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SPLITTING OF QUADRATIC FORMS UNDER SOME GENERIC FIELD EXTENSIONS

ABSTRACT. Let F be a field, char$F \neq 2$, φ and ψ be anisotropic quadratic forms over F. Let L be the generic field extension of F such that $i(\psi_L) \geqslant 2$. Under which conditions is the form φ_L isotropic? We give the answer to this question in the cases where dim $\varphi = 5$, dim $\psi = 6$ and dim $\varphi = 6$, dim $\psi = 7$.

1. Notation and basic facts

Let F be a field of characteristic different from 2, let φ and ψ be anisotropic quadratic forms over F. There is a great number of papers devoted to the question of splitting the form φ over the field $F(\psi)$, i.e., the generic field extension of F over which ψ becomes isotropic. Clearly, $F(\psi)$ can be regarded as the generic field extension K over F such that $i(\psi_K) \geqslant 1$. In the present paper we consider an analogous question concerning the generic field extension L/F such that $i(\psi_L) \geqslant 2$. Which conditions does the form φ have to satisfy to become isotropic over L? In general this difficult problem can hardly be expected to be solved. We give an answer to this question if dim $\varphi = 5$, dim $\psi = 6$ and if dim $\varphi = 6$, dim $\psi = 7$. It turns out that the answer depends on ind φ and ind ψ.

All the fields in the paper have characteristic different from 2 and all the quadratic forms are nondegenerate. By a form we always mean a quadratic form. A form φ over a field F is called an F-form. By φ_{an} we denote its anisotropic part, i.e., $\varphi \simeq \varphi_{\text{an}} \perp k\mathbb{H}$ for some $k \geqslant 0$ and φ_{an} is anisotropic. Here, the number $k = i(\varphi)$ is the Witt index of the form φ. By definition $d_{\pm}(\varphi) = (-1)^{\frac{n(n-1)}{2}} \det \varphi$, where $n = \dim \varphi$. $W(F)$ is the Witt ring of the field F, $I(F)$ is the ideal of classes of evendimensional forms in $W(F)$. The symbol $\langle\langle a_1, \dots, a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes \dots \otimes \langle 1, -a_n\rangle$ stands for an n-fold Pfister form. $P_n(F)$ is the set of all n-fold Pfister forms, $GP_n(F)$ is the set of all forms similar to n-fold Pfister forms. If $\pi \in GP_n(F)$, τ is

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similar to a subform of π and $\dim \tau > 2^{n-1}$, then we say that τ is an n-fold Pfister neighbor of π. $[\varphi]$ is the class of φ in $W(F)$. We use the signs $\simeq$ and $\sim$ for isomorphism and similarity. For example, $\varphi_1 \sim \varphi_2$ means that there is $a \in F^*$ such that $\varphi_1 \simeq a\varphi_2$. If K/F is a field extension and φ is a form over F, then φ_K is the extension of φ to K, i.e. $\varphi \otimes_F K$. Given a form φ, we denote by $F(\varphi)$ the function field of the projective quadric X_φ defined by the equation $\varphi = 0$. For an anisotropic form φ we set $i_1(\varphi) = i(\varphi_{F(\varphi)})$. By definition $H^n(F) = H^n(\mathrm{Gal}(F_{\mathrm{sep}}/F), \mathbb{Z}/2\mathbb{Z})$. In particular, $H^2(F) \simeq {}_2\mathrm{Br}(F)$ is 2-torsion in the Brauer group of F. For a form φ we denote by $C(\varphi)$ the Clifford algebra of φ and by $C_0(\varphi)$ the even part of $C(\varphi)$. If $\varphi \in I^2(F)$, then $C(\varphi)$ is a central simple algebra over F, and we set $\mathrm{ind}\ \varphi = \mathrm{ind}\ C(\varphi)$, i.e. the Schur index of $C(\varphi)$. If $\dim \varphi$ is odd, then $C_0(\varphi)$ is a central simple algebra over F and we set $\mathrm{ind}\ \varphi = \mathrm{ind}\ C_0(\varphi)$. Finally, if $\dim \varphi = 2m$ but $\varphi \notin I^2(F)$, then $C_0(\varphi)$ is a central simple algebra over $F(\sqrt{d_\pm(\varphi)})$ and we set $\mathrm{ind}\ \varphi = \mathrm{ind}\ C_0(\varphi)$. There exists an isomorphism $I^2(F)/I^3(F) \longrightarrow_2 \mathrm{Br}(F)$, which takes any $[\varphi] \in I^2(F)$ to $[C(\varphi)] \in_2 \mathrm{Br}(F)$ ([14]). If $[D] \in_2 \mathrm{Br}(F)$ and $\mathrm{ind}\ D = 4$, then any form $\varphi \in I^2(F)$ with $\dim \varphi = 6$ corresponding to $[D]$ under this isomorphism is called Albert form. If A is a central simple algebra over F, then $F(A)$ is the function field of the Severi-Brauer variety of A. It is well known that $\mathrm{ind}\ A_{F(A)} = 1$.

For a given form ψ by L we always denote the generic splitting field for ψ such that $i(\psi_L) \geqslant 2$. If $i_1(\psi) \geqslant 2$, then $L = F(\psi)$. If $i_1(\psi) = 1$, then $L = K(\tau)$, where $K = F(\psi)$ and $\tau \simeq (\psi_{F(\psi)})_{\mathrm{an}}$.

The proofs of the main results of the paper are somewhat technical and founded on difficult theorems of many authors. For convenience of the reader we give the list of assertions which we shall use in the sequel.

(1.1) Arason-Pfister Hauptsatz ([1]). Let $\varphi \in I^n(F)$. If $\dim \varphi < 2^n$, then φ is hyperbolic.

(1.2) Theorem ([18]). 1) If φ and ψ are 4-dimensional forms over F such that $d_\pm(\varphi) = d_\pm(\psi) = d \notin F^{*2}$ and $[C_0(\varphi)] = [C_0(\psi)] \in \mathrm{Br}\ F(\sqrt{d})$, then $\varphi \sim \psi$.

2) If φ and ψ are 4-dimensional anisotropic forms over F such that $\varphi_F(\psi)$ is isotropic and $\psi \notin GP_2(F)$, then $\varphi \sim \psi$.

(1.3) Theorem ([4]). Let ψ be an anisotropic form over a field F, $\dim \psi = 6$, $\mathrm{ind}\ \psi = 2$, $\psi \sim \langle\langle a, b\rangle\rangle \perp \langle -c, cd\rangle$, $\widehat{\psi} \sim \langle ab, -a, -b, d\rangle$.

1) Since $[C_0(\widehat{\psi})] = [C_0(\psi)]$ it follows from theorem (1.2) that the form $\widehat{\psi}$ is defined up to similarity and $(\psi_{F(\psi)})_{\mathrm{an}} \sim \widehat{\psi}_{F(\psi)}$.

2) The form $\psi_{F(\widehat{\psi})}$ is anisotropic and is a Pfister neighbor of $\langle\langle a,b,c\rangle\rangle_{F(\widehat{\psi})}$. In particular, the Pfister form $\langle\langle a,b,c\rangle\rangle_{F(\widehat{\psi})}$ is uniquely defined.

(1.4) Theorem ([9]). Any two Albert forms corresponding to the same biquaternion algebra are similar.

(1.5) Theorem ([3]). Let φ and ψ be anisotropic forms over F and $\dim\varphi \leqslant 5$. Suppose that φ is not a 3-fold Pfister neighbor and $\psi \notin GP_2(F)$. Then $\varphi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform of φ.

(1.6) Theorem ([10]). Let φ and ψ be anisotropic forms over F, $\dim\varphi = 6$, $\dim\psi \geqslant 5$ and ψ is not a 3-fold Pfister neighbor. Suppose that $\varphi_{F(\psi)}$ is isotropic. Then φ has a subform similar to ψ.

(1.7) Theorem ([5]). Let φ be an anisotropic form, $\dim\varphi = 7$. Then $i_1(\varphi) \neq 2$.

(1.8) Theorem ([12], [17]). Let F be a field, D a division algebra over F, q an F-form. Then the following conditions are equivalent:

a) $D_{F(q)}$ is not a division algebra.

b) There exists an F- algebra homomorphism $C_0(q) \longrightarrow D$.

In particular, if $q \in I^3(F)$, then $D_{F(q)}$ is a division algebra.

(1.9) Theorem ([16]). Let D be a division algebra over F of exponent 2, $a \in F^*$, t an indeterninate variable. Then $D_{F(\sqrt{a})}$ is a division algebra if and only if $D_{F(t)} \otimes (a,t)$ is a division algebra over $F(t)$.

(1.10) Theorem ([13]). For any field F there is an isomorphism $H^3(F) \simeq I^3(F)/I^4(F)$ such that for any $a,b,c \in F^*$ the cup product $l_a \cup l_b \cup l_c \in H^3(F)$ goes to $\langle\langle a,b,c\rangle\rangle \mod I^4(F)$.

(1.11) Theorem ([15]). Let D be a central division F-algebra of exponent 2. Suppose that D is decomposable (into the tensor product of two proper subalgebras). Then $H^3(F(D)/F) = H^1(F) \cup [D]$.

(1.12) Theorem ([7]). Let A be a central simple F-algebra, and let σ be an anisotropic form over F. If $\dim\sigma \leqslant 5$, then $(\sigma_{F(A)})_{\mathrm{an}}$ is defined over F.

(1.13) Theorem ([6]). Let φ be an anisotropic form over F, $\dim\varphi = 8$, $\varphi \notin I^2(F)$ or $\varphi \in I^2(F)$ and $\operatorname{ind}\varphi = 8$. Then for any F-form σ the form $(\varphi_{F(\varphi)})_{\mathrm{an}}$ is not similar to $\sigma_{F(\varphi)}$.

(1.14) Theorem ([6]). Let φ be an anisotropic form, $\dim \varphi = 8$, $\varphi \notin I^2(F)$. Then $i_1(\varphi) = 1$.

(1.15) Theorem ([4]). Let φ be an anisotropic form, $\dim \varphi \leqslant 6$, let π be an anisotropic 3-fold Pfister form. Suppose that $\varphi_{F(\pi)}$ is isotropic. Then there is an F-form τ such that $(\varphi_{F(\pi)})_{\mathrm{an}} \simeq \tau_{F(\pi)}$.

(1.16) Theorem ([2], [8]). Let β be an anisotropic form, $\dim \beta = 8$, $\beta \in I^2(F)$, $\operatorname{ind} \beta = 8$. Let further D be a division algebra corresponding to β under the isomorphism $I^2(F)/I^3(F) \simeq_2 \operatorname{Br}(F)$. Then the form $\beta_{F(D)}$ is anisotropic.

(1.17) Theorem ([11]). Let β be an anisotropic form, $\dim \beta = 8$, $\beta \in I^2(F)$, $\operatorname{ind} \beta = 4$. Let further D be a division algebra corresponding to β under the isomorphism $I^2(F)/I^3(F) \simeq_2 \operatorname{Br}(F)$. Then the form $\beta_{F(D)}$ is anisotropic.

2. Splittings of forms

In this section we solve the problem in question for two cases: Propositions 2.1 and 2.3 deal with the case where $\dim \varphi = 5$, $\dim \psi = 6$ and Proposition 2.4 treats the case where $\dim \varphi = 6$, $\dim \psi = 7$. Recall that we reserve the letter L for the generic field over the ground field F such that $i(\psi_L) \geqslant 2$. In the sequel we use a few times the following rather obvious statement.

Let φ and ψ be anisotropic F-forms. Suppose φ_L is isotropic and K/F is such a field extension that $i(\psi_K) \geqslant 2$. Then φ_K is isotropic.

(Indeed, it is easy to see that the field extension LK/K is purely transcendental, so since φ is isotropic over LK, it remains isotropic over K).

Proposition 2.1. *Let F be a field, let ψ and φ be anisotropic forms over F such that $\dim \psi = 6$, $\dim \varphi = 5$, $\operatorname{ind} \varphi = 2$, φ is a neighbor of $\pi \in P_3(F)$.*

1) Suppose $\operatorname{ind} \psi = 1$, ψ is a neighbor of $\rho \in P_3(F)$. Then φ_L is isotropic iff $\pi \simeq \rho$.

2) Suppose $\operatorname{ind} \psi = 2$, $\psi_{F(\widehat{\psi})}$ is a neighbor of $\rho_{F(\widehat{\psi})}$ where $\rho \in P_3(F)$. Then φ_L is isotropic iff either $[(\pi \perp -\rho)]_{F(\widehat{\psi})} = 0$ or $[\pi_{F(\widehat{\psi})}] = 0$.

3) Suppose $\operatorname{ind} \psi = 4$. Then φ_L is isotropic iff there exists $a \in F^*$ such that $\pi \equiv \langle\langle a\rangle\rangle \otimes \psi \pmod{I^4(F)}$. In this case $[\langle\langle a, d_\pm(\psi)\rangle\rangle] = 0$.

Proof.

1) Obvious.

2) Recall that if $\psi \sim \langle\langle a, b\rangle\rangle \perp \langle -c, cd\rangle$, then we set $\widehat{\psi}$ to be any form similar to $\langle ab, -a, -b, d\rangle$. Since $(\psi_{F(\psi)})_{\mathrm{an}} \sim \widehat{\psi}_{F(\psi)}$ we have the following chain of equivalent conditions:

φ_L is isotropic $\Leftrightarrow \varphi_{F(\psi,\widehat{\psi})}$ is isotropic $\Leftrightarrow \varphi_{F(\widehat{\psi},\rho)}$ is isotropic $\Leftrightarrow \pi_{F(\widehat{\psi},\rho)}$ is isotropic $\Leftrightarrow$ either $\pi_{F(\widehat{\psi})} = 0$, or $\pi_{F(\widehat{\psi})} = \rho_{F(\widehat{\psi})}$.

3) This case is the most difficult one. If $\pi \equiv \langle\langle a\rangle\rangle \otimes \psi \pmod{I^4(F)}$, then $\pi_L \equiv \langle\langle a\rangle\rangle \otimes (\psi_L)_{\mathrm{an}} \pmod{I^4(L)}$. Since $\dim\,(\psi_L)_{\mathrm{an}} = 2$ we obviously have $\dim\,(\pi_L \perp -\langle\langle a\rangle\rangle \otimes \psi_L)_{\mathrm{an}} \leqslant 8 + 4 = 12$, therefore, it follows from Theorem 1.1 that $(\pi_L)_{\mathrm{an}} \simeq (\langle\langle a\rangle\rangle \otimes \psi_L)_{\mathrm{an}}$, so $\dim\,(\pi_L)_{\mathrm{an}} \leqslant 4$. Applying theorem 1.1 again we get $[\pi_L] = 0$, and so the form φ_L is isotropic. Moreover, since $(\psi_L)_{\mathrm{an}} \sim \langle\langle d_\pm(\psi)\rangle\rangle$ we get $[\langle\langle a, d_\pm(\psi)\rangle\rangle_L] = 0$, hence $[\langle\langle a, d_\pm(\psi)\rangle\rangle] = 0$ as well.

Assume now that φ_L is isotropic. If $\psi \in I^2(F)$, then the assertion 3) follows from Theorem 1.11. Assume now that $\psi \notin I^2(F)$. Put $d = d_\pm(\psi)$. Take any $c \in D(\psi)$ and let $\psi \simeq \psi' \perp \langle c\rangle$. Then $\alpha \simeq \psi' \perp \langle -d_\pm(\psi')\rangle$ is an Albert form. Let A be a division algebra corresponding to α. If t is an indeterminate, then by Theorem 1.9 we get $D = A \otimes_F (d, t)$ is a division algebra as well. Put $K = F(t)$.

Lemma 2.2. $[\pi_{K(D)}] = 0$.

Proof. It is evident that

$$[\psi]_{K(\sqrt{d},D)} = [\alpha]_{K(\sqrt{d},D)} = [(a, d)]_{K(\sqrt{d},D)} = 0.$$

So $\dim\,(\psi_{K(D)})_{\mathrm{an}}$ is either 2 or 6. If $\dim\,(\psi_{K(D)})_{\mathrm{an}} = 2$, then $i(\psi_{K(D)}) = 2$, and since φ_L is isotropic the form $\varphi_{K(D)}$ is isotropic too, hence $[\pi_{K(D)}] = 0$. If $\dim\,(\psi_{K(D)})_{\mathrm{an}} = 6$, then $\psi_{K(D)} \simeq \langle 1, -d\rangle \otimes \tau$ for some $K(D)$-form τ, so $\psi_{K(D)}$ is a Pfister neighbor and $(\psi_{K(\psi,D)})_{\mathrm{an}} = 2$. Hence $[\pi_{K(\psi,D)}] = 0$. Suppose now that $[\pi_{K(D)}] \neq 0$. Since $[\pi_{K(D,\psi)}] = 0$ we get $\psi_{K(D)}$ is a neighbor of $\pi_{K(D)}$. Since $c \in D(\psi)$ we have $\psi_{K(D)} \subset c\pi_{K(D)}$. Consider the form $\beta = \psi \perp \langle -ct, ctd_\pm(\psi)\rangle$. Obviously $\beta \in I^2(K)$ and β corresponds to D, hence $\operatorname{ind}\beta = 8$. By Theorem 1.16 the form $\beta_{K(D)} \in I^3(K(D))$ is anisotropic. Then

$$\dim\,(\beta_{K(D)} \perp -c\pi_{K(D)})_{\mathrm{an}} \leqslant$$
$$\dim\,(\beta_{K(D)} \perp -\psi_{K(D)})_{\mathrm{an}} + \dim\,(\psi_{K(D)} \perp -c\pi_{K(D)})_{\mathrm{an}} = 2 + 2 = 4 < 8.$$

Thus $\beta_{K(D)} \simeq c\pi_{K(D)}$. Therefore, $[(\beta_{K(\pi)})_{K(\pi)(D)}] = 0$. Besides $C(\beta_{K(\pi)}) \sim D_{K(\pi)}$ and by Theorem 1.8 we obtain $\operatorname{ind} D_{K(\pi)} = 8$. Hence

$\beta_{K(\pi)}$ is anisotropic. By Theorem 1.16 we get a contradiction. The lemma is proved.

On the other hand, by Theorems 1.11 and 1.10 we have $H^3(K(D)/K) = [D] \cup H^1(K)$ and $H^3(K) = I^3(K)/I^4(K)$. Thus, in view of Lemma 2.2 we have

$$\pi \equiv \langle\langle p \rangle\rangle \otimes (\alpha \perp \langle\langle d, t \rangle\rangle) \pmod{I^4(K)} \qquad (*)$$

for some $p \in F[t]$. Clearly, we may assume that p is squarefree. Now take into consideration the first residue map $d_t : I^3(K)/I^4(K) \longrightarrow I^2(F)/I^3(F)$ which is well defined by the following rule:

$$d_t(\langle\langle p_1, p_2, p_3 t^n \rangle\rangle \pmod{I^4(K)}) = n\langle\langle p_1(0), p_2(0) \rangle\rangle \pmod{I^3(F)}$$

if $p_1, p_2, p_3 \in F[t]$ and $p_1(0), p_2(0), p_3(0) \neq 0$. Since d_t takes π to zero, $p(0) \neq 0$, for otherwise applying d_t to the both parts of $(*)$ we would get $\alpha \equiv \langle\langle d, -\frac{p}{t}(0) \rangle\rangle \mod I^3(F)$, a contradiction. Hence $[\langle\langle p(0), d \rangle\rangle] = d_t([\pi - \langle\langle p \rangle\rangle \otimes \alpha]) = 0$. Besides, $[\alpha] = [\psi] + [\langle x, -xd \rangle]$ for $x = -d_{\pm}(\psi')$, and so

$$\pi \equiv \langle\langle p \rangle\rangle \otimes (\psi \perp \langle x, -xd \rangle \perp \langle\langle d, t \rangle\rangle) = \langle\langle p \rangle\rangle \otimes \psi \perp \langle\langle p, d \rangle\rangle \otimes \langle 1, x, -t \rangle \equiv$$
$$\langle\langle p \rangle\rangle \otimes \psi \perp xt\langle\langle p, d \rangle\rangle \pmod{I^4(K)}. \qquad (**)$$

Finally consider the map $s_t : W(K) \longrightarrow W(F)$ which is well defined by the following equation: $s_t([\langle pt^n \rangle]) = [\langle p(0) \rangle]$ if $p \in F[t]$ and $p(0) \neq 0$. Applying s_t to $(**)$ we obtain the desirable result. The proposition is proved.

Proposition 2.3. *Let F be a field, let ψ and φ be anisotropic forms over F such that* $\dim \psi = 6$, $\dim \varphi = 5$, $\operatorname{ind} \varphi = 4$.

1) *Suppose* $\operatorname{ind} \psi = 1$. *Then φ_L is anisotropic.*

2) *Suppose* $\operatorname{ind} \psi = 2$. *Then φ_L is isotropic iff φ has a subform similar to $\widehat{\psi}$ or φ is similar to a subform of ψ.*

3) *Suppose* $\operatorname{ind} \psi = 4$, $\psi \notin I^2(F)$. *Then φ_L is isotropic iff φ is similar to a subform of ψ.*

4) *Suppose* $\operatorname{ind} \psi = 4$, $\psi \in I^2(F)$. *Then φ_L is isotropic iff for some $\lambda \in F^*$ the forms φ and $\lambda\psi$ have a common 4- dimensional subform.*

Proof. 1) Since by Theorem 1.8 the algebra $C_0(\varphi)_{F(\psi)}$ is a skewfield, the form $\varphi_{F(\psi)}$ is anisotropic.

2) If φ contains a subform similar to $\widehat{\psi}$ or φ is similar to a subform of ψ, then evidently φ_L is isotropic. Now suppose that φ_L is isotropic.

Let $K = F(\psi)$, $\tau = (\psi_K)_{\text{an}}$. By Theorem 1.8 φ_K is anisotropic and $\operatorname{ind}\varphi_K = 4$. On the other hand, $\varphi_{K(\tau)}$ is isotropic. Hence it follows from Theorem 1.5 that τ is similar to a subform of φ_K, that is $x\tau \perp \langle y\rangle \simeq \varphi_K$ for some $x, y \in K$. Comparing the determinants in the last equality we get $y = d_\pm(\psi)d_\pm(\varphi)$. Let $\beta \simeq \varphi \perp \langle -d_\pm(\psi)d_\pm(\varphi)\rangle$. We obtain that β_K is isotropic. If β is anisotropic, then, since ψ is not a Pfister neighbor we have by Theorem 1.6 that $\beta \sim \psi$, and so φ is similar to a subform of ψ. If β is isotropic, then $\varphi \simeq \tau_1 \perp \langle d_\pm(\psi)d_\pm(\varphi)\rangle$ for some form τ_1. Now notice that by Theorem 1.3 $\tau \sim \widehat{\psi}_K$. Hence it follows that $\varphi_{K(\widehat{\psi})}$ is isotropic and so by Theorem 1.5 $\varphi_K \simeq z\widehat{\psi} \perp \langle d_\pm(\psi)d_\pm(\varphi)\rangle$ for some $z \in K$. Therefore, we conclude that the forms $\widehat{\psi}_K$ and τ_{1K} are similar, hence $C_0(\widehat{\psi})_K \simeq C_0(\tau_1)_K$. Then by Theorem 1.8 $[C_0(\widehat{\psi})] = [C_0(\tau_1)]$, and so by Theorem 1.2 $\widehat{\psi} \sim \tau_1$ over F.

3) We preserve the notation in the proof of 2). Let α be any Albert form over F such that $\alpha \simeq \psi_{F(\sqrt{d_\pm(\psi)})}$. Then by Theorem 1.6 the form $\psi_{F(\alpha)}$ is anisotropic and $\operatorname{ind}\psi_{F(\alpha)} = 2$. Suppose φ_L is isotropic. Following the proof of 2) it suffices to show that the form $\beta \simeq \varphi \perp \langle -d_\pm(\psi)d_\pm(\varphi)\rangle$ is anisotropic. If this is not the case, then as was shown in the proof of 2), over $F(\alpha)$ the form $\widehat{\psi_{F(\alpha)}}$ is similar to some F- form τ_1. Then $\psi_{F(\alpha,\tau_1)}$ is a Pfister neighbor, hence $[\psi_{F(\alpha,\tau_1,\sqrt{d_\pm(\psi)})}] = 0$ which in turn implies that $[\psi_{F(\tau_1,\sqrt{d_\pm(\psi)})}] = 0$ and therefore, $\operatorname{ind}\psi_{F(\tau_1)} = 1$. But this is impossible because $\operatorname{ind}\psi = 4$.

4) Since $[\psi_L] = 0$ it is clear that if φ and $\lambda\psi$ have a common 4-dimensional subform, then φ_L is isotropic.

Let D be a skewfield corresponding to ψ, $K = F(D)$. By Theorem 1.12 the form $(\varphi_K)_{\text{an}}$ is defined over F. Now suppose φ_L is isotropic. Since $i(\psi_K) = 3$ it follows that φ_K is isotropic. Let $\tau \simeq (\varphi_K)_{\text{an}}$. If $\dim\tau = 1$ then there is $a \in F^*$ such that $[(\varphi \perp \langle a\rangle)]_K = 0$. Then $(\varphi \perp \langle a\rangle)_{F(\psi)}$ is isotropic and it follows from Theorem 1.6 that $\varphi \perp \langle a\rangle \sim \psi$. Suppose $\dim\tau = 3$. It is clear that $C([\varphi \perp -\tau]) \sim D$. We have $[(\varphi \perp -\tau)_K] = 0$. Hence by Theorem 1.17 $\dim(\varphi \perp -\tau)_{\text{an}} = 6$ and so $\psi \sim (\varphi \perp -\tau)_{\text{an}}$. Therefore, there exists $\lambda \in F^*$ such that $\varphi \perp -\lambda\psi = \tau \perp 4\mathbb{H}$ which proves this part of the proposition.

Proposition 2.4. *Let* $\dim\psi = 7$, $\dim\varphi = 6$.

1) *Suppose* $\operatorname{ind}\psi = 1$. *Then* φ_L *is isotropic if and only if* φ *has a subform isomorphic to a neighbor of* $\psi \perp d_\pm(\psi) \in GP_3(F)$.

2) *Suppose* $\operatorname{ind}\psi = 2$, $\operatorname{ind}\varphi = 1$, φ *is a neighbor of* $\pi \in P_3(F)$,

$C_0(\psi) \sim (u,v)$. Then φ_L is isotropic iff $\dim\,(\psi_{F(\pi)})_{\mathrm{an}} = 3$, or equivalently iff there exists $s \in F^*$ such that $\psi \perp -d_\pm(\psi)\langle u,v,-uv\rangle \simeq s\pi \perp \mathbb{H}$.

3) *Suppose* $\operatorname{ind}\psi = 2$, $\operatorname{ind}\varphi = 2$, $\varphi \sim \langle\langle a,b\rangle\rangle \perp \langle -c, cd\rangle$, $C_0(\psi) \sim (u,v)$. *Then $\psi \perp -d_\pm(\psi)\langle u,v,-uv\rangle \simeq s\rho \perp \mathbb{H}$ for some $s \in F^*$ and 3-fold Pfister form ρ. Moreover φ_L is isotropic iff $\rho_{F(\widehat{\varphi})} \simeq \langle\langle a,b,c\rangle\rangle_{F(\widehat{\varphi})}$.*

4) *Suppose* $\operatorname{ind}\psi \le 2$, $\operatorname{ind}\varphi \ge 4$. *Then φ_L is anisotropic.*

5) *Suppose* $\operatorname{ind}\psi = 4$, *$\alpha$ be an Albert form corresponding to ψ. Then φ_L is isotropic iff φ is similar to either α or a subform of ψ.*

6) *Suppose* $\operatorname{ind}\psi = 8$. *Then φ_L is isotropic iff φ is similar to a subform of ψ.*

Proof. 1) This follows from Theorem 1.15.

2) Suppose φ_L is isotropic. Then by Theorem 1.7 $\dim\,(\psi_{F(\psi)})_{\mathrm{an}} = 5$, so $(\psi_{F(\psi)})_{\mathrm{an}}$ is a neighbor of π and $\dim\,(\psi_{F(\psi,\pi)})_{\mathrm{an}} = 3$. If $\psi_{F(\pi)}$ is anisotropic, then $i_1(\psi_{F(\pi)}) = 2$, which is impossible in view of Theorem 1.7. Hence $\psi_{F(\pi)}$ is isotropic, hence $\dim\,(\psi_{F(\pi)})_{\mathrm{an}} = \dim\,(\psi_{F(\pi,\psi)})_{\mathrm{an}} = 3$ and therefore, $(\psi_{F(\pi)})_{\mathrm{an}} \simeq d_\pm(\psi)\langle u,v,-uv\rangle$. By Theorem 1.7 the first case is impossible. It follows that

$$\psi \perp -d_\pm(\psi)\langle u,v,-uv\rangle \simeq s\pi \perp \mathbb{H}$$

for some $s \in F^*$. Conversely, if $\dim\,((\psi_{F(\pi)})_{\mathrm{an}}) = 3$, then $(\psi_{F(\psi)})_{\mathrm{an}}$ is isotropic over $F(\psi,\pi)$, and so $(\psi_{F(\psi)})_{\mathrm{an}}$ is a neighbor of $(\pi_{F(\psi)})_{\mathrm{an}}$. Therefore, φ_L is isotropic.

3) It is clear that the form $\overline{\psi} \simeq \psi \perp -d_\pm(\psi) \in I^2(F)$ and $\operatorname{ind}\overline{\psi} = 2$. It is easy to see that there is $a \in F^*$ such that $\overline{\psi} \simeq \langle\langle a\rangle\rangle \otimes \tau$ for some form τ. Hence there are forms γ and $\rho \in P_3(F)$ and $s \in F^*$ such that $\dim\gamma = 3$ and $\psi' \perp -\gamma \simeq s\rho$. Since $C_0(\psi') \sim (u,v)$ we have $\gamma \simeq d_\pm(\psi)\langle u,v,-uv\rangle$.

Suppose φ_L is isotropic. Since $\varphi_{F(\widehat{\varphi})}$ is anisotropic, $\dim\,(\psi_{F(\widehat{\varphi})})_{\mathrm{an}} \ge 5$, that is $\rho_{F(\widehat{\varphi})} \ne 0$. If $\dim\,(\psi_{F(\widehat{\varphi})})_{\mathrm{an}} = 7$, then replacing ψ and φ by $\psi_{F(\widehat{\varphi})}$ and $\varphi_{F(\widehat{\varphi})}$ we reduce this case to case 2). If $\dim\,(\psi_{F(\widehat{\varphi})})_{\mathrm{an}} = 5$, then $(\psi_{F(\widehat{\varphi})})_{\mathrm{an}}$ is a neighbor of both of nonzero Pfister forms $\langle\langle a,b,c\rangle\rangle_{F(\widehat{\varphi})}$ and $\rho_{F(\widehat{\varphi})}$. Consequently, we have $[\langle\langle a,b,c\rangle\rangle \perp -\rho]_{F(\widehat{\varphi})} = 0$.

Conversely, suppose that φ_L is anisotropic and $\langle\langle a,b,c\rangle\rangle_{F(\widehat{\varphi})} = \rho_{F(\widehat{\varphi})}$. Then

$$0 = [\rho_{L(\widehat{\varphi})}] = [\langle\langle a,b,c\rangle\rangle_{L(\widehat{\varphi})}] = [(\varphi_L)_{L(\widehat{\varphi_L})} \perp \beta],$$

where $\dim\beta = 2$, and so $(\varphi_L)_{L(\widehat{\varphi_L})}$ is isotropic, a contradiction to Theorem 1.3.

4) If $\operatorname{ind}\psi = 2$, then by Theorem 1.8 the form $\varphi_1 = \varphi_{F(\psi)}$ is anisotropic and $\operatorname{ind}\varphi_1 \ge 4$. On the other hand, the form $\tau = (\psi_{F(\psi)})_{\mathrm{an}}$ is a Pfister

neighbor. Applying Theorem 1.8 again we get that ${\varphi_1}_{F(\psi)(\tau)}$ is anisotropic, and so φ_L is anisotropic.

5) It is evident that if φ is a subform of ψ, then φ_L is isotropic. Since $\operatorname{ind}\psi_L = 2$ it follows that φ_L also isotropic if $\varphi \sim \alpha$. Let, conversely, φ_L is isotropic. Consider two possible cases:

a) $\psi \perp \langle -d_\pm(\psi)d_\pm(\varphi)\rangle$ is isotropic.

b) $\psi \perp \langle -d_\pm(\psi)d_\pm(\varphi)\rangle$ is anisotropic.

In case a) there exists a subform φ_1 of ψ such that $d_\pm(\varphi_1) = d_\pm(\varphi)$ and $\dim \varphi_1 = 6$. Since φ_L and ${\varphi_1}_L$ are isotropic and $\operatorname{ind}\,(\psi_{F(\psi)})_{\mathrm{an}} = 4$ by Theorem 1.6 there exist $a, a_1 \in F(\psi)^*$ such that $a(\psi_{F(\psi)})_{\mathrm{an}} \subset \varphi_{F(\psi)}$ and $a_1(\psi_{F(\psi)})_{\mathrm{an}} \subset {\varphi_1}_{F(\psi)}$. It follows that $\tau = (aa_1\varphi_{F(\psi)} \perp -{\varphi_1}_{F(\psi)}) \in I^2(F(\psi))$ and $\dim \tau_{\mathrm{an}} \le 2$. Therefore, $aa_1\varphi_{F(\psi)} \simeq {\varphi_1}_{F(\psi)}$. Consider the form $\sigma = {\varphi_1}_{F(\varphi)}$. Since $\sigma_{F(\varphi,\psi)}$ is isotropic and $\psi_{F(\varphi)}$ is not a Pfister neighbor one can see that by Theorem 1.6 σ is itself isotropic. But then, since φ is not a Pfister neighbor we have also by Theorem 1.6 $\varphi \sim \varphi_1$ and the proof is complete.

Case b) can be reduced to case a). Let $\beta = \psi \perp \langle -d_\pm(\psi)d_\pm(\varphi)\rangle$ and $K = F(\beta)$. Notice that by Theorem 1.14 we have $i_1(\beta) = 1$. The subform $\varphi_1 = (\beta_K)_{\mathrm{an}}$ of ψ_K is such that $\dim \varphi_1 = 6$, and $d_\pm(\varphi_1) = d_\pm(\varphi)$. Replacing the field F by K we are under conditions of case a), so $\varphi_K \sim \varphi_1$. By Theorem 1.13 this is possible only if $\beta \in I^2(F)$ and $\operatorname{ind}\beta = 4$. In this case $\varphi_K \sim \varphi_1 \sim \alpha_K$. But then $C(\varphi)_K \simeq C(\alpha)_K$. Hence by Theorem 1.8 $C(\varphi) \simeq C(\alpha)$, and by Theorem 1.4 $\varphi \sim \alpha$.

6) The proof of this part coincides with that of part 5) except that the proof of part 5) shows that the case b) is impossible.

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