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BEREZIN KERNELS AND MAXIMAL DEGENERATE REPRESENTATIONS ASSOCIATED WITH RIEMANNIAN SYMMETRIC SPACES OF HERMITIAN TYPE

ABSTRACT. We introduce the Berezin kernels for Riemannian Symmetric Spaces of Hermitian type by restricting the maximal degenerate representations of the corresponding non-compactly causal Lie groups.

INTRODUCTION

There are several ways to introduce Berezin kernels. From a historical point of view the theory of quantization would be the right context (see [1], [2]). We will make a more modern choice however and take the theory of group representations as a starting point.

There are essentially two ways of introduction, each having its own merits. An a priori question is about the class of group we will consider. It appears that the class of "compactly causal" groups is the best choice. We include therefore the table (Table 1), which contains the list of these groups (Lie algebras, more precisely). So let us work with a connected, simply connected, complex Lie group G_c with Lie algebra $\mathfrak{g}_c$, the complexification of $\mathfrak{g}$. Let G, G_c, H be the connected subgroups of G_c with Lie algebras $\mathfrak{g}$, $\mathfrak{g}^c$ and $\mathfrak{h}$ respectively. Observe that G/K, where K is a maximal compact subgroup of G, is a Hermitian symmetric space. So it make sense to consider holomorphic discrete series representations π_ℓ of G and to restrict them to H. The problem one poses is: how does π_ℓ, after restriction to H, decompose into irreducible unitary representations of H? The restriction of π_ℓ to H is called a canonical representation. The problem can be reformulated in terms of positive-definite (spherical) functions, like similar questions for finite groups can be formulated in terms of characters. The positive-definite functions which arise are coming from certain G-invariant kernels, called Berezin kernels. These kernels depend of course on $\ell \in \mathbb{Z}$ but turn out to be defined for all $\ell \in \mathbb{R}$. We thus get a one-parameter family of kernels and taking the limit $\ell \to \infty$

corresponds to taking the limit to the "classical case" in the theory of quantization after Berezin.

The problem was solved by Berezin [1], Upmeier and Unterberger [18] (upper part of Table 1) and by Van Dijk, Pevzner (see [5]), Neretin [13], Ørsted and Zhang [20] (lower part of Table 1).

Table 1. Irreducible compactly causal pairs.

$\mathfrak{g}$ compactly causal	$\mathfrak{g}^c$ non-compactly causal	$\mathfrak{h}$
$\mathfrak{su}(p,q)\oplus\mathfrak{su}(p,q)$	$\mathfrak{sl}(p+q;\mathbf{C})$	$\mathfrak{su}(p,q)$
$\mathfrak{so}^*(2n)\oplus\mathfrak{so}^*(2n)$	$\mathfrak{so}(2n;\mathbf{C})$	$\mathfrak{so}^*(2n)$
$\mathfrak{so}(2,n)\oplus\mathfrak{so}(2,n)$	$\mathfrak{so}(2+n;\mathbf{C})$	$\mathfrak{so}(2,n)$
$\mathfrak{sp}(n,\mathbf{R})\oplus\mathfrak{sp}(\mathbf{n},\mathbf{R})$	$\mathfrak{sp}(n,\mathbf{C})$	$\mathfrak{sp}(n,\mathbf{R})$
$\mathfrak{e}_{6(-14)}\oplus\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(-14)}$
$\mathfrak{e}_{7(-25)}\oplus\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_7$	$\mathfrak{e}_{7(-25)}$
$\mathfrak{su}(p,q)$	$\mathfrak{sl}(p+q;\mathbf{R})$	$\mathfrak{so}(p,q)$
$\mathfrak{su}(n,n)$	$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n;\mathbf{C})\oplus\mathbf{R}$
$\mathfrak{su}(2p,2q)$	$\mathfrak{su}^*(2(p+q))$	$\mathfrak{sp}(p,q)$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}(n,n)$	$\mathfrak{so}(n;\mathbf{C})$
$\mathfrak{so}^*(4n)$	$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n)\oplus\mathbf{R}$
$\mathfrak{so}(2,p+q)$	$\mathfrak{so}(p+1,q+1)$	$\mathfrak{so}(p,1)\times\mathfrak{so}(1,q)$
$\mathfrak{sp}(n,\mathbf{R})$	$\mathfrak{sp}(n,\mathbf{R})$	$\mathfrak{sl}(n;\mathbf{R})\oplus\mathbf{R}$
$\mathfrak{sp}(2n,\mathbf{R})$	$\mathfrak{sp}(n,n)$	$\mathfrak{sp}(n,\mathbf{C})$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(2,2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)}\oplus\mathbf{R}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{su}^*(8)$

Berezin kernels arise also in a natural way by restricting representations of the "non-compactly causal" group G^c, induced from a (specific) maximal parabolic subgroup P^c, to H, and to subspaces of functions living on open H-orbits in G^c/P^c. These restricted representations are then provided with an invariant scalar product in the same style as is known from the construction of the "complementary series". The method was discussed for the group case (upper part of Table 1) in [3]. This alternative introduction of Berezin kernels has a great advantage: it can be generalized even to groups far beyond those in Table 1, in particular to para-Hermitian groups (Molchanov [12], Hille [8]).

This short note extends the methods from [3] to all cases of Table 1, i.e. all non-compactly causal pairs.

1. Berezin kernels

In this section we define the Bergman and Berezin kernels for Hermitian symmetric spaces and more generally, for symmetric spaces of Hermitian type.

1.1. For Hermitian symmetric spaces.

We begin with recalling some structure theory, mainly following [17] and [7], Ch. VIII. So, let $\mathfrak{g}$ be a non-compact simple real Lie algebra with complexification $\mathfrak{g}_c$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition and let θ denote the corresponding Cartan involution. Let $\mathfrak{z}$ denote the center of $\mathfrak{k}$. The Lie algebra $\mathfrak{g}$ is said to be Hermitian if the centralizer of $\mathfrak{z}$ in $\mathfrak{g}$ is equal to $\mathfrak{k}$. The center of $\mathfrak{k}$ is one-dimensional and there is an element $Z_0 \in \mathfrak{z}$ such that $(\mathrm{ad}Z_0)^2 = -1$ on $\mathfrak{p}$. Fixing i a square root of -1, one has $\mathfrak{p}_c = \mathfrak{p} + i\mathfrak{p} = \mathfrak{p}_+ + \mathfrak{p}_-$ where $\mathrm{ad}Z_0\big|_{\mathfrak{p}+} = i$, $\mathrm{ad}Z_0\big|_{\mathfrak{p}_-} = i$. Then

$$\mathfrak{g}_c = \mathfrak{p}_+ \oplus \mathfrak{k}_c \oplus \mathfrak{p}_-. \tag{1}$$

and $[p_\pm, \mathfrak{p}_\pm] = 0$, $[\mathfrak{p}_+, \mathfrak{p}_-] = \mathfrak{k}_c$ and $[\mathfrak{k}_c, \mathfrak{p}_\pm] = \mathfrak{p}_\pm$.

Let G_c be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}_c$ and K_c, P_+, P_-, G, K the analytic subgroups corresponding to $\mathfrak{k}_c$, $\mathfrak{p}_+$, $\mathfrak{p}_-$, $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Then K_cP_- (and K_cP_+) is a maximal parabolic subgroup of G_c with split component $A = \exp i\mathbb{R}Z_0$. G is closed in G_c.

Moreover, the exponential mapping is a diffeomorphism of $\mathfrak{p}_-$ onto P_- and of $\mathfrak{p}_+$ onto P_+ ([7], Ch. VIII, Lemma 7.8). Furthermore:

Lemma 1. a. *The mapping $(q, k, p) \mapsto qkp$ is a diffeomorphism of $P_+ \times K_c \times P_-$ onto an open dense submanifold of G_c containing G.*
b. *The set GK_cP_- is open in $P_+K_cP_-$ and $G \cap K_cP_- = K$.*

(See [7], Ch. VIII, Lemmas 7.9 and 7.10).

Thus G/K is mapped on a open, bounded domain $\mathcal{D}$ in $\mathfrak{p}_+$. G acts on $\mathcal{D}$ via holomorphic transformations.

Example. Let $\mathfrak{g} = \mathfrak{su}(1,1)$. Then $G_c = SL(2,\mathbb{C})$ and $G = SU(1,1)$. Clearly $Z_0 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, $\mathfrak{p}_\pm$ are one-dimensional and generated by

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ respectively. Let $g = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \in G$. Then the decomposition $g = qkp$ (see Lemma 1.a) is given by

$$g = \begin{pmatrix} 1 & \beta\bar\alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar\alpha^{-1} & 0 \\ 0 & \bar\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar\alpha^{-1}\bar\beta & 1 \end{pmatrix}$$

where $|\alpha|^2 - |\beta|^2 = 1$. The embedding of G/K into $\mathbb{C}$ is given by

$$g \mapsto \beta\bar\alpha^{-1} = \zeta.$$

Since $|\alpha|^2 - |\beta|^2 = 1$, it follows $|\zeta| < 1$. Conversely, let $|\zeta| < 1$. Take then α such that $|\alpha|^2 = (1 - |\zeta|^2)^{-1}$ and let $\beta = \zeta\bar\alpha$. Then $\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$ is mapped onto ζ. So $\mathcal{D}$ is the unit disk "$|\zeta| < 1$". G acts on $\mathcal{D}$ by means of fractional linear transformations

$$g.\zeta = \frac{\alpha\zeta + \beta}{\bar\beta\zeta + \bar\alpha}, \qquad g = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \in G.$$

Everywhere we shall denote $\bar g$ the complex conjugate of $g \in G_c$ with respect to G. So, for example, if $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL(2,\mathbb{C})$, then its conjugate with respect to $SU(1,1)$ is given by $\bar g = \begin{pmatrix} \bar a^{-1} & 0 \\ 0 & \bar a \end{pmatrix}$. Notice that $\bar P_+ = P_-$.

For $g \in P_+K_cP_-$ we shall write $g = (g)_+(g)_0(g)_-$, where $(g)_\pm \in P_\pm$, $(g)_0 \in K_c$. For $g \in G_c$, $z \in \mathfrak{p}_+$ such that $g.\exp z \in P_+K_cP_-$ we define

$$\exp g(z) = (g.\exp z)_+ \tag{2}$$
$$J(g,z) = (g.\exp z)_0. \tag{3}$$

$J(g,z) \in K_c$ is called the *canonical automorphic factor* of G_c (terminology of Satake).

Lemma 2. *The canonical automorphic factor J satisfies*

(i) $J(g,o) = (g)_0$, *for* $g \in P_+K_cP_-$,

(ii) $J(k,z) = k$, *for* $k \in K_c$, $z \in \mathfrak{p}_+$.

If for $g_1, g_2 \in G_c$ and $z \in \mathfrak{p}_+$, $g_1(g_2(z))$ and $g_2(z)$ are defined, then $(g_1g_2)(z)$ is also defined and

(iii) $J(g_1g_2,z) = J(g_1,g_2(z))J(g_2,z)$.

(see [17], Ch. II, Lemma 5.1).

For $z, w \in \mathfrak{p}_+$ satisfying $(\exp \bar{w})^{-1}.\exp z \in P_+K_cP_-$ we define

$$K(z,w) = J\big((\exp \bar{w})^{-1}, z)\big)^{-1} \tag{4}$$

$$= \big((\exp \bar{w})^{-1}.\exp z\big)_0^{-1}. \tag{5}$$

This expression is always defined for $z, w \in \mathcal{D}$, for then

$$(\exp \bar{w})^{-1}.\exp z \in \overline{(GK_cP_-)}^{-1} GK_cP_- = P_+K_cGK_cP_- = P_+K_cP_-.$$

$K(z,w)$, defined on $\mathcal{D}\times\mathcal{D}$, is called the canonical kernel on $\mathcal{D}$ (by Satake). $K(z,w)$ is holomorphic in z, anti-holomorphic in w, with values in K_c. Here are a few properties:

Lemma 3. (i) $K(z,w) = \overline{K(w,z)}^{-1}$ *if $K(z,w)$ is defined,*

(ii) $K(o,w) = K(z,o) = 1$ *for $z, w \in \mathfrak{p}_+$.*

When $g(z), \bar{g}(w)$ and $K(z,w)$ are defined, then $K(g(z), \bar{g}(w))$ is also defined and one has:

(iii) $K(g(z), \bar{g}(w)) = J(g,z)K(z,w)\overline{J(\bar{g},w)}^{-1}$,

(see [17], Ch. II, Lemma 5.2).

Lemma 4. *For $g \in G_c$ the Jacobian of the holomorphic mapping $z \mapsto g(z)$, when it is defined, is given by*

$$Jac(z \mapsto g(z)) = Ad_{\mathfrak{p}_+}(J(g,z)).$$

(see [17], Ch. II, Lemma 5.3).

For any holomorphic character $\chi : K_c \mapsto \mathbb{C}$ we define:

$$j_\chi(g,z) = \chi(J(g,z)), \tag{6}$$

$$k_\chi(z,w) = \chi(K(z,w)). \tag{7}$$

Since $\chi(\bar{k}) = \overline{\chi(k)}^{-1}$ we have:

$$k_\chi(z,w) = \overline{k_\chi(w,z)}, \tag{8}$$

$$k_\chi(g(z), \bar{g}(w)) = j_\chi(g,z)k_\chi(z,w)\overline{j_\chi(\bar{g},w)} \tag{9}$$

in place of Lemma 3 (i) and (iii).

The character $\chi_1(k) = \det \mathrm{Ad}_{\mathfrak{p}+}(k)$, $(k \in K_c)$ is of particular importance. We call the corresponding j_{χ_1}, k_{χ_1}: j_1 and k_1. Notice that

$$j_1(g,z) = \det Jac(z \mapsto g(z)). \tag{10}$$

Example. $\mathfrak{g} = \mathfrak{su}(1,1)$. For $g = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$ in $SU(1,1)$ one has

$$J(g,z) = \begin{pmatrix} (\bar\beta z + \bar\alpha)^{-1} & 0 \\ 0 & (\bar\beta z + \bar\alpha) \end{pmatrix}, \ K(z,w) = \begin{pmatrix} (1 - z\bar w) & 0 \\ 0 & (1 - z\bar w)^{-1} \end{pmatrix}$$

and $\chi_1 \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \alpha^2$, $(\alpha \in \mathbb{C}^*)$, so

$$j_1(g,z) = (\bar\beta z + \bar\alpha)^{-2}, \qquad k_1(z,w) = (1 - z\bar w)^2.$$

Because of (10), $|k_1(z,z)|^{-2} d\mu(z)$, where $d\mu(z)$ is the Euclidean measure on $\mathfrak{p}_+$, is a G invariant measure on $\mathcal{D}$. Indeed:

$$d\mu(g(z)) = |j_1(g,z)|^2 d\mu(z),$$
$$k_1(g(z), g(z)) = j_1(g,z) k_1(z,z) j_1(g,z), \quad \text{for } g \in G.$$

(See (9)). One can actually show that $k_1(z,z) > 0$ on $\mathcal{D}$. ([17], Ch. II, Lemma 5.8).

We are now introducing the Bergman kernel of $\mathcal{D}$. Therefore consider $\mathcal{H}^2(\mathcal{D})$: the space of holomorphic functions f on $\mathcal{D}$ with $f \in L^2(\mathcal{D}, d\mu)$. The space $\mathcal{H}^2(\mathcal{D})$ is closed in $L^2(\mathcal{D})$ and has a reproducing kernel, the *Bergman kernel* of $\mathcal{D}$. Call it $k_{\mathcal{D}}(z,w)$. So

- for any $w \in \mathcal{D}$ the map $z \mapsto k_{\mathcal{D}}(z,w)$ is in $\mathcal{H}^2(\mathcal{D})$,
- $k_{\mathcal{D}}(z,w) = \overline{k_{\mathcal{D}}(w,z)}$,
- for every $f \in \mathcal{H}^2(\mathcal{D})$, $f(z) = \int_{\mathcal{D}} k_{\mathcal{D}}(z,w) f(w)\, d\mu(w)$ for $z \in \mathcal{D}$.

Satake shows:

$$k_{\mathcal{D}}(z,w) = \frac{1}{\mathrm{vol}(\mathcal{D})} \cdot \frac{1}{k_1(z,w)} \tag{11}$$

(see [17], Ch. II, p. 73).

For any $\lambda \in \mathbb{R}$, a Berezin kernel on $\mathcal{D} \times \mathcal{D}$ is given by:

$$\mathcal{B}_\lambda(z,w) = \left[\frac{k_{\mathcal{D}}(z,z) k_{\mathcal{D}}(w,w)}{k_{\mathcal{D}}(z,w) k_{\mathcal{D}}(w,z)}\right]^\lambda, \qquad (z, w \in \mathcal{D}). \tag{12}$$

$\mathcal{B}_\lambda$ is G-invariant: $\mathcal{B}_\lambda(g(z), g(w)) = \mathcal{B}_\lambda(z,w)$ for $g \in G$; $z, w \in D$. So $\mathcal{B}_\lambda$ is associated with a bi-K-invariant function φ_λ given by

$$\varphi_\lambda(g) = \mathcal{B}_\lambda(g(o), o) = k_1^\lambda(z,z), \tag{13}$$

where $z = g(o) \in \mathcal{D}$.

It is known that φ_λ is positive-definite for λ/p in the *Berezin–Wallach* set, where p is the genus of $\mathcal{D}$. For the list of groups G considered here, we refer to the upper part of Table 1, right column.

1.2. For symmetric spaces of Hermitian type.

Let $\mathfrak{g}, \mathfrak{g}_c, G, G_c, \dots$ be as in section 1.1. We add to $\mathfrak{g}$ an involutive automorphism σ, commuting with the Cartan involution θ. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition of $\mathfrak{g}$ into $+1$ and -1 eigenspaces of σ. The Lie algebra $\mathfrak{g}$ is said to be of Hermitian type if $\mathfrak{g}$ is Hermitian and, in addition, $Z_0 \in \mathfrak{q} \cap \mathfrak{k}$. There are several other terminologies in use; the most closely to us is: $\mathfrak{g}$ is a compactly causal Lie algebra.

The involution σ is extended to $\mathfrak{g}_c$ and G_c and leaves G invariant. Let H denote the closed subgroup of G consisting of the fixed points of σ. The Lie algebra of H is $\mathfrak{h}$. Now observe that, since $\sigma(Z_0) = Z_0$, $\sigma(\mathfrak{p}_+) = \mathfrak{p}_-$. Since $\bar{\mathfrak{p}}_+ = \mathfrak{p}_-$, we see that $\bar{\sigma}$ leaves $\mathfrak{p}_+$ and $\mathfrak{p}_-$ invariant. Set

$$\mathfrak{p}_\pm^{\bar{\sigma}} = \{X \in \mathfrak{p}_\pm \ : \ \bar{\sigma}(X) = X\}.$$

Then clearly $\dim_{\mathbb{R}} \mathfrak{p}_+^{\bar{\sigma}} = \dim_{\mathbb{R}} \mathfrak{p}_-^{\bar{\sigma}} = \dim_{\mathbb{C}} \mathfrak{p}_+$, since $\bar{\sigma}$ is a conjugation.

It is clear that $\bar{\sigma}(\mathcal{D}) = \mathcal{D}$. Set $\mathcal{D}^{\bar{\sigma}}$ for the set of fixed points of $\bar{\sigma}$ in $\mathcal{D}$. Since $\bar{\sigma}(H) = H$ it easily follows that $H/H \cap K$ can be identified with an open submanifold of $\mathcal{D}^{\bar{\sigma}}$. The proof is according to the same lines as in Lemma 1. The real "ball" $\mathcal{D}^{\bar{\sigma}}$ is an interesting object; one can actually show that H acts transitively on it.

Example. $\mathfrak{g} = \mathfrak{su}(1,1)$, $\sigma(X) = \bar{X}$, $\mathfrak{h} = \mathfrak{so}(1,1)$. $H = SO(1,1)$, $h \in H$ is of the form $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, $t \in \mathbb{R}$. Now $\mathcal{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$, so $\mathcal{D}^{\bar{\sigma}} = (0,1) \subset \mathbb{R}$. This is clearly the same as $H.o = \{\tanh t \ : \ t \in \mathbb{R}\}$.

It is clear that $|k_1(z,z)|^{-1} d\nu(z)$, where $d\nu(z)$ is a Euclidean measure on $\mathcal{D}^{\bar{\sigma}}$, is a H-invariant measure on $\mathcal{D}^{\bar{\sigma}}$. The proof is along the same line as in Section 1. Moreover, one can restrict the Berezin kernel to $\mathcal{D}^{\bar{\sigma}}$, which lead to a function φ_λ on H, which is bi-$K \cap H$-invariant. It is a positive-definite function, at least for λ/p in the Berezin–Wallach set of G/K. Let us denote the restriction of $\mathcal{B}_\lambda$ to $\mathcal{D}^{\bar{\sigma}} \times \mathcal{D}^{\bar{\sigma}}$ by $\mathcal{B}_\lambda^{\bar{\sigma}}$.

For the spaces we are talking about, see Table 1 (lower part). This table also includes a list of non-compactly causal Lie algebras $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$ which will be discussed in the next section. Table 1 is taken from [6].

2. Maximal degenerate representations

2.1. The group G^c and its standard maximal parabolic subgroups.

We keep the setting of Section 1.2 and let

$$\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q} \tag{14}$$

which is usually called the "compact dual" of $\mathfrak{g}$, with respect to $\mathfrak{h}$ (or to σ). Clearly $\mathfrak{g}^c$ is the set of fixed points of the anti-involution $\bar{\sigma}$ on $\mathfrak{g}^c$. We refer to Table 1 for the possible $\mathfrak{g}^c$.

Let G^c be the connected Lie subgroup of G_c with Lie algebra $\mathfrak{g}^c$. G^c is a closed subgroup. Set $\mathfrak{l} = \mathfrak{k}_c^{\bar{\sigma}} = \mathfrak{h} \cap \mathfrak{k} + i\mathfrak{q} \cap \mathfrak{k}$ and let $L = G^c \cap K_c$. The Lie algebra of L is $\mathfrak{l}$. Notice that $\mathfrak{p}_{\pm}^{\bar{\sigma}}$ is contained in $\mathfrak{g}^c$, even in $\mathfrak{h} \cap \mathfrak{p} + i\mathfrak{q} \cap \mathfrak{p}$. Set $\mathfrak{n}_{\pm} = \mathfrak{p}_{\pm}^{\bar{\sigma}}$ and $N_{\pm} = \exp \mathfrak{n}_{\pm}$.

Then $Q^+ = LN_+$ and $Q^- = LN_-$ are (standard) maximal parabolic subgroups of G^c. In both cases $A = \exp i\mathbb{R}Z_0$ is the split component. Furthermore,

$$H \cap Q^+ = H \cap Q^- = H \cap L^{\sigma} = H \cap K. \tag{15}$$

2.2. The maximal degenerate principal series.

For $\mu \in \mathbb{C}$ define the character ω_μ of $Q^{\pm}$ by the formula

$$\omega_\mu(q) = \left|\det \mathrm{A}d(\ell)_{\mathfrak{n}_{\pm}}\right|^{\mu}, \tag{16}$$

if $q = \ell n$, ($\ell \in L$, $n \in N_{\pm}$). The "det" is the determinant over $\mathbb{R}$.

Consider the representations $\pi_\mu^{\pm}$ of G^c induced from $Q^{\pm}$:

$$\pi_\mu^{\pm} = \operatorname{Ind} \omega_{\mp\mu}. \tag{17}$$

These representations are said to belong to the maximal degenerate principal series. One can describe these representations in several pictures. We shall choose here a "non-compact picture". Since N_+LN_- is open and dense in G^c, the continuous functions in the space of π_μ^- are completely determined by their restriction to N_+, so to $\mathfrak{n}_+$. The formulæ are:

$$\pi_\mu^-(g)f(z) = f(g^{-1}(z))\,\omega_\mu(J(g^{-1}, z)) \tag{18}$$

($z \in \mathfrak{n}_+$, $g \in G^c$) and

$$\pi_\mu^+(g)f(z) = f(\bar{g}^{-1}(z))\omega_\mu(J(\bar{g}^{-1}, z)) \tag{19}$$

(see, e.g., [19], Section 8.4). Notice that these formulae have a disadvantage: the expressions $g^{-1}(z)$ and $J(g^{-1}, z)$ are not always defined. They make however perfectly sense if $f \in C_c^\infty(\mathcal{D}^{\bar{\sigma}})$ and $g \in H$, since then

$$g^{-1}.\exp z \subset HQ^- \subset N_+LN_-$$

and hence $g^{-1}(z) \in \mathcal{D}^{\bar{\sigma}}$. We shall consider later on the restriction of π_μ^- to H on the subspace $C_c^\infty(\mathcal{D}^{\bar{\sigma}})$.

2.3. Irreducibility, unitarizability, equivalence.

It is quite interesting to study the usual questions, mentioned in the title of this section, for π_μ^- (and π_μ^+). Let us follow Table 1 and just mention what is known (to our knowledge), by giving references. Interesting new cases:

$\mathfrak{g}^c$	Reference
$\mathfrak{su}(n,n), \mathfrak{so}^*(4n), \mathfrak{sp}(n,\mathbb{R}), \mathfrak{so}(n,2), \mathfrak{e}_{7(-25)}$	[10]
$\mathfrak{su}(n,n)$	[11]
$\mathfrak{su}(n,n), \mathfrak{sp}(n,\mathbb{R})$	[14], [15]
$\mathfrak{sl}(p+q,\mathbb{C}); \mathfrak{sl}(p+q,\mathbb{R})$	[9], [4], [16]

$\mathfrak{sp}(n,\mathbb{C})$, $\mathfrak{so}(n,n)$, $\mathfrak{sp}(n,n)$. The most efficient way to study π_μ^- turns out to be using the "*compact picture*". Let K^c be the subgroup of G^c with Lie algebra $\mathfrak{h}\cap\mathfrak{k}\oplus i\mathfrak{q}\cap\mathfrak{p}$. Then K^c is a maximal compact subgroup of G^c, acting irreducibly on $\mathfrak{p}^c = \mathfrak{h}\cap\mathfrak{p}\oplus i\mathfrak{q}\cap\mathfrak{k}$. So the smallest real subalgebra of $\mathfrak{g}^c$ containing $\mathfrak{k}^c$ and $L_0 = iZ_0$ is precisely $\mathfrak{g}^c$. For the study of the irreducibility of $\pi_\mu^\pm$ it should be useful to consider their K^c-types. So, we have to decompose $\pi_{mu}^\pm$ into K^c-types, which can be done multiplicity free. A crucial role is played by the operators $\pi_\mu^\pm(L_0)$. One will notice that these operators send spherical functions to spherical functions (on K^c with respect to $K\cap H$), since L_0 commutes with $K\cap H$. We will neither pursue the irreducibility problem here, nor the questions of unitarizability and equivalence. See for the illustration of our (very simplified) method the papers mentioned above, where (except the papers of Molchanov) rather complicated computations occur.

2.4. Restriction to H; Berezin kernels revisited.

Denote by R_μ the restriction of π_μ^- to H on the subspace $C_c^\infty(\mathcal{D}^{\bar\sigma})$. We will perform a construction with R_μ similar to the construction of a "complementary series". Notice that $K(z,w)$ is well defined for $z,w\in\mathcal{D}^{\bar\sigma}$. Let $\mu\in\mathbb{R}$ from now on.

For $\phi_1,\phi_2\in C_c^\infty(\mathcal{D}^{\bar\sigma})$ consider

$$\langle\phi_1,\phi_2\rangle = \int_{\mathfrak{n}_+}\int_{\mathfrak{n}_+} \big|k_1(z,w)\big|^{-\mu-1}\phi_1(z)\bar\phi_2(w)\,d\nu(z)d\nu(w). \qquad (20)$$

By Lemma 3 (iii) this is clearly a H-invariant Hermitian form on $C_c^\infty(\mathcal{D}^{\bar\sigma})$:

$$\langle R_\mu(g)\phi_1, R_\mu(g)\phi_2\rangle = \langle\phi_1,\phi_2\rangle$$

for all $g \in H$.

Let $\phi \in C_c^\infty(\mathcal{D}^{\bar{\sigma}})$. Define

$$\psi(z) = \phi(z) k_1(z,z)^{-\frac{\mu}{2}}, (z \in \mathcal{D}^{\bar{\sigma}}). \tag{21}$$

Then

$$\begin{aligned}\psi(g^{-1}(z)) &= \phi(g^{-1}(z)) k_1(g^{-1}(z), g^{-1}(z))^{-\frac{\mu}{2}} \\ &= \phi(g^{-1}(z)) j_1(g^{-1}, z)^{\mu} k_1(z,z)^{-\frac{\mu}{2}} \\ &= \big(R_\mu(g)\phi\big)(z) k_1(z,z)^{-\frac{\mu}{2}}\end{aligned}$$

for $g \in H$. So the linear map $\phi \mapsto \psi$ intertwines R_μ with the left action of H on $C_c^\infty(\mathcal{D}^{\bar{\sigma}}) \simeq C_c^\infty(H/H \cap K)$. An H-invariant measure on $\mathcal{D}^{\bar{\sigma}}$ is given by $\left|k_1(z,z)\right|^{-1} d\nu(z)$ (see Section1.2). Applying (21) we obtain for the form (20):

$$\langle \psi_1, \psi_2 \rangle = \int_{\mathfrak{n}_+}\int_{\mathfrak{n}_+} \left[\frac{k_1(z,w)k_1(w,z)}{k_1(z,z)k_1(w,w)}\right]^{-\frac{\mu}{2}-1} \psi_(z)\bar{\psi}_2(w) \frac{d\nu(z)}{k_1(z,z)} \cdot \frac{d\nu(w)}{k_1(w,w)}. \tag{22}$$

We obtain the Berezin kernel $\mathcal{B}_{\frac{\mu}{2}+1}$ on $\mathcal{D}^{\bar{\sigma}} \simeq H/H \cap K$ and see that R_μ is unitarizable for $\frac{\mu}{2}+1$ in the Berezin–Wallach set.

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