})^+\|^r_{\dot B^{1-2/r}_{q,r}(R^n)}\Big),$$

where v^+ is an even extension of v into R^n_-. It follows from the estimates obtained in [5] that

$$\|(v_{kx_n})^+\|_{\dot B^{1-2/r}_{q,r}(R^n)} \le c\|v_{kx_n}\|_{\dot B^{1-2/r}_{q,r}(R^n_+)}$$

hence,

$$\|V_k\|^r_{W^{2,1}_{q,r}(\Pi^T_+)} \le c\Big(\|v_k\|^{(2-2/r)}_{R^n_+}\Big)^r.$$

We define $V_0(x,t)$, $x \in \Omega$, as

$$V_0(x,t) = \sum_{k=1}^{k_1} \chi_k(x) V_k(x,t) + \sum_{k=k_1+1}^{k_2} \chi_k(T^{-1}_k y) V_k(T^{-1}_k y, t)$$

where $\chi_k(x)$ are smooth functions with compact supports such that $\chi_k(x)\psi_k(x) = \psi_k(x)$. Putting the above estimates together we arrive at (2.4). The proposition is proved.

We also present the theorem of imbedding of $W^{2,1}_{q,r}(Q^T)$ into Hölder spaces of functions.

Proposition 2.3. *If $u \in W^{2,1}_{q,r}(Q^T)$ with $2-n/q-2/r \equiv b > 0$, then $u \in B^{b,b/2}_{\infty,\infty}(Q^T)$, and*

$$\|u\|_{B^{b,b/2}_{\infty,\infty}(Q^T)} \le c\|u\|_{W^{2,1}_{q,r}(Q^T)}.$$

Proof. It is enough to consider the case $\Omega = R^n$, since every function $u \in W^{2,1}_{q,r}(Q^T)$ can be extended into Π^T with preservation of class (see [2]). We also extend u into the half-space $\{t < 0\}$ with preservation of class in such a way that $u(x,t) = 0$ for $t < -2T$. From the representation formula

$$u(x,t) = \int_{-\infty}^{t} \int_{R^n} \Gamma(x-y, t-\tau)\big(u_\tau(y,\tau) - \nu\Delta u(y,\tau)\big)dyd\tau$$

it follows that

$$|\Delta_i^{[b]+1}(h)u(x,t)| \le c\|u\|_{W^{2,1}_{q,r}(\Pi^T)} \|\Delta_i^{[b]+1}(h)\Gamma\|_{L_{q',r'}(R^{n+1})},$$

$$|\Delta_t(h)u(x,t)| \le c\|u\|_{W^{2,1}_{q,r}(\Pi^T)} \|\Delta_t(h)\Gamma\|_{L_{q',r'}(R^{n+1})},$$

where $[b]$ is an integral part of b and $\Delta_t(h)$ denotes finite difference with respect to t. Easy computations show that

$$\|\Delta_i^{[b]+1}(h)\Gamma\|_{L_{q',r'}(R^{n+1})} \le ch^b,$$

$$\|\Delta_t(h)\Gamma\|_{L_{q',r'}(R^{n+1})} \le ch^{b/2},$$

hence,

$$\|u\|_{\dot{B}^{b,b/2}_{\infty,\infty}(\Pi^T)} \le c\|u\|_{W^{2,1}_{q,r}(\Pi^T)}.$$

Using also (2.1) with $p_1 = s_1 = \infty$ we arrive at the desired estimate. The proposition is proved.

The space $B^{b,b/2}_{\infty,\infty}(\Pi^T)$ coinsides with the Hölder space $C^{b,b/2}(Q^T)$ for $b < 1$ and for $b > 1$.

Finally, we consider an auxiliary stationary problem

$$-\nu\Delta\vec{u}(x) + \nabla s(x) = 0, \quad \nabla\cdot\vec{u} = g(x), \quad x \in \Omega, \quad \vec{u}|_S = 0. \tag{2.5}$$

Proposition 2.4. *If $g \in W^1_q(\Omega)$, $q > 1$, and $\int_\Omega g(x)dx = 0$, then problem (2.5) has a solution $\vec{u} \in W^2_q(\Omega)$, $s \in W^1_q(\Omega)$, which is unique, up to an additive constant for the pressure, and satisfies the inequalities*

$$\|\vec{u}\|_{W^2_q(\Omega)} + \|\nabla s\|_{L_q(\Omega)} \le c_1\|g\|_{W^1_q(\Omega)} \tag{2.6}$$

and

$$\|\vec{u}\|_{L_q(\Omega)} \le c_2\|g\|_{W^{-1}_q(\Omega)}. \tag{2.7}$$

In particular, if

$$g(x) = \nabla \cdot \vec{h}(x) + h_0(x), \tag{2.8}$$

then

$$\|\vec{u}\|_{L_q(\Omega)} \leq c_2 \Big(\|\vec{h}\|_{L_q(\Omega)} + \|h_0\|_{L_{q_1}(\Omega)} + \|\vec{h} \cdot \vec{n}\|_{L_{q_2}(S)}\Big), \tag{2.9}$$

where $q_1, q_2 \leq q$ *satisfy the conditions*

$$\frac{1}{q_1} \leq \frac{1}{q} + \frac{1}{n}, \quad \frac{1}{q_2} \leq \frac{n}{(n-1)q}. \tag{2.10}$$

If $\operatorname{supp} \vec{h}$, $\operatorname{supp} h_0 \in \omega_\rho(x_0) \equiv \bar{\Omega} \cap \{x - x_0| \leq \rho\}$, $\rho \in (0,1)$, *then*

$$\|\vec{u}\|_{L_q(\Omega)} \leq c_2 \Big(\|\vec{h}\|_{L_q(\Omega)}$$
$$+\rho^{1+n/q-n/q_1} \|h_0\|_{L_{q_1}(\Omega)} + \rho^{n/q-(n-1)/q_2} \|\vec{h} \cdot \vec{n}\|_{L_{q_2}(S)}\Big). \tag{2.11}$$

Proof. The first statement of the theorem and the estimate (2.6) are well known. (2.7) is proved by duality arguments (see [7, 12]). We consider an auxiliary problem

$$-\nu \Delta \vec{\varphi}(x) + \nabla \pi(x) = \vec{\psi}(x), \quad \nabla \cdot \vec{\varphi}(x) = 0,$$
$$\vec{\varphi}(x)|_S = 0$$

with arbitrary smooth $\vec{\psi}(x)$. Integrating by parts we easily obtain

$$\int\limits_\Omega \vec{u}(x) \cdot \vec{\psi}(x) dx = -\int\limits_\Omega g \pi dx = \int\limits_\Omega g(\pi - \bar{\pi}) dx, \tag{2.12}$$

where $\bar{\pi} = |\Omega|^{-1} \int_\Omega \pi(x) dx$. From this relation and from the Poincaré inequality it follows that

$$\Big| \int\limits_\Omega \vec{u}(x) \cdot \vec{\psi}(x) dx \Big| \leq \|g\|_{W_q^{-1}(\Omega)} \|\pi - \bar{\pi}\|_{W^1_{q_1'}(\Omega)} \leq c \|g\|_{W_q^{-1}(\Omega)} \|\nabla \pi\|_{L_{q'}(\Omega)}$$
$$\leq c \|g\|_{W_q^{-1}(\Omega)} \|\vec{\psi}\|_{L_{q'}(\Omega)}, \quad q' = q/(q-1).$$

Since $\vec{\psi}$ is arbitrary, this implies (2.4). If $g(x) = \nabla \cdot \vec{h}(x) + h_0(x)$, then

$$\int\limits_\Omega g(\pi - \bar{\pi}) dx = -\int\limits_\Omega \vec{h} \cdot \nabla \pi dx + \int\limits_\Omega h_0 (\pi - \bar{\pi}) dx + \int\limits_S \vec{h} \cdot \vec{n} (\pi - \bar{\pi}) dx$$

and

$$\Big|\int\limits_\Omega \vec u(x)\cdot\vec\psi(x)dx\Big| \le c\Big(\|\vec h\|_{L_q(\Omega)}\|\nabla\pi\|_{L_{q'}(\Omega)} + \|h_0\|_{L_{q_1}(\Omega)}\|\pi-\bar\pi\|_{L_{q_1'}(\Omega)}$$

$$+\|\vec h\cdot\vec n\|_{L_{q_2}(S)}\|\pi-\bar\pi\|_{L_{q_2'}(S)}\Big)\|\vec\psi\|_{L_{q'}(\Omega)}.$$

with q_1' and $q_2'\ge q'$ satisfying the conditions

$$1-\frac{n}{q'}+\frac{n}{q_1'}\ge 0,\ 1-\frac{n}{q'}+\frac{n-1}{q_2'}\ge 0$$

equivalent to (2.10). This implies (2.9). Assume finally that $\operatorname{supp}\vec h$, $\operatorname{supp} h_0 \in \omega_\rho(x_0)\equiv\bar\Omega\cap\{|x-x_0|\le\rho\}$. It is clear that (2.12) holds with the constant $\bar\pi$ replaced by $\bar\pi_\rho = |\omega_\rho(x_0)|^{-1}\int_{\omega_\rho(x_0)}\pi(x)dx$.

We have

$$\Big|\int\limits_\Omega h_0(\pi-\bar\pi_\rho)dx\Big| \le \|h_0\|_{L_{q_1}(\omega_\rho(x_0))}\|\pi-\bar\pi_\rho\|_{L_{q_1'}(\omega_\rho(x_0))}$$

$$\le c\rho^{1-n/q'+n/q_1'}\|h_0\|_{L_{q_1}(\omega_\rho(x_0))}\|\nabla\pi\|_{L_{q'}(\omega_\rho(x_0))}$$

and, by virtue of an interpolation inequality,

$$\Big|\int\limits_S \vec h\cdot\vec n(\pi-\bar\pi_\rho)dS\Big| \le \|\vec h\cdot\vec n\|_{L_{q_2}(s_\rho)}\|\pi-\bar\pi_\rho\|_{L_{q_2'}(s_\rho)}$$

$$\le c\|\vec h\cdot\vec n\|_{L_{q_2}(s_\rho)}\Big(\rho^{1-n/q'+(n-1)/q_2'}\|\nabla\pi\|_{L_{q'}(\omega_\rho(x_0))}$$

$$+\rho^{-n/q'+(n-1)/q_2'}\|\pi-\bar\pi_\rho\|_{L_{q'}(\omega_\rho(x_0))}\Big)$$

$$\le c\rho^{n/q-(n-1)/q_2}\|\vec h\cdot\vec n\|_{L_{q_2}(s_\rho)}\|\nabla\pi\|_{L_{q'}(\omega_\rho(x_0))}.$$

Hence, in this case

$$\|g\|_{W_q^{-1}(\Omega)} \le c\Big(\|\vec h\|_{L_q(\omega_\rho)} + \rho^{1+n/q-n/q_1}\|h_0\|_{L_{q_1}(\omega_\rho(x_0))}$$

$$+\rho^{n/q-(n-1)/q_2}\|\vec h\cdot\vec n\|_{L_{q_2}(s_\rho)}\Big). \tag{2.13}$$

Collecting the estimates we arrive at (2.11). The proposition is proved.

3. Proof of Theorems 1.1–1.4

Proof of Theorem 1.1. We reduce problem (1.1) to a similar problem with $g = 0$ and $\vec{v}_0 = 0$ by construction of some auxiliary functions. Let $\vec{V}_0(x,t)$ be a vector field from $W^{2,1}_{q,r}(Q^T)$ satisfying the conditions

$$\vec{V}_0(x,0) = \vec{v}_0(x), \quad \vec{V}_0(x,t)|_{x\in S} = 0$$

and the inequality

$$\|\vec{V}_0\|_{W^{2,1}_{q,r}(Q^T)} \le c\|\vec{v}_0\|^{(2-2/r)}_{\Omega}. \tag{3.1}$$

Such a vector field can be constructed, as indicated in the proposition 2.2. Now, let $(\vec{u}, s)$ and $(\vec{w}, \sigma)$ be solutions to the problems

$$-\nu\Delta\vec{u}(x,t) + \nabla s(x,t) = 0, \quad \nabla\cdot\vec{u} = g(x,t) - \nabla\cdot\vec{V}_0(x,t),$$
$$\vec{u}|_{x\in S} = 0, \tag{3.2}$$

$$\vec{w}_t - \nu\Delta\vec{w} + \nabla\sigma = \vec{f}_1(x,t), \quad \nabla\cdot\vec{w} = 0, \qquad x\in\Omega,\ t\in(0,T),$$
$$\vec{w}(x,0) = 0, \qquad \vec{w}(x,t)|_{x\in S} = 0 \tag{3.3}$$

with $\vec{f}_1 = \vec{f} - \vec{V}_{0t} + \nu\Delta\vec{V}_0 - \vec{u}_t$. By virtue of (2.6) and (2.7), we have

$$\|\vec{u}\|_{W^{2,1}_{q,r}(Q^T)} + \|\nabla s\|_{L_{q,r}(Q^T)} \le c\Big(\|\vec{V}_0\|_{W^{2,1}_{q,r}(Q^T)}$$
$$+\|g\|_{L_r(0,T;W^1_q(\Omega))} + \|g_t\|_{L_r(0,T;W^{-1}_q(\Omega))}\Big). \tag{3.4}$$

As shown in [4, 5], problem (3.3) in uniquely solvable, and

$$\|\vec{w}\|_{W^{2,1}_{q,r}(Q^T)} + \|\nabla\sigma\|_{L_{q,r}(Q^T)} \le c\|\vec{f}_1\|_{L_{q,r}(Q^T)}$$
$$\le c\Big(\|\vec{f}\|_{L_{q,r}(Q^T)} + \|\vec{V}_0\|_{W^{2,1}_{q,r}(Q^T)} + \|\vec{u}_t\|_{L_{q,r}(Q^T)}\Big) \tag{3.5}$$

Further, from (1.5) it follows that

$$\int_\Omega \nabla\varphi(x)\cdot\vec{u}(x,t)dx = \int_\Omega \nabla\varphi(x)\cdot(\vec{u}(x,t) + \vec{V}_0(x,t) - \vec{v}_0(x))dx$$
$$-\int_\Omega \nabla\varphi(x)\cdot(\vec{V}_0(x,t) - \vec{v}_0(x))dx \to 0 \quad \text{as} \quad t\to 0,$$

which shows that $\vec{u}(x,0)=0$. Hence, $\vec{v}=\vec{V}_0+\vec{u}+\vec{w}$, $\nabla p=\nabla s+\nabla\sigma$ is a solution of (1.1). The estimates obtained above imply (1.6). Clearly, the solution is unique. The theorem is proved.

Proof of Theorem 1.2. Let $\lambda\in(0,(1-\theta)\rho)$ and let $\zeta(x,t)$ be a smooth cut-off function such that $0\le\zeta(x,t)\le 1$, $\zeta(x,t)=1$ for $(x,t)\in Q_{\rho-\lambda}$ and $\zeta(x,t)=0$ for $(x,t)\in Q_\rho\setminus Q_{\rho-\lambda/2}$, $|D_x^jD_t^a\zeta(x,t)|\le c(|j|,a)\lambda^{-|j|-2a}$. The functions $\vec{w}(x,t)=\vec{v}(x,t)\zeta(x,t)$ and $s(x,t)=p(x,t)\zeta(x,t)$ extended by zero into $Q^{t_0}\setminus Q_\rho$ can be considered as a solution of initial boundary-value problem

$$\begin{gathered}\vec{w}_t-\nu\Delta\vec{w}+\nabla s=\vec{f}_1(x,t),\quad \nabla\cdot\vec{w}=g(x,t),\\ \vec{w}(x,0)=0,\quad \vec{w}|_{x\in S}=0,\end{gathered}\tag{3.6}$$

with

$$\begin{gathered}\vec{f}_1(x,t)=\zeta(x,t)\vec{f}(x,t)-2\nu\nabla\vec{v}(x,t)\nabla\zeta-\nu\vec{v}\Delta\zeta+p\nabla\zeta+\vec{v}\zeta_t\\ g(x,t)=\vec{v}\cdot\nabla\zeta.\end{gathered}$$

According to (1.6),

$$\begin{aligned}&\|\vec{v}\|_{W^{2,1}_{q,r}(Q_{\rho-\lambda})}+\|\nabla p\|_{L_{q,r}(Q_{\rho-\lambda})}\le\|\vec{w}\|_{W^{2,1}_{q,r}(Q^{t_0})}+\|\nabla s\|_{L_{q,r}(Q^{t_0})}\\ &\le c\Big(\|\vec{f}_1\|_{L_{q,r}(Q^{t_0})}+\|g\|_{L_r(0,t_0;W^1_q(\Omega))}+\|g_t\|_{L_r(0,t_0;W^{-1}_q(\Omega))}\Big).\end{aligned}\tag{3.7}$$

The norms in the right-hand side are estimated as follows:

$$\begin{aligned}&\|\vec{f}_1\|_{L_{q,r}(Q^{t_0})}+\|g\|_{L_r(0,t_0;W^1_q(\Omega))}\le\|\vec{f}\|_{L_{q,r}(Q_{\rho-\lambda/2})}\\ &+c\lambda^{-1}\Big(\|\nabla\vec{v}\|_{L_{q,r}(Q_{\rho-\lambda/2})}+\|p\|_{L_{q,r}(Q_{\rho-\lambda/2})}\Big)+c\lambda^{-2}\|\vec{v}\|_{L_{q,r}(Q_{\rho-\lambda/2})}.\end{aligned}$$

Since

$$\begin{aligned}g_t&=\nabla\zeta\cdot(\vec{f}+\nu\Delta\vec{v}-\nabla p)=\nabla\cdot\vec{h}+h_0,\\ h_k&=\nu\nabla\zeta\cdot\frac{\partial\vec{v}}{\partial x_k}-p\frac{\partial\zeta}{\partial x_k},\quad k=1,2,\dots,n,\\ h_0&=\nabla\zeta\cdot\vec{f}-\nu\sum_{k=1}^n\frac{\partial\nabla\zeta}{\partial x_k}\frac{\partial\vec{v}}{\partial x_k}+p\Delta\zeta,\end{aligned}$$

we can estimate the norm of g_t in (3.7) by inequality (2.13):

$$\|g_t(\cdot,t)\|_{W^{-1}_q(\Omega)}\le c\Big(\|\vec{h}(\cdot,t)\|_{L_q(\omega_{\rho-\lambda/2})}$$

$$+\rho^{1/q}\|\vec{h}\cdot\vec{n}\|_{L_q(s_{\rho-\lambda/2})}+\rho\|h_0\|_{L_q(\omega_{\rho-\lambda/2})}\Big)$$

which implies

$$\|g_t\|_{L_r(0,t_0;W_q^{-1}(\Omega))}\le c\Big(\lambda^{-1}\rho\|\vec{f}\|_{L_{q,r}(Q_{\rho-\lambda/2})}+\lambda^{-2}\|\vec{v}\|_{L_{q,r}(Q_{\rho-\lambda/2})}$$

$$+\lambda^{-1}(1+\rho\lambda^{-1})\Big(\|\nabla\vec{v}\|_{L_{q,r}(Q_{\rho-\lambda/2})}+\|p\|_{L_{q,r}(Q_{\rho-\lambda/2})}\Big)$$

$$+\rho^{1/q}\lambda^{-1}\Big(\|\nabla\vec{v}\|_{L_{q,r}(\sigma_{\rho-\lambda/2})}+\|p\|_{L_{q,r}(\sigma_{\rho-\lambda/2})}\Big)\Big), \tag{3.8}$$

where $\sigma_\mu=s_\mu\times(t_0-\mu^2,t_0)$. We estimate the norms in the right-hand side by inequalities (2.1)–(2.3), i.e.,

$$\|\vec{v}\|_{L_{q,r}(Q_{\rho-\lambda/2})}\le\eta_1^2\|\vec{v}\|_{W^{2,1}_{q,r}(Q_{\rho-\lambda/2})}$$

$$+c\eta_1^{n/q-n/q_1+2/r-2/r_1}\|\vec{v}\|_{L_{q_1,r_1}(Q_{\rho-\lambda/2})},$$

$$\|\nabla\vec{v}\|_{L_{q,r}(Q_{\rho-\lambda/2})}\le\eta_2\|\vec{v}\|_{W^{2,1}_{q,r}(Q_{\rho-\lambda/2})}$$

$$+c\eta_2^{-1+n/q-n/q_1+2/r-2/r_1}\|\vec{v}\|_{L_{q_1,r_1}(Q_{\rho-\lambda/2})},$$

$$\|\nabla\vec{v}\|_{L_{q,r}(\sigma_{\rho-\lambda/2})}\le\eta_3^{1-1/q}\|\vec{v}\|_{W^{2,1}_{q,r}(Q_{\rho-\lambda/2})}$$

$$+c\eta_3^{-1+(n-1)/q-n/q_1+2/r-2/r_1}\|\vec{v}\|_{L_{q_1,r_1}(Q_{\rho-\lambda/2})},$$

and, in addition,

$$\|p\|_{L_q(\omega_{\rho-\lambda/2})}\le\eta_4\|\nabla p\|_{L_q(\omega_{\rho-\lambda/2})}+c\eta_4^{n/q-n/q_2}\|p\|_{L_{q_2}(\omega_{\rho-\lambda/2})},$$

$$\|p\|_{L_q(\omega'_{\rho-\lambda/2})}\le\eta_5^{1-1/q}\|\nabla p\|_{L_q(\omega_{\rho-\lambda/2})}+c\eta_5^{(n-1)/q-n/q_2}\|p\|_{L_{q_2}(\omega_{\rho-\lambda/2})}.$$

We set $\eta_1=\varepsilon_1\lambda$, $\eta_2=\varepsilon_2\lambda^2\rho^{-1}$, $\eta_3=(\varepsilon_3\lambda\rho^{-1/q})^{q'}$, $\eta_4=\varepsilon_4\lambda^2\rho^{-1}$, and $\eta_5=\varepsilon_5(\lambda\rho^{-1/q})^{q'}$ with sufficiently small ε_i. Choosing ε_i in an appropriate way we obtain

$$N(\lambda)\le\varepsilon N(\lambda/2)+c(1+\lambda^{-1}\rho)\|\vec{f}\|_{L_{q,r}(Q_{\rho-\lambda/2})}$$

$$+\phi_1(\lambda,\rho,\varepsilon)\|\vec{v}\|_{L_{q_1,r_1}(Q_{\rho-\lambda/2})}+\phi_2(\lambda,\rho,\varepsilon)\|p\|_{L_{q_2,r}(Q_{\rho-\lambda/2})}, \tag{3.9}$$

where

$$N(\mu)=\|\vec{v}\|_{W^{2,1}_{q,r}(Q_{\rho-\mu})}+\|\nabla p\|_{L_{q,r}(Q_{\rho-\mu})}$$

and ϕ_i are linear conbinations of the expressions $c(\varepsilon)\lambda^{a_i}\rho^{b_i}$ such that

$$a_i + b_i = -\mu_i, \qquad i = 1, 2. \tag{3.10}$$

It is clear that for a certain $\kappa \geq 1$ $\lambda^\kappa \phi_i(\lambda, \rho, \varepsilon)$ are increasimg functions of λ. We multiply (3.9) by λ^κ and require that

$$2^\kappa \varepsilon \leq \frac{1}{2}.$$

Then (3.9) implies

$$f(\lambda) \leq \frac{1}{2} f(\lambda/2) + K(\lambda), \quad \lambda \in (0, \rho(1-\theta)),$$

where $f(\lambda) = \lambda^\kappa N(\lambda)$ and

$$K(\lambda) = \lambda^\kappa \Big(\phi_1(\lambda, \rho, \varepsilon)\|\vec{v}\|_{L_{q_1,r_1}(Q_\rho)} + \phi_2(\lambda, \rho, \varepsilon)\|p\|_{L_{q_2,r}(Q_\rho)}\Big)$$
$$+ c\lambda^\kappa (1 + \lambda^{-1}\rho)\|\vec{f}\|_{L_{q,r}(Q_\rho)}.$$

Since K is a monotone increasing function of λ, we conclude that

$$f(\lambda) \leq K(\lambda) + \frac{1}{2} K(\lambda/2) + \ldots \leq 2K(\lambda),$$

or

$$N(\lambda) \leq \phi_1(\lambda, \rho, \varepsilon)\|\vec{v}\|_{L_{q_1,r_1}(Q_\rho)} + \phi_2(\lambda, \rho, \varepsilon)\|p\|_{L_{q_2,r}(Q_\rho)}$$
$$+ c(1 + \lambda^{-1}\rho)\|\vec{f}\|_{L_{q,r}(Q_\rho)}.$$

Setting $\lambda = \rho(1-\theta)$ and taking (3.10) into account we arrive at (1.9).
Let us prove the second statement of the theorem.

The functions $\vec{w} = \zeta\vec{v} \in W^{2,1}_{q_0,r}(Q^{t_0})$, $s = \zeta p$, where by ζ we mean now a smooth cut-off function with $\operatorname{supp}\zeta \in Q_\rho$ and $\zeta(x,t) = 1$ in $Q_{\rho''}$, $\rho'' \in (\rho', \rho)$ is a solution of problem (3.6) whose data, $\vec{f_1}$ and g, satisfy the hypotheses of the Theorem 1.1 with a certain exponent greater than q_0. Indeed, according to imbedding theorems for the spaces $W^{2,1}_{q,r}(Q^{t_0})$ (see Proposition 2.1), $\vec{v}, \nabla\vec{v} \in L_{m_1,r}(Q_\rho)$, $\nabla\vec{v} \in L_{m_2,r}(\sigma_\rho)$, $\sigma_\rho = s_\rho \times (t_0 - \rho^2, t_0)$, where m_1, $m_2 > q_0$ satisfy the conditions

$$1 - \frac{n}{q_0} + \frac{n}{m_1} \geq 0, \quad 1 - \frac{n}{q_0} + \frac{n-1}{m_2} \geq 0; \tag{3.11}$$

and also $p \in L_{m_1,r}(Q_\rho)$, $p \in L_{m_2,r}(\sigma_\rho)$. We fix m_1, $m_2 > q_0$ such that

$$\frac{n}{n-1}\Big(\frac{1}{q_0}-\frac{1}{n}\Big) \le \frac{1}{m_2} \le \frac{n}{(n-1)m_1}.$$

Then both condition (3.11) and (2.10) (with $q = q_1 = m_1$, $q_2 = m_2$) are satisfied; as a consequence, $g \in L_r(0,t_0;W^1_{m_1}(\Omega))$, $g_t \in L_r(0,t_0;W^{-1}_{m_1}(\Omega))$. By Theorem 1.1, problem (3.6) has a solution $\vec{w} \in W^{2,1}_{m,r}(Q^{t_0})$, $\nabla s \in L_{m,r}(Q^{t_0})$, $m = \min(q, m_1) > q_0$. Clearly, it coinsides with $\zeta\vec{v}$ and ζp. Repeating these arguments, if necessary, we conclude the proof.

Let us consider a nonlinear problem

$$\vec{v}_t - \nu\Delta\vec{v} + \nabla p = \vec{f}(x,t) - (\vec{v}\cdot\nabla)\vec{v}, \quad \nabla\cdot\vec{v} = 0, \tag{3.12}$$

$$\vec{v}(x,0) = \vec{v}_0(x), \qquad \vec{v}(x,t)|_{x\in S} = 0. \tag{3.13}$$

It is well known that a generalized solution of this problem possesses the following properties: $\vec{v} \in L_{2,\infty}(Q^T)$, $\nabla\vec{v} \in L_{2,2}(Q^T)$, and the nonlinear term belongs to $L_{q_0,r_0}(Q^T)$ with

$$\frac{n}{q_0} + \frac{2}{r_0} = 1 + n, \quad q_0 \in [1, \frac{n}{n-1}], \quad r_0 \in [1,2].$$

It follows that $\vec{v} \in W^{2,1}_{q_0,r_0}(Q^T)$, $\nabla p \in L_{q_0,r_0}(Q^T)$.

We fix $q_0 > 1$ and $r_0 < 2$ sufficiently close to 2.

Proof of Theorem 1.3. We estimate the nonlinear term as follows:

$$\|(\vec{v}\cdot\nabla)\vec{v}\|_{L_{m_1,m_0}(Q_\rho)} \le \|\nabla\vec{v}\|_{L_{2,r_0}(Q_\rho)}\|\vec{v}\|_{L_{\sigma,\tau}(Q_\rho)}, \tag{3.14}$$

$$\frac{1}{m_1} = \frac{1}{\sigma} + \frac{1}{2}, \quad \frac{1}{m_0} = \frac{1}{\tau} + \frac{1}{r_0}. \tag{3.15}$$

Since $r_0 < 2$, the right-hand side in (3.14) is bounded. It is clear that $m_1 \in (1,2)$, $m_0 \in (1, r_0)$, and

$$\frac{n}{m_1} + \frac{2}{m_0} = \frac{n}{\sigma} + \frac{2}{\tau} + \frac{n}{2} + \frac{2}{r_0} = \frac{n+4}{2} - \epsilon + \epsilon_1$$

with $\epsilon_1 = 2/r_0 - 1$, We require that $\epsilon_1 \in (0,\epsilon)$. By Theorem 1.2, $\vec{v} \in W^{2,1}_{m_1,m_0}(Q_{\rho_1})$, $\forall \rho_1 < \rho$, and

$$\|\vec{v}\|_{W^{2,1}_{m_1,m_0}(Q_{\rho_1})} + \|\nabla p\|_{L_{m_1,m_0}(Q_{\rho_1})} \le c(\rho,\rho_1)\Big(\|\vec{f}\|_{L_{m_1,m_0}(Q_\rho)}$$

$$+\|\nabla\vec{v}\|_{L_{2,r_0}(Q_\rho)}\|\vec{v}\|_{L_{\sigma,\tau}(Q_\rho)}+\|\vec{v}\|_{L_{s_1,m_0}(Q_\rho)}+\|p\|_{L_{s_2,m_0}(Q_\rho)}\Big),\ s_1,\ s_2\le m_1.$$

By virtue of imbedding theorem (Proposition 2.1), $\nabla\vec{v}\in L_{q_1,r_0}(Q_{\rho_1})$ with $q_1\ge m_1$ satisfying the condition $1-n/m_1-2/m_0+n/q_1+2/r_0\ge 0$, i.e.,

$$\frac{n}{q_1}+\frac{2}{r_0}\ge -1+\frac{n}{m_1}+\frac{2}{m_0}=\frac{n+2}{2}-\epsilon+\epsilon_1=\frac{n}{2}+\frac{2}{r_0}-\epsilon. \qquad (3.16)$$

It follows that $1/q_1-1/2\ge -\epsilon/n$, in particular, q_1 may be chosen greater than 2.

Now, we repeat the estimate of a nonlinear term. Instead of (3.14)–(3.16), we have

$$\|(\vec{v}\cdot\nabla)\vec{v}\|_{L_{m_2,m_0}(Q_{\rho_1})}\le\|\nabla\vec{v}\|_{L_{q_1,r_0}(Q_{\rho_1})}\|\vec{v}\|_{L_{\sigma,\tau}(Q_{\rho_1})}, \qquad (3.17)$$

$$\frac{1}{m_2}=\frac{1}{\sigma}+\frac{1}{q_1},\quad \frac{n}{m_2}+\frac{2}{m_0}=\frac{n}{\sigma}+\frac{2}{\tau}+\frac{n}{q_1}+\frac{2}{r_0}\ge\frac{n+2}{2}-2\epsilon+\epsilon_1,$$

$$\vec{v}\in W^{2,1}_{m_2,m_0}(Q_{\rho_2}),\quad \nabla\vec{v}\in L_{q_2,r_0}(Q_{\rho_2}),$$

$$1/q_2-1/2\ge -2\epsilon/n.$$

If we continue in this way, we show after a finite number of steps that $\nabla\vec{v}\in L_{q_l,r_0}(Q_{\rho_l})$ and $\vec{v}\in W^{2,1}_{m_{l+1},m_0}(Q_{\rho_{l+1}})$ with $\rho_{l+1}<\rho_1<\ldots<\rho$ and with arbitrarily large q_l and m_{l+1}. Since $2-n/m_{l+1}-2/m_0=2-n/m_{l+1}-2(1/\tau+1/r_0)=1-2/r_0-n/m_{l+1}+n/\sigma+\epsilon>0$ if m_{l+1} is large enough and r_0 is close enough to 2, $\vec{v}(x,t)$ is bounded in $Q_{\rho_{l+1}}$. As a consequence, $(\vec{v}\cdot\nabla)\vec{v}\in L_{M,r_0}(Q_l)$, and $\vec{v}\in W^{2,1}_{M,r_0}(Q_{\rho'})$, $\rho'<\rho_{l+1}$ The theorem is proved.

Proof of Theorem 1.4. We reduce the theorem to the preceding one by showing that $\vec{v}\in L_{\sigma_1,\tau_1}(Q_{\rho'})$, $\forall\rho'<\rho$, with some $\sigma_1>\sigma$, $\tau_1>\tau$. We again start with the estimate of the nonlinear term:

$$\|(\vec{v}\cdot\nabla)\vec{v}\|_{L_{m,\mu}(Q_\rho)}\le\|\vec{v}\|_{L_{\sigma,\tau}(Q_\rho)}\|\nabla\vec{v}\|_{L_{2,2}(Q_\rho)},$$

$$\frac{1}{m}=\frac{1}{\sigma}+\frac{1}{2},\quad \frac{1}{\mu}=\frac{1}{\tau}+\frac{1}{2},$$

and we conclude that, by virtue of Theorem 1.2, $\vec{v}\in W^{2,1}_{m,\mu}(Q_{\rho_1})$, $\forall\rho_1<\rho$. Now, we want to show that $\vec{v}\in W^{2,1}_{q_1,\mu}(Q_{\rho_2})$, $\forall\rho_2<\rho_1$, with some $q_1>m$.

Let $\zeta(x,t)$ be a smooth cut-off function with $\operatorname{supp}\zeta\in Q_{\rho_1}$, $\zeta(x,t)=1$ in Q_{ρ_2}, and let ζ_1 be another cut-off function supported in Q_{ρ_1} and such that $0 \le \zeta_1 \le 1$, $\zeta_1(x,t)\zeta(x,t) = \zeta(x,t)$. The functions $\vec{w} = \zeta\vec{v}$, $s = \zeta p$ extended by zero into $Q^{t_0} \setminus Q_{\rho_1}$ are a solution of the problem

$$\vec{w}_t - \nu\Delta\vec{w} + \nabla s = \vec{f}_2(x,t) - (\zeta_1\vec{v}\cdot\nabla)\vec{w}, \qquad \nabla\cdot\vec{w} = g(x,t),$$
$$\vec{w}(x,0) = 0, \qquad \vec{w}(x,t)|_{x\in S} = 0 \tag{3.18}$$

with

$$g = \vec{v}\cdot\nabla\zeta \qquad \vec{f}_2 = \vec{f}_1 + (\vec{v}\cdot\nabla\zeta)\vec{v}$$

and with $\vec{f}_1$ defined in (3.6). Moreover, $g_t = g' + g''$ where

$$g'' = \nabla\zeta\cdot(\vec{f} + \nu\Delta\vec{v} - \nabla p), \quad g' = \nabla\zeta\cdot(\vec{v}\cdot\nabla)\vec{v} = \nabla\cdot\vec{h}' + h_0',$$
$$h_k' = v_k\nabla\zeta\cdot\vec{v}, \qquad h_0' = -\vec{v}\cdot(\vec{v}\cdot\nabla)\nabla\zeta.$$

We have seen above in the proof of Theorem 1.2 that $\vec{f}_1 \in L_{m_1,\mu}(Q^{t_0})$, $g \in L_\mu(0,t_0;W^1_{m_1}(\Omega))$, $g'' \in L_\mu(0,t_0;W^{-1}_{m_1}(\Omega))$ with $m_1 > m$ satisfying the inequality $1/m_1 \ge 1/m - 1/n$. In addition, we have $\vec{h}' \in L_{\sigma/2,\tau/2}(Q_{\rho_1})$, $h_0' \in L_{\sigma/2,\tau/2}(Q_{\rho_1})$. From $1/\mu = 1/\tau + 1/2 > 2/\tau$ it follows that $\mu < \tau/2$, hence, $\vec{h}' \in L_{\sigma/2,\mu}(Q_{\rho_1})$, $h_0' \in L_{\sigma/2,\mu}(Q_{\rho_1})$, and as a consequence (since $\vec{v}$ vanishes at the boundary), $g' \in L_\mu(0,t_0;W^{-1}_{\sigma/2}(\Omega))$.

Next, we observe that for arbitrary $\tilde{m} < \sigma$, $\tilde{\mu} < \tau$ the estimate

$$\|(\zeta_1\vec{v}\cdot\nabla)\vec{w}\|_{L_{\tilde{m},\tilde{\mu}}(Q_\rho)} \le \|\vec{v}\|_{L_{\sigma,\tau}(Q_\rho)}\|\nabla\vec{w}\|_{L_{\tilde{q},\tilde{r}}(Q_\rho)}$$
$$\le c\|\vec{v}\|_{L_{\sigma,\tau}(Q_\rho)}\|\vec{w}\|_{W^{2,1}_{\tilde{m},\tilde{\mu}}(Q_\rho)}$$

holds with $1/\tilde{q} = 1/\tilde{m} - 1/\sigma$ and $1/\tilde{r} = 1/\tilde{\mu} - 1/\tau$. If the norm $\|\vec{v}\|_{L_{\sigma,\tau}(Q_\rho)}$ is sufficiently small, which is the case for small ρ, then the term $(\zeta_1\vec{v}\cdot\nabla)\vec{w}$ determines a continuous linear operator from $W^{2,1}_{\tilde{m},\tilde{\mu}}(Q^{t_0})$ into $L_{\tilde{m},\tilde{\mu}}(Q^{t_0})$ with a small norm. This makes it possible to conclude that problem (3.18) has a unique solution $\vec{w} \in W^{2,1}_{q_1,\mu}(Q^{t_0})$, $\nabla s \in L_{q_1,\mu}(Q^{t_0})$ where $q_1 = \min(m_1,\sigma/2) > m$. It is clear that the solution coincides with $(\zeta\vec{v},\zeta p)$. If $m_1 < \sigma/2$, then we can repeat our arguments and show that $\vec{v} \in W^{2,1}_{q_2,\mu}(Q_{\rho_2})$, $q_2 = \min(m_2,\sigma/2)$, $1/m_2 > 1/m - 2/n$, in a smaller cylinder Q_{ρ_2}. After a finite number of steps we arrive at the conclusion that $\vec{v} \in W^{2,1}_{\sigma/2,\mu}(Q_{\rho'})$, $\rho' < \rho$. Now we notice that

$$2 - 2n/\sigma - 2/\mu + n/\sigma + 2/\tau = 1 - n/\sigma > 0,$$

so according to the Proposition 2.1 $\vec{v} \in L_{\sigma_1,\tau_1}(Q_{\rho'})$ with some $\sigma_1 > \sigma$ and $\tau_1 > \tau$ such that

$$2 - 2n/\sigma - 2/\mu + n/\sigma_1 + 2/\tau_1 \ge 0.$$

Since $n/\sigma_1 + 2/\tau_1 < 1$, the theorem is reduced to Theorem 1.3.

4. On problem (1.13)

We recall that the solution of problem (1.13) is representable in the form

$$\vec{v}(x,t) = \int_0^t \int_{R^n_+} \mathcal{G}(x,y,t-\tau)\vec{f}'(y,\tau)dyd\tau, \tag{4.1}$$

$$p(x,t) = \phi(x,t) + \int_0^t \int_{R^n_+} \vec{P}(x,y,t-\tau)\cdot\vec{f}'(y,\tau)dyd\tau, \tag{4.2}$$

where

$$\phi(x,t) = -\int_{R^n_+} \nabla_y(E(x-y)+E(x-y^*))\cdot\vec{f}(y,t)dy,$$

and

$$\vec{f}'(x,t) = \vec{f}(x,t) - \nabla\phi(x,t) \equiv \mathcal{Q}\vec{f} \tag{4.3}$$

is a projection of $\vec{f}$ onto the space of divergence free vector fields with $f_n|_{x_n=0} = 0$ (see [12]).

The elements of the Green's tensor $\{\mathcal{G}(x,y,t),\vec{P}(x,y,t)\}$ of problem (1.13) with $\vec{f} = \vec{f}'$ are given by

$$G_{ij}(x,y,t) = \delta_{ij}\Big(\Gamma(x-y,t) - \Gamma(x-y^*,t)\Big)$$

$$+4(1-\delta_{jn})\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{R^{n-1}} \frac{\partial E(x-z)}{\partial x_i}\Gamma(z-y^*,t)dz, \quad i,\, j = 1,\dots,n,$$

$$P_j(x,y,t) = 4\nu(1-\delta_{nj})\frac{\partial}{\partial x_j}\Big(\int_{R^{n-1}} \frac{\partial E(x'-z',x_n)}{\partial x_n}\Gamma(z'-y',y_n,t)dz'$$

$$+\int_{R^{n-1}} E(x'-z',x_n)\frac{\partial\Gamma(z'-y',y_n,t)}{\partial y_n}dz'\Big), \quad j = 1,\dots,n. \tag{4.4}$$

By $E(x)$ we mean the fundamental solution of the Laplace equation, i.e., $E(x) = -((n-2)|S_1|)^{-1}|x|^{-(n-2)}$ if $n > 2$ and $E(x) = (2\pi)^{-1}\log|x|$, if $n = 2$, and $|S_1|$ is the surface area of the unit sphere in R^n, $y^* = (y_1,\dots,-y_n)$ and $y' = (y_1,\dots,y_{n-1})$.

Proposition 4.1. *Functions* (4.1) *and* (4.2) *satisfy the inequalitiy*

$$\|\vec{v}_t\|^r_{L_{q,r}(\Pi^T_+)} + \sum_{|j|=2} \|D^j \vec{v}\|^r_{L_{q,r}(\Pi^T_+)} + \|\nabla p\|^r_{L_{q,r}(\Pi^T_+)} \leq c_1 \|\vec{f}\|^r_{L_{q,r}(\Pi^T_+)}, \quad (4.5)$$

with the constant c_1 *independent of* T *and*

$$\|\vec{v}\|^r_{L_{q,r}(\Pi^T_+)} \leq c_2(T)\|\vec{v}_t\|^r_{L_{q,r}(\Pi^T_+)} \leq c_1 c_2(T)\|\vec{f}\|^r_{L_{q,r}(\Pi^T_+)}. \quad (4.6)$$

Proof. We write (4.1) in the form

$$\vec{v}(x,t) = \vec{U}(x,t) - \vec{U}^*(x,t) + 4\sum_{j=1}^{n-1} \mathcal{S}_j (U^*_j)^{(0)}(x,t),$$

where

$$\vec{U}(x,t) = \int_0^t \int_{R^n_+} \Gamma(x-y, t-\tau)\vec{f}'(y,\tau)dyd\tau,$$

$$\vec{U}^*(x,t) = \int_0^t \int_{R^n_+} \Gamma(x-y^*, t-\tau)\vec{f}'(y,\tau)dyd\tau, \qquad x \in R^n_+,$$

$(\vec{U}^*)^{(0)}$ is a zero extension of $\vec{U}^*$ into R^n_-, and $\mathcal{S}_j$ is a singular integral operator with the vector-valued kernel equal to $\frac{\partial \nabla E(x)}{\partial x_j}$, if $x_n > 0$, and to zero, if $x_n < 0$. By virtue of inequality

$$\|u_t\|^r_{L_{q,r}(\Pi^T_+)} + \sum_{|j|=2} \|D^j u\|^r_{L_{q,r}(\Pi^T_+)} \leq c\|f\|^r_{L_{q,r}(\Pi^T_+)}$$

for the heat potential

$$u(x,t) = \int_0^t \int_{R^n} \Gamma(x-y, t-\tau) f(y,\tau)dyd\tau$$

(see [5]), and of the Calderon–Zygmund theorem, we have

$$\|\vec{v}_t\|^r_{L_{q,r}(\Pi^T_+)} \leq c\|\vec{f}'\|^r_{L_{q,r}(\Pi^T_+)}.$$

Applying again the Calderon–Zygmund theorem, this time to the operator $\nabla\phi$ in (4.3), we obtain

$$\|\vec{f}'\|^r_{L_{q,r}(\Pi^T_+)} \leq c\|\vec{f}\|^r_{L_{q,r}(\Pi^T_+)}.$$

Now, we can consider $\vec{v}$ as a solution of a stationary Stokes problem

$$-\nu\Delta\vec{v}(x,t)+\nabla p(x,t)=\vec{f}(x,t)-\vec{v}_t(x,t),\quad \nabla\cdot\vec{v}=0,$$

$$\vec{v}(x,t)|_{x_n=0}=0.$$

From coercive L_p-estimate for this problem it follows that

$$\sum_{|j|=2}\|D^j\vec{v}\|^r_{L_{q,r}(\Pi^T_+)}+\|\nabla p\|^r_{L_{q,r}(\Pi^T_+)}\le c_1\Big(\|\vec{f}\|^r_{L_{q,r}(\Pi^T_+)}+\|\vec{v}_t\|^r_{L_{q,r}(\Pi^T_+)}\Big).$$

Together with the above estimate for $\vec{v}_t$, this implies (4.5), and (4.6) follows easily from

$$\begin{aligned}\|\vec{v}(\cdot,t)\|^r_{L_q(R^n_+)}&\le r\int_0^t\|\vec{v}_\tau(\cdot,\tau)\|_{L_q(R^n_+)}\|\vec{v}(\cdot,\tau)\|^{r-1}_{L_q(R^n_+)}d\tau\\&\le\int_0^t\|\vec{v}_\tau(\cdot,\tau)\|^r_{L_q(R^n_+)}d\tau+(r-1)\int_0^t\|\vec{v}(\cdot,\tau)\|^r_{L_q(R^n_+)}d\tau.\end{aligned}$$

The proposition is proved.

Proof of Theorem 1.5. Let $\rho'=(\rho+\rho_1)/2$ and let $\omega(t)$ and $\psi(x)$ be two smooth cut-off functions such that $0\le\omega(t),\psi(x)\le1$, $\psi(x)=1$ for $|x-x_0|\le\rho'$, $\psi(x)=0$ for $|x-x_0|\ge\rho$, $\omega(t)=1$ for $t\in(t_0-\rho'^2,t_0)$, $\omega(t)=0$ for $t<t_0-\rho^2$. We set $\zeta(x,t)=\omega(t)\psi(x)$, $\vec{f}_1(x,t)=\vec{f}(x,t)\zeta(x,t)$, $\vec{f}_2(x,t)=\vec{f}(x,t)(1-\psi(x))\omega(t)$, $\vec{f}_3(x,t)=\vec{f}(x,t)(1-\omega(t))$, and we define $\vec{v}_i$, $p_i(x,t)$ by the formulas (4.1) and (4.2) with $\vec{f}=\vec{f}_i$, $i=1,2,3$. It is clear that $\vec{f}_1+\vec{f}_2+\vec{f}_3=\vec{f}$ and that (1.14) holds. We also set $\vec{f}'_i=\mathcal{Q}\vec{f}_i$,

$$\phi_i(x,t)=-\int_{R^n_+}\nabla(E(x-y)-E(x-y^*))\cdot\vec{f}_i(y,t)dy,$$

$$p'_i(x,t)=\int_0^t\int_{R^n_+}\vec{P}(x,y,t-\tau)\cdot\vec{f}'_i(y,\tau)dyd\tau.$$

We start the estimates of p_i with the proof of (1.21). If $(x,t)\in Q_{\rho_1}$, then $\phi_3(x,t)=0$, and $p_3(x,t)=p'_3(x,t)$. Let $\mathcal{U}^{(i)}(x',t)$, $i=1,2,3$ be matrices with the elements

$$U_{kj}^{(i)}(x',t)=\int_0^t\int_{R^n_+}\frac{\partial\Gamma(x'-y',y_n,t-\tau)}{\partial y_k}f_{ij}(y,\tau)dyd\tau.\quad k,j=1,\dots,n,$$

where $f_{ij}=\vec{f_i}\cdot e_j$. If $t\in(t_0-\rho_1^2,t_0)$ and $\tau<t_0-\rho'^2$, then the kernel $\nabla\Gamma(x-y,t-\tau)$ has no singularity. This implies

$$|D^j_{x'}D^h_tU^{(3)}_{kj}(x',t)|\le c\|\vec{f_3}\|_{L_{q_0,r_0}(\Pi^{t_0-\rho'^2})},\quad x'\in R^{n-1},\quad t\in(t_0-\rho_1^2,t_0),$$

$$\|D^h_t\,U^{(3)}(\cdot,t)\|_{L_{q_0}(R^{n-1})}\leqslant c\int_0^t\int_0^\infty e^{-\frac{c_1y_n^2}{t-\tau}}\left\|f_3(\cdot,y_n,\tau)\right\|_{L_{q_0}(R^{n-1})}\frac{dy_n\,d\tau}{(t-\tau)^{1+h}}$$

$$\leqslant c\left\|\vec{f_3}\right\|_{L_{q_0,r_0}(\Pi^{t_0-\rho'^2})},$$

and, due to elementary estimates of harmonic simple layer potential,

$$|D^j_xD^h_tp'_3(x,t)|\le c\Big(\sum_{|\beta|=|j|}^{|j|+1}\sup_{x'\in R^{n-1}}|D^\beta_{x'}D^h_tU^{(3)}(x',t)|$$

$$+\|D^h_tU^{(3)}(\cdot,t)\|_{L_{q_0}(R^{n-1}\setminus s_{\rho_1}(x_0))}\Big)\le c\|\vec{f}\|_{L_{q_0,r_0}(\Pi^{t_0-\rho'^2})},\quad (x,t)\in Q_{\rho_1}.$$

Hence, (1.21) is proved. Inequalities (1.15) and (1.18) follow from (4.5). Let us estimate $L_{q,r}(Q_{\rho_1})$-norm of $p_1=\phi_1+p'_1$. Clearly,

$$\|\phi_1\|_{L_{q,r}(Q_{\rho_1})}\le c\|\vec{f}\|_{L_{q,r}(Q_\rho)}.$$

Further, for sufficiently small $\varepsilon>0$ we have

$$|p'_1(x,t)|=\le c\int_{R^{n-1}}|\mathcal{U}^{(1)}(y',t)|\frac{dy'}{(x'-y'|^2+x_n^2)^{(n-1)/2}}$$

$$\le c\Big(\int_{R^{n-1}}|\mathcal{U}^{(1)}(y',t)|^q\frac{dy'}{(x'-y'|^2+x_n^2)^{(n-1)/2-\varepsilon q/2}}\Big)^{1/q},$$

$$\Big(\int_{R^{n-1}}\frac{dy'}{(x'-y'|^2+x_n^2)^{(n-1)/2+\varepsilon q'/2}}\Big)^{1/q'}$$

$$\le\frac{c}{x_n^\varepsilon}\Big(\int_{R^{n-1}}|\mathcal{U}^{(1)}(y',t)|^q\frac{dy'}{(x'-y'|^2+x_n^2)^{(n-1)/2-\varepsilon q/2}}\Big)^{1/q}$$

from which it follows that

$$\|p_1'\|_{L_q(\omega_{\rho_1}(x_0))}$$
$$\leq c\Big(\int\limits_{R^{n-1}} |\mathcal{U}^{(1)}(y',t)|^q dy' \int\limits_{\omega_{\rho_1}(x_0)} \frac{dx}{x_n^{\varepsilon q_1}(x'-y'|^2+x_n^2)^{(n-1)/2-\varepsilon q/2}}\Big)^{1/q}$$
$$\leq c\|\mathcal{U}^{(1)}(\cdot,t)\|_{L_q(R^{n-1})}.$$

By virtue of the Young inequality,

$$\|\mathcal{U}^{(1)}(\cdot,t)\|_{L_q(R^{n-1})} \leq c\int\limits_0^t\int\limits_0^\infty \|\vec{f}_1'(\cdot,y_n,\tau)\|_{L_q(R^{n-1})} e^{-\frac{c_1 y_n^2}{t-\tau}}\frac{dy_n\,d\tau}{t-\tau}$$
$$\leq c\int\limits_0^t \|\vec{f}_1(\cdot,\tau)\|_{L_q(R^n_+)} \frac{d\tau}{(t-\tau)^{1-1/2q'}}$$

and, as a consequence,

$$\|p_1'\|_{L_{q,r}(Q_{\rho_1})} \leq c\|\mathcal{U}^{(1)}\|_{L_{q,r}(R^{n-1}\times(t_0-\rho^2,t_0))} \leq c\|\vec{f}_1\|_{L_{q,r}(Q_\rho)}.$$

This completes the proof of (1.16). Let us pass to (1.17). If $(x,t)\in Q_{\rho_1}$, then the function ϕ_2 is harmonic, and the kernel of the integral operator in

$$D^j\phi_2(x,t) = -\int\limits_{R^n_+\setminus\omega_{\rho'}(x_0)} D_x^j\nabla_y(E(x-y)-E(x-y^*))\cdot\vec{f}_2(y,t)dy$$

is bounded by a constant depending on $\rho-\rho_1$, hence,

$$|D^j\nabla\phi_2(x,t)| \leq c\|\vec{f}(\cdot,t)\|_{L_{q_0}(R^n_+\setminus\omega_{\rho'}(x_0))},$$

which implies

$$\|D^j\nabla\phi_2\|_{L_{\infty,r_0}(Q_{\rho_1})} \leq c\|\vec{f}\|_{L_{q_0,r_0}(\pi(\rho,\rho_1)}.$$

To estimate p_2', we introduce a new smooth cut-off function $\psi_1(x)$ such that $0\leq\psi_1(x)\leq 1$, $\psi_1(x)=1$, if $x\in\omega_{\rho_2}(x_0)$, $\psi(x)=0$, if $x\in R^n\setminus\omega_{\rho_3}(x_0)$ where $\rho_1<\rho_2<\rho_3<\rho'$, and we write p_2' in the form $p_2'=q'+q''$ with

$$q'(x,t) = 4\nu(1-\delta_{nj})\Big(\int\limits_{R^{n-1}} \frac{\partial E(x'-z',x_n)}{\partial x_n}\psi_1(z',0)U_{jj}^{(2)}(z',t)dz'$$

$$+\int\limits_{R^{n-1}} \frac{\partial E(x'-z',x_n)}{\partial x_j}\psi_1(z',0)U^{(2)}_{nj}(z',t)dz'\Big),$$

$$q^{''}(x,t)=4\nu(1-\delta_{nj})\Big(\int\limits_{R^{n-1}} \frac{\partial E(x'-z',x_n)}{\partial x_n}(1-\psi_1(z',0))U^{(2)}_{jj}(z',t)dz'$$

$$+\int\limits_{R^{n-1}} \frac{\partial E(x'-z',x_n)}{\partial x_j}(1-\psi_1(z',0))U^{(2)}_{nj}(z',t)dz'\Big)$$

Since q' is harmonic, arbitrary derivative of $D^k_x q'$, $|k|\geq 2$, is a linear combination of $D^{k'}_{x'}D^m_{x_n}q'$, $m=0,1$, $|k'|=|k|-m$. By a standard estimate of the harmonic simple layer potential, we have

$$|D^j\nabla q'(x,t)|\leq c\sum_{|k|=|j|}^{|j|+1}|D^k_{x'}\psi_1\mathcal{U}^{(2)}(\cdot,t)|_{C^\alpha(R^{n-1})},$$

$\alpha\in(0,1)$. For $(z,t)\in Q_{\rho_2}$ and $(y,\tau)\in\pi(\rho,\rho_1)$ the kernel $\nabla\Gamma(z-y,t-\tau)$ in $\mathcal{U}^{(2)}$ is bounded together with all its derivatives, so we easily obtain

$$|D^j\nabla q'(x,t)|\leq c\int\limits_0^t\|\vec f_2(\cdot,\tau)\|_{L_{q_0}(R^n_+)}d\tau$$

and, as a consequence,

$$\|D^j\nabla q'\|_{L_{\infty,r_0}(Q_{\rho_1})}\leq c\|\vec f_2\|_{L_{q_0,r_0}(\pi(\rho,\rho_1))}. \tag{4.7}$$

Further, if $x\in\omega_{\rho_1}(x_0)$ and $y\in R^{n-1}\setminus\omega_{\rho_2}$, then $E(x-y)$ and its derivatives are bounded, hence,

$$|D^j\nabla q^{''}(x,t)|\leq c\|(1-\psi_1)\mathcal{U}^{(2)}(\cdot,t)\|_{L_{q_0}(R^{n-1})}$$

$$\leq c\int\limits_0^t\|\vec f_2(\cdot,\tau)\|_{L_{q_0}(R^n_+)}\frac{d\tau}{(t-\tau)^{1-1/2q_0'}};$$

from which it follows that (4.7) holds, also for $q^{''}$ and (1.17) is proved.

Finally, we prove (1.19) and (1.20). We have for $(x,t)\in Q_{\rho_1}$:

$$|\phi_2(x,t)|\leq c\int\limits_{R^n_+}|x-y|^{-n-1}|\vec f_2(y,t)|dy\leq c\|\vec f_2(\cdot,t)\|_{L_{\tilde q}(R^n)}$$

with $\tilde{q} = \min(q_0, n-\epsilon)$. In addition, we should estimate $p_2'(x,t)$. It is easily seen that for $(x,t) \in Q_{\rho_1}$

$$|p_2'(x,t)| \le c\Big(|\mathcal{U}^{(2)}(\cdot,t)|_{C^\alpha(s_{\rho_3}(x_0))} + \|\mathcal{U}^{(2)}(\cdot,t)\|_{L_{q_0}(R^{n-1}\setminus s_{\rho_2})}\Big)$$

$$\le c\int_0^t \|\vec{f}_2(\cdot,\tau)\|_{L_{q_0}(R^n_+)} \frac{d\tau}{(t-\tau)^{1-1/2q_0'}}.$$

Hence,

$$\|p_2'\|_{L_{\infty,r_0}(Q_{\rho_1})} \le c\|\vec{f}\|_{L_{q_0,r_0}(\pi(\rho,\rho_1))}.$$

which completes the proof of (1.19) and (1.20).

As for $\vec{v}_i(x,t)$, these vector fields can be considered as solutions to parabolic problems

$$\vec{v}_{it} - \nu\Delta\vec{v}_i = \vec{f}_i - \nabla p_i,$$
$$\vec{v}_i|_{t=0} = 0, \quad \vec{v}_i\big|_{x_n=0} = 0, \quad i = 1,2,3.$$

We make use of local estimates for these problems in Sobolev and Hölder spaces and obtain

$$\|\vec{v}_1\|_{W^{2,1}_{q,r}(Q_{\theta\rho_1})} \le \Big(\|\vec{f}_1\|_{L_{q,r}(Q_{\rho_1})} + \|\nabla p_1\|_{L_{q,r}(Q_{\rho_1})} + \rho^{-\lambda_1}\|\vec{v}_1\|_{L_{q,r}(Q_{\rho_1})}\Big),$$

$$\|\vec{v}_2\|_{W^{2,1}_{M,r_0}(Q_{\theta\rho_1})} \le \Big(\|\nabla p_2\|_{L_{q_0,r_0}(Q_{\rho_1})} + \rho^{-\lambda_2}\|\vec{v}_2\|_{L_{q_0,r_0}(Q_{\rho_1})}\Big),$$

$$|\vec{v}_3|_{C^{m+2+\alpha,(m+2+\alpha)/2}(Q_{\theta\rho_1})}$$
$$\le c\Big(|\nabla p_3|_{C^{m+\alpha,(m+\alpha)/2}(Q_{\rho_1})} + \rho^{-\lambda_3}\|\vec{v}_3\|_{L_{q_0,r_0}(Q_{\rho_1})}\Big)$$

with $\alpha \in (0,1)$ and with some positive λ_i. We estimate the norms of ∇p_i and of $\vec{v}_i$ by inequalities (1.15), (1.18), (1.21), and (4.6), and arrive at (1.22)-(1.24) (in a smaller cylinder $Q_{\theta\rho_1}$, which is not essential). Theorem 1.5 is proved.

Finally, let us estimate $\nabla\vec{v} = \nabla\vec{v}_1 + \nabla\vec{v}_2 + \nabla\vec{v}_3$. By virtue of (1.22) and (1.24),

$$\sup_{Q_{\theta\rho_1}} |\nabla\vec{v}_1(x,t)| \le c\|\vec{f}_1\|_{L_{q,r}(Q_\rho)}$$

if $n/q + 2/r < 1$, and

$$\sup_{Q_{\theta\rho_1}} |\nabla\vec{v}_3(x,t)| \le c\|\vec{f}_3\|_{L_{q_0,r_0}(Q^{t_0-\rho'^2})}.$$

For $(x,t) \in Q_{\theta\rho_1}$ the vector field $\vec{v}_2(x,t)$ is given by

$$\vec{v}_2(x,t) = -\int\limits_{t_0-\rho'^2}^{t}\int\limits_{R^n_+} G(x,y,t-\tau)\nabla p_2(y,\tau)dyd\tau,$$

where $G(x,y,t-\tau) = \Gamma(x-y,t-\tau) - \Gamma(x-y^*,t-\tau)$, hence,

$$\frac{\partial}{\partial x_k}\vec{v}_2(x,t) = -\int\limits_{t_0-\rho'^2}^{t}\int\limits_{R^n_+} \frac{\partial}{\partial x_k}G(x,y,t-\tau)(1-\psi_1(y))\nabla p_2(y,\tau)dyd\tau$$

$$-\int\limits_{t_0-\rho'^2}^{t}\int\limits_{R^n_+}(\Gamma(x-y,t-\tau) \pm \Gamma(x-y^*,t-\tau))\frac{\partial}{\partial y_k}\psi_1(y)\nabla p_2(y,\tau)dyd\tau$$

$$-2\delta_{kn}\int\limits_{t_0-\rho'^2}^{t}\int\limits_{R^{n-1}} \Gamma(x'-y',x_n,t-\tau)\psi(y)\nabla p_2(y',0,\tau)dy'd\tau$$

$$\equiv \vec{w}_k^{(1)}(x,t) + \vec{w}_k^{(2)}(x,t) + \vec{w}_k^{(3)}(x,t)$$

where the signs "$+''$ and "$-''$ correspond to $k=n$ and $k<n$, respectively. It is clear that

$$|\vec{w}_k^{(1)}(x,t)| \le c\int\limits_{t_0-\rho'^2}^{t} \|\nabla p_2(\cdot,\tau\|_{L_{q_0}(R^n_+)}d\tau \le c\|\vec{f}_2\|_{L_{q_0,r_0}(\pi(\rho,\rho_1))},$$

$$|\vec{w}_k^{(2)}(x,t)| \le c\int\limits_{t_0-\rho'^2}^{t} \sup_{R^n_+}|(\psi_1(y)\nabla p_2(y,\tau))_{y_k}|d\tau \le c\|\vec{f}_2\|_{L_{q_0,r_0}(\pi(\rho,\rho_1))}.$$

Assume that $\partial\omega_q(x_0)\cap\{x_n=0\} \ne \varnothing$. Then for arbitrary $x \in \partial\omega_q(x_0)$ with $x_n = 0$ we have

$$|\vec{w}_n^{(3)}(x,t)| \le c\int\limits_{t_0-\rho'^2}^{t} (t-\tau)^{-1/2}\sup_{R^n_+}|\psi_1(y)\nabla p_2(y,\tau)|d\tau$$

$$\leqslant c\int\limits_{t_0-\rho'^2}^{t} (t-\tau)^{-1/2}\|\vec{f}(\cdot,\tau)\|_{L_{q_0}(R^n_+\setminus\omega_{\rho'})}d\tau,$$

which implies

$$|\vec{w}_n^{(3)}(x,t)| \leq c\|\vec{f}_2\|_{L_{q_0,r_0}(\pi(\rho,\rho_1))}$$

only in the case $r_0 > 2$. This condition is not satisfied for the nonlinear term in Propositions 1.3 and 1.4.

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