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MODULI SPACE OF SYMMETRIC CONNECTIONS

Abstract. The action of origin-preserving diffeomorphisms on a space of jets of symmetric connections is considered. Dimensions of moduli spaces of generic connections are calculated. Poincaré series of the geometric structure of connection is constructed, and shown to be a rational function.

1. Introduction

A problem of finding functional moduli or at least establishing their finitness in various local differential-geometric settings was discussed by Arnold in [1]. Here we are interested in local differential invariants of a geometric structure consisting of a symmetric connection, under smooth coordinate changes. The structure of the resulting generic moduli space is reflected in the Poincaré series which we explicitly calculate, cf. (5.17). This series turns out to be a rational function, cf. (5.18), indicating a finite number of invariants. This confirms the finitness assertion of Tresse [7], which he stated for any "natural" differential-geometric structure. Similar results for Riemannian, Kähler and Hyper–Kähler structures were obtained in [5], and an explicit normal form for Riemannian structure in [6], by Shmelev. Earlier Vershik and Gershkovich investigated jet asymptotic dimension of moduli spaces of jets of generic distributions at $0 \in \mathbb{R}^n$ in [3], and their normal form in [2].

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2. Preliminaries and main result.

Let $\mathcal{F}$ and $\mathcal{F}_k$ be the spaces of germs and k-jets of symmetric C^∞-connections at a point on $\mathbb{R}^n$. From now on all connections we consider are assumed symmetric. As usual, two C^∞-functions on $\mathbb{R}^n$ have the same k-jet at a point if their first k derivatives are equal in any local coordinates. We say that two connections ∇ and $\widetilde{\nabla}$ have the same k-jet at 0 if for any two C^∞-vector fields X,Y and any C^∞-function f, the functions $\nabla_X Y(f)$ and $\widetilde{\nabla}_X Y(f)$ have the same k-jet at 0. (This is equivalent to Cristoffel symbols of ∇ and $\tilde{\nabla}$ having the same k-jet.) We will frequently denote connection and its Cristoffel symbol with the same letter, e.g., Γ; $j^k\Gamma$ would stand for its k-jet.

There is an action of the group of germs of origin-preserving diffeomorphisms $G := \mathrm{Diff}(\mathbb{R}^n, 0)$ on $\mathcal{F}$ and $\mathcal{F}_k$. For $\varphi \in G$, ∇(or Γ) $\in \mathcal{F}$ and $j^k\Gamma \in \mathcal{F}_k$:

$$\Gamma \mapsto \varphi^*\Gamma, \quad j^k\Gamma \mapsto j^k(\varphi^*\Gamma),$$

where

$$(\varphi^*\nabla)_X Y = \varphi_*^{-1}(\nabla_{\varphi_* X} \varphi_* Y).$$

Let us introduce a filtration of G by normal subgroups:

$$G = G_1 \rhd G_2 \rhd G_3 \rhd \dots ,$$

where

$$G_k = \{\varphi \in G \,|\, \varphi(x) = x + (\varphi_1(x), \dots \varphi_n(x)),\ \varphi_i = O(|x|^k),\ i = 1, \dots , n\}.$$

The subgroup G_k acts trivially on $\mathcal{F}_p$ for $k \geq p + 3$. It means that the action of G coincides with that of G/G_{p+3} on each $\mathcal{F}_p$. Now G/G_{p+3} is a finite-dimensional Lie group, which we will call K_p. $j^k\Gamma \in \mathcal{F}_k$. Denote by $\mathrm{Vect}_0(\mathbb{R}^n)$ the Lie algebra of C^∞-vector fields, vanishing at the origin. It acts on $\mathcal{F}$ as follows:

Definition 2.1. *For $V \in \mathrm{Vect}_0(\mathbb{R}^n)$ generating a local 1-parameter subgroup g^t of $\mathrm{Diff}(\mathbb{R}^n, 0)$, the Lie derivative of a connection ∇ in the direction V is a $(1, 2)$-tensor:*

$$\mathcal{L}_V \nabla = \frac{d}{dt}\bigg|_{t=0} g^{t\,*}\nabla.$$

Lemma 2.2.

$$(\mathcal{L}_V \nabla)(X, Y) = [V, \nabla_X Y] - \nabla_{[V,X]} Y - \nabla_X [V, Y].$$

Proof. Below the composition $\circ$ is understood as that of differential operators acting on functions.

$$\begin{aligned}(\mathcal{L}_V \nabla)(X, Y) &= \frac{d}{dt}\bigg|_{t=0} g_*^{-t}[\nabla_{g_*^t X} g_*^t Y] \\ &= \frac{d}{dt}\bigg|_{t=0} \left[(g^t)^* \circ [\nabla_{g_*^t X} g_*^t Y] \circ (g^{-t})^*\right] =\end{aligned}$$

$$\left.\frac{d}{dt}\right|_{t=0}(g^t)^*\circ\nabla_X Y+\nabla_X Y\circ\left.\frac{d}{dt}\right|_{t=0}(g^{-t})^*$$

$$+\nabla_{\left.\frac{d}{dt}\right|_{t=0}g_*^t X}Y+\nabla_X\left.\frac{d}{dt}\right|_{t=0}g_*^t Y=$$

$$V\circ\nabla_X Y-\nabla_X Y\circ V-\nabla_{\left.\frac{d}{dt}\right|_{t=0}g_*^{-t}X}Y-\nabla_X\left.\frac{d}{dt}\right|_{t=0}g_*^{-t}Y=$$

$$\mathcal{L}_V(\nabla_X Y)-\nabla_{\mathcal{L}_V X}Y-\nabla_X(\mathcal{L}_V Y).$$

$\square$

This defines the action on the germs of connections. Now we can define the action of $\mathrm{Vect}_0(\mathbb{R}^n)$ on jets $\mathcal{F}_k$. For $V\in\mathrm{Vect}_0(\mathbb{R}^n)$:

$$\mathcal{L}_V(j^k\Gamma)=j^k(\mathcal{L}_V\Gamma),$$

where Γ on the right hand, is an arbitrary representative of the $j^k\Gamma$ on the left.

This is well-defined, since in the coordinate version of (2.1):

$$(\mathcal{L}_V\Gamma)^l_{ij}=V^k\frac{\partial\Gamma^l_{ij}}{\partial x^k}-\Gamma^k_{ij}\frac{\partial V^l}{\partial x^k}+\Gamma^l_{kj}\frac{\partial V^k}{\partial x^i}+\Gamma^l_{ik}\frac{\partial V^k}{\partial x^j}+\frac{\partial^2 V^l}{\partial x^i\partial x^j}\tag{2.2}$$

elements of kth order and less are only coming from $j^k\Gamma$, because $V(0)=0$. Einstein summation convention in (2.2) above and further on is assumed.

Consequently, the action is invariantly defined. This can also be expressed as commutativity of the following diagram:

$$\begin{array}{ccccccccccc}
j^0\mathcal{F} & \longleftarrow & \dots & \longleftarrow & j^{k-1}\mathcal{F} & \xleftarrow{\pi_k} & j^k\mathcal{F} & \longleftarrow & \dots & \longleftarrow & \mathcal{F}\\
\downarrow\mathcal{L}_V & & & & \downarrow\mathcal{L}_V & & \downarrow\mathcal{L}_V & & & & \downarrow\mathcal{L}_V,\\
j^0\Pi & \longleftarrow & \dots & \longleftarrow & j^{k-1}\Pi & \xleftarrow{\pi_k} & j^k\Pi & \longleftarrow & \dots & \longleftarrow & \Pi
\end{array}$$

where π_k is projection from k-jets onto $(k-1)$-jets, $\mathcal{F}$ and Π denote spaces of germs of connections and that of (1,2)-tensors respectively, at 0.

Poincaré series will encode information about these actions for all k. The space $\mathcal{M}=\mathcal{F}/\mathrm{Diff}(\mathbb{R}^n,0)$ of $\mathrm{Diff}(\mathbb{R}^n,0)$-orbits on $\mathcal{F}$ is called the *moduli space* of connections at 0 on $\mathbb{R}^n$. We do not introduce any topology on $\mathcal{M}$. Similarly, the orbit space $\mathcal{M}_k=\mathcal{F}_k/\mathrm{Diff}(\mathbb{R}^n,0)=\mathcal{F}_k/K_k$ is called the moduli space of connection k-jets.

The action of K_k is algebraic, a subspace $\mathcal{F}^0_k\subset\mathcal{F}_k$ of points on generic orbits (those of largest dimension) is a smooth manifold, open and dense in $\mathcal{F}_k$. Subspace of points on orbits of any other given dimension is a manifold as well, albeit of a lesser dimension. We could consider the G-quotient for each of those subspaces, and have a moduli space of its own for each of the orbit types. Let $\mathcal{O}_k$ denote a generic orbit. Denote by $\mathcal{M}_k^0$ the moduli space of generic connections:

$$\mathcal{M}_k^0 = \mathcal{F}_k^0/\mathrm{Diff}(\mathbb{R}^n, 0) = \mathcal{F}_k^0/K_k$$

or generic subspace of moduli space F_k^0. Its dimension is found as:

$$\dim \mathcal{M}_k^0 = \dim \mathcal{F}_k^0 - \dim \mathcal{O}_k. \tag{2.3}$$

Unfortunately it is not true that the generic subspace has the maximal dimension for general Lie group actions, even those that are algebraic, for an explicit counterexample see [4]. Thus we define

$$\dim \mathcal{M}_k = \dim \mathcal{M}_k^0$$

for mere simplicity of notation. One more piece of notation:

$$a_k = \begin{cases} \dim \mathcal{M}_k, & k = 0 \\ \dim \mathcal{M}_k - \dim \mathcal{M}_{k-1}, & k \geq 1, \end{cases}$$

and we can introduce our main object of interest.

Definition 2.3. *The formal power series*

$$p_\Gamma(t) = \sum_{k=0}^{\infty} a_k t^k$$

is called the Poincaré series for the moduli space $\mathcal{M}$.

Our main result is the following

Theorem 2.4. *Poncaré series coefficients $a_k = a(k)$ are polynomial in k, and the series has the form:*

$$p_\Gamma(t) = (t - t^2)\delta_2^n + n \sum_{k=1}^{\infty} \left[\frac{n(n+1)}{2} \binom{n+k-1}{n-1} - n \binom{n+k+1}{n-1} \right] t^k$$

$$(\delta \text{ is Kronecker symbol}).$$

It represents a rational function.

Remark. This complies with Tresse' assertion that algebras of "natural" differential-geometric structures be finitely-generated.

Proof of this theorem is relegated to Sec. 5.

To explain significance of Poincaré series represented by a rational function, we make the following:

Remark. If a geometric structure is described by a finite number of functional moduli, then its Poncaré series is rational. In particular, if there are m functional invariants in n variables, then

$$p(t) = \frac{m}{(1-t)^n}.$$

Indeed, dimension of moduli spaces of k-jets is just the number of monomials up to the order k in the formal power series of the m given invariants:

$$\dim \mathcal{M}_k = m\binom{n+k}{n}.$$

For more details and slightly more general formulation see Theorem 2.1 in [6].

3. Stabilizer of a generic k-jet

In order to calculate Poincaré series, we need to find $\dim \mathcal{O}_k = \dim K_k - \dim G_\Gamma$, and hence, the size of a stabilizer G_Γ for a generic connection Γ.

Let us find the subalgebra generating G_Γ – the stabilizer of a k-jet Γ. It consists of such $V \in \mathrm{Vect}_0(\mathbb{R}^n)$, that:

$$\mathcal{L}_V(j^k\Gamma) = 0. \tag{3.4}$$

We will argue in local coordinates. Let us introduce grading in homogeneous components:

$$V = V_1 + V_2 + \dots$$

($V_0 = 0$, since we require V to preserve the origin),

$$\Gamma = \Gamma_0 + \Gamma_1 + \dots$$

is analogous grading on the connection Γ.

Note that each Γ_k has the same symmetry on indexes as the original Γ. Using these decompositions we can rewrite (3.4) as follows:

$$\mathcal{L}_V(j^k\Gamma) = j^k\mathcal{L}_V(\Gamma) = j^k\mathcal{L}_{V_1+V_2+\dots}(\Gamma_0 + \Gamma_1 + \dots + \Gamma_k + \dots) =$$

$$= \underbrace{\mathcal{L}_{V_1}\Gamma_0 + \frac{\partial^2 V_2}{\partial x^2}}_{\text{0th order}} + \underbrace{\mathcal{L}_{V_1}\Gamma_1 + \widetilde{\mathcal{L}}_{V_2}\Gamma_0 + \frac{\partial^2 V_3}{\partial x^2}}_{\text{1st order}} + \ldots$$

$$\ldots + \underbrace{\widetilde{\mathcal{L}}_{V_{k+1}}\Gamma_0 + \widetilde{\mathcal{L}}_{V_k}\Gamma_1 + \ldots + \mathcal{L}_{V_1}\Gamma_k + \frac{\partial^2 V_{k+2}}{\partial x^2}}_{k\text{th order}},$$

where

$$\left(\frac{\partial^2 V_2}{\partial x^2}\right)^l_{ij} = \frac{\partial^2 V_2^l}{\partial x^i x^j} \quad \text{and} \quad \widetilde{\mathcal{L}}_V\Gamma = \mathcal{L}_V\Gamma - \frac{\partial^2 V}{\partial x^2},$$

the same with indexes:

$$(\widetilde{\mathcal{L}}_V\Gamma)^l_{ij} = V^k\frac{\partial\Gamma^l_{ij}}{\partial x^k} - \Gamma^k_{ij}\frac{\partial V^l}{\partial x^k} + \Gamma^l_{kj}\frac{\partial V^k}{\partial x^i} + \Gamma^l_{ik}\frac{\partial V^k}{\partial x^j}$$

are just the first 4 terms from (2.2) of $(\mathcal{L}_V\Gamma)^l_{ij}$. The stabilizer condition therefore results in a system:

$$\begin{cases} \mathcal{L}_{V_1}\Gamma_0 + \dfrac{\partial^2 V_2}{\partial x^2} = 0 \\ \mathcal{L}_{V_1}\Gamma_1 + \widetilde{\mathcal{L}}_{V_2}\Gamma_0 + \dfrac{\partial^2 V_3}{\partial x^2} = 0 \\ \quad \vdots \\ \widetilde{\mathcal{L}}_{V_{k+1}}\Gamma_0 + \widetilde{\mathcal{L}}_{V_k}\Gamma_1 + \ldots + \mathcal{L}_{V_1}\Gamma_k + \dfrac{\partial^2 V_{k+2}}{\partial x^2} = 0. \end{cases} \tag{3.5}$$

Our present goal is finding all $(V_1, V_2, \ldots, V_{k+2})$ solving the above system for a generic Γ. Assume V_1 is arbitrary, to find V_2 from the first equation we need the following

Proposition 3.1. *Given a family $\{f_{ij}\}_{1\le i,j\le n}$ of smooth functions, solution u for the system:*

$$\begin{cases} u_{,kl} = f_{kl} \\ 1 \le k,\ l \le n \end{cases}$$

(indexes after a comma henceforth will denote differentiations in corresponding variables) exists if and only if

$$\begin{cases} f_{ij} = f_{ji} \\ f_{ij,k} = f_{kj,i}. \end{cases} \tag{3.6}$$

If f_{ij} are homogeneous polynomials of degree $s \geqslant 0$, then u can be uniquely chosen as a polynomial of degree $s+2$.

Proof is a straightforward integration of the right-hand sides. □

Therefore, if we treat highest-order V_k in each equation of (3.5) as an unknown, we see that various (combinations of) $\mathcal{L}_V\Gamma$ must satisfy (3.6). The first condition is satisfied automatically. The second one gives:

$$(\mathcal{L}_V\Gamma)^l_{ij,p} = (\mathcal{L}_V\Gamma)^l_{pj,i}.$$

This condition for the 1st equation in (3.5) is satisfied trivially, since V_1 is of the 1st degree, and Γ_0 - 0^{th} degree in x. Hence, V_2 exists, and since it must be of the second degree, is unique. Let us now find V_3 from the next equation:

$$\mathcal{L}_{V_1}\Gamma_1 + \widetilde{\mathcal{L}}_{V_2}\Gamma_0 + \frac{\partial V_3}{\partial x^2} = 0.$$

It follows from the Proposition 3.1, that for existence of V_3 it is necessary (and sufficient) to have the following condition:

$$(\mathcal{L}_{V_1}\Gamma_1 + \widetilde{\mathcal{L}}_{V_2}\Gamma_0)_{ij,p} = (\mathcal{L}_{V_1}\Gamma_1 + \widetilde{\mathcal{L}}_{V_2}\Gamma_0)_{pj,i}. \tag{3.7}$$

We assert that, except for cases $n \leqslant 2$ considered in the next section, it is not satisfied for a generic connection unless $V_1 = 0$. In other words, (3.7) considered as a condition on V_1 implies $V_1 = 0$ (and hence $V_2 = V_3 = \ldots = 0$ too). Indeed, write:

$$V_1^k = \sum_{s=1}^{n} b_s^k x^s \tag{3.8}$$

and let us consider (3.7) as a homogeneous linear system on components $\{b_s^k\}$ of V_1. In what follows we will see that for a generic connection this system is nondegenerate. More precisely, we will find a connection, and a suitable minor of the system (since there are more equations than variables, we will choose a subset of equations to obtain a square matrix), and show that it is nondegenerate. Since it is an open condition, it would be generically true. We will construct connection and minor step by step, trying to obtain one as close to diagonal as possible. Let us look for such connection Γ among those with Γ_0=0. Then we see (3.7) shrink to:

$$(\mathcal{L}_{V_1}\Gamma_1)^l_{ij,p} = (\mathcal{L}_{V_1}\Gamma_1)^l_{pj,i}.$$

Let us expand it using (2.2):

$$V^k_{1,p}\Gamma^l_{1ij,k} - V^l_{1,k}\Gamma^k_{1ij,p} + V^k_{1,i}\Gamma^l_{1kj,p} + V^k_{1,j}\Gamma^l_{1ik,p} = (i \leftrightarrow p),$$

where the right-hand side is the same as the left-hand side with the two indexes swapped. After simplifying we get:

$$(\Gamma^l_{1ij,k} - \Gamma^l_{1kj,i})V^k_{1,p} + (\Gamma^l_{1kj,p} - \Gamma^l_{1pj,k})V^k_{1,i} +$$
$$(\Gamma^k_{1pj,i} - \Gamma^k_{1ij,p})V^l_{1,k} + (\Gamma^l_{1ik,p} - \Gamma^l_{1pk,i})V^k_{1,j} = 0, \tag{3.9}$$

where $1 \leqslant i < p \leq n$, and j, l vary from 1 to n.

That makes for $n^2 \cdot \dfrac{n(n-1)}{2}$ equations on n^2 variables.

Also write:

$$\Gamma^l_{1ij} = \sum_{m=1}^{n} c^{lm}_{ij} x^m,$$

$$c^{lm}_{ij} = c^{lm}_{ji} \quad \text{(recall that the connection is assumed symmetric)} .$$

Then, recalling also (3.8), (3.9) becomes:

$$(c^{lk}_{ij} - c^{li}_{kj})b^k_p + (c^{lp}_{kj} - c^{lk}_{pj})b^k_i + (c^{ki}_{pj} - c^{kp}_{ij})b^l_k + (c^{lp}_{ik} - c^{li}_{pk})b^k_j = 0. \tag{3.10}$$

Recall that summation over repeated indexes above is assumed.

Let us now impose a further restriction on Γ, namely that

$$c^{lp}_{ij} \neq 0 \;\text{ only if }\; \{i,j,l,p\} = \{\alpha,\beta\}, \alpha \neq \beta. \tag{3.11}$$

In other words, nonzero coefficients may only occur among those with indexing set consisting of two distinct numbers, and must be zero otherwise.

Lemma 3.2. *The homogeneous linear system* (3.10) *with coefficients as restricted in* (3.11) *has a* $n^2 \times n^2$ *nondegenerate minor.*

Proof. Let us specify the minor by letting j and l in the index set $\{i,j,l,p\}$ be arbitrary. Accordingly, we will be labeling equations in the system by this pair of indexes (jl), in lexicographic order. The remaining two indexes i and p will be determined by $\{j,l\}$ in the following manner:

i) if $j < l$ set $i = j$, $p = l$.

Equation (jl) becomes then:

$$(jl) \qquad (c^{lk}_{jj} - c^{lj}_{kj})b^k_l + (c^{ll}_{kj} - c^{lk}_{lj})b^k_j + (c^{kj}_{lj} - c^{kl}_{jj})b^l_k + (c^{ll}_{jk} - c^{lj}_{lk})b^k_j = 0.$$

Diagonal coefficients (at b^l_j) are:

$$(c^{jj}_{jl} - c^{jl}_{jj}) + (c^{ll}_{lj} - c^{lj}_{ll}) =: A_{jl} = A_{lj}.$$

Using (3.11), the only nondiagonal terms that remain are:

$$(c^{ll}_{jj} - c^{lj}_{lj})b^l_l + (c^{lj}_{jj} - c^{lj}_{jj})b^j_l + (c^{ll}_{jj} - c^{lj}_{lj})b^j_j +$$

$$+(c^{lj}_{lj} - c^{ll}_{jj})b^l_l + (c^{ll}_{jj} - c^{lj}_{lj})b^j_j = 2(c^{ll}_{jj} - c^{lj}_{lj})b^j_j.$$

To eliminate them we need to set $c^{ll}_{jj} = c^{lj}_{lj}$ for $j < l$.

ii) if $j = l < n$ set $i = l$, $p = l + 1$.

Then the corresponding equations have the form:

$$(ll) \qquad (c^{lk}_{ll} - c^{ll}_{kl})b^k_{l+1} + (c^{l(l+1)}_{kl} - c^{lk}_{(l+1)l})b^k_l + (c^{kl}_{(l+1)l} - c^{k(l+1)}_{ll})b^l_k +$$

$$+(c^{l(l+1)}_{lk} - c^{ll}_{(l+1)k})b^k_l = 0$$

with diagonal coefficients (at b^l_l): $(c^{l(l+1)}_{ll} - c^{ll}_{(l+1)l})$, and suspicious off-diagonal:

$$(c^{ll}_{ll} - c^{ll}_{ll})b^l_{l+1} + (c^{l(l+1)}_{ll} - c^{ll}_{(l+1)l})b^{l+1}_{l+1} + (c^{l(l+1)}_{(l+1)l} - c^{l(l+1)}_{(l+1)l})b^{l+1}_l +$$

$$+(c^{(l+1)l}_{(l+1)l} - c^{(l+1)(l+1)}_{ll})b^l_{l+1} + (c^{l(l+1)}_{l(l+1)} - c^{ll}_{(l+1)(l+1)})b^{l+1}_l =$$

$$= (c^{l(l+1)}_{ll} - c^{ll}_{(l+1)l})b^{l+1}_{l+1} + (c^{l(l+1)}_{(l+1)l} - c^{ll}_{(l+1)(l+1)})b^{l+1}_l +$$

$$+(c^{(l+1)l}_{(l+1)l} - c^{(l+1)(l+1)}_{ll})b^l_{l+1}.$$

Hence to nullify them, we could require that:

$$c^{l(l+1)}_{ll} = c^{ll}_{(l+1)l}, \quad c^{(l+1)l}_{(l+1)l} = c^{(l+1)(l+1)}_{ll},$$

$$c^{l(l+1)}_{l(l+1)} = c^{ll}_{(l+1)(l+1)} \quad \text{for any} \quad l : 1 \le l \le n-1$$

iii) if $j > l$ set $i = l$ and $p = j$.

These are the equations:

$$(jl) \qquad (c^{lk}_{lj} - c^{ll}_{kj})b^k_j + (c^{lj}_{kj} - c^{lk}_{jj})b^k_l + (c^{kl}_{jj} - c^{kj}_{lj})b^l_k + (c^{lj}_{lk} - c^{ll}_{jk})b^k_j = 0.$$

The diagonal coefficients (at b^l_j) are:

$$(c^{jl}_{jj} - c^{jj}_{jl}) + (c^{lj}_{ll} - c^{ll}_{lj}) = -A_{jl}.$$

Off-diagonal coefficients are: $2(c^{lj}_{lj} - c^{ll}_{jj})b^j_j$, hence it's necessary to require that

$$c^{lj}_{lj} = c^{ll}_{jj} \quad \text{for} \quad j > l.$$

iv) Finally, if $j = l = n$ set:

$$\begin{cases} i = 1, & p = n \quad \text{for} \quad n\text{-odd} \\ i = 2, & p = n \quad \text{for} \quad n\text{-even}. \end{cases}$$

We have for n-odd:

$$(nn) \qquad (c^{nk}_{1n} - c^{n1}_{kn})b^k_n + (c^{nn}_{kn} - c^{nk}_{nn})b^k_1 + (c^{k1}_{nn} - c^{kn}_{1n})b^n_k + (c^{nn}_{1k} - c^{n1}_{nk})b^k_n = 0.$$

The diagonal coefficients (at b^n_n) are: $c^{nn}_{1n} - c^{n1}_{nn}$, the off-diagonal ones are:

$$(c^{nn}_{1n} - c^{n1}_{nn}) \;\text{ at }\; b^1_1, \quad (c^{11}_{nn} - c^{1n}_{1n}) \;\text{ at }\; b^n_1, \quad (c^{nn}_{11} - c^{n1}_{n1}) \;\text{ at }\; b^1_n.$$

Hence, we would need:

$$c^{nn}_{1n} = c^{n1}_{nn}, \quad c^{11}_{nn} = c^{1n}_{1n}, \quad c^{nn}_{11} = c^{n1}_{n1}$$

to get rid of them.

For n-even just exchange index 1 for index 2 in all equations throughout iv) above.

From this it is clear that all equations can be "diagonalized," except those labeled (ll) (the requirements above are contradictory in ii) and iv)). Each of these equations then will have a single off-diagonal coefficient.

Summarizing, we set:

$$\begin{aligned} &c^{ll}_{jj} = c^{lj}_{lj}, \\ &c^{jl}_{jj} - c^{jj}_{jl} = 1, \quad j \neq l. \end{aligned}$$

That implies $A_{jl} = -2$ for $j \neq l$, for $l < n$ in $(ll)^{\text{th}}$ equation there is a unit off-diagonal coefficient at b^{l+1}_{l+1}, and for $l = n$ in $(nn)^{\text{th}}$ equation there is a negative unit off-diagonal coefficient at b^1_1 for n-odd, or at b^2_2 for n-even.

Now we switch the order of equations and variables, so that $(kk)^{\text{th}}$ equations and variables appear first, followed by all the rest in the preset lexicographic order. Then the minor for n-odd will assume the form:

$$\begin{pmatrix}
1 & 1 & & & & & & & & \\
 & \ddots & \ddots & & & & & & & \\
 & & \ddots & \ddots & & & & 0 & & \\
 & & & 1 & 1 & & & & & \\
-1 & & & & -1 & & & & & \\
 & & & & & -2 & & & & \\
 & & & & & & \ddots & & & \\
 & & 0 & & & & & \pm 2 & & \\
 & & & & & & & & \ddots & \\
 & & & & & & & & & 2
\end{pmatrix}.$$

For n-even, just relocate -1 in the nth row from the first position to the second. The minor above is easily seen to be nondegenerate, as required. □

This argument proves

Proposition 3.3. *The stabilizer of a k-jet of a generic connection for $n \geq 3$ is: G_1/G_2 for $k = 0$, and 0 for $k \geq 1$.*

4. Exceptions: the stabilizer in low dimensions

Let us start with the case $n = 1$. In this case any index can assume only a single possible value: 1. Then the compatibility conditions (3.6) are vacuous in all k-jets. Hence the stabilizer for k-jet is determined by V_1 with no restrictions and is equal to G_1/G_2 for all k. That is a 1-dimensional stabilizer, resulting in a $k+1$-dimensional orbit and 0-dimensional moduli space. Poincaré series then is identically equal to 0.

For the case $n = 2$, the compatibility conditions arising from second equation in (3.5) have nontrivial solutions, that is, unlike the higher n's, the stabilizer of the 1st jet is nontrivial. The reason is that the analogue of equation (3.7), considered as a (4 by 4) homogeneous linear system on coefficients b_k^l of V_1 is degenerate. Compatibility conditions for the third equation however do make up a nondegenerate linear system and have only trivial solution. Hence for $n = 2$ stabilizer is trivial, starting with second jet (for higher n, trivial starting with first jet, and for $n = 1$, always nontrivial).

Let us start with considering (3.7) for $n = 2$, showing it is degenerate, and finding its rank. Notice that we consider (3.7) in the most general form, with arbitrary Γ_0 and Γ_1. Since it involves $\widetilde{\mathcal{L}}_{V_2}\Gamma_0$, we need to express V_2 from the first equation of the system:

$$\mathcal{L}_{V_1}\Gamma_0 + \frac{\partial^2 V_2}{\partial x^2} = 0.$$

Setting $\Gamma^k_{0ij} =: \gamma^k_{ij}$, it can be rewritten in index form as:

$$V^l_{2,ij} = \gamma^k_{ij} b^l_k - \gamma^l_{kj} b^k_i - \gamma^l_{ik} b^k_j =: v^l_{ij}.$$

Actually, second derivatives of V_2 is all we need in (3.7), where they appear in

$$(\widetilde{\mathcal{L}}_{V_2}\Gamma_0)^l_{ij,p} = -\gamma^k_{ij} V^l_{2,kp} + \gamma^l_{kj} V^k_{2,ip} + \gamma^l_{ik} V^k_{2,jp},$$

which we can now rewrite as:

$$(\widetilde{\mathcal{L}}_{V_2}\Gamma_0)^l_{ij,p} = -\gamma^k_{ij}(\gamma^s_{kp} b^l_s - \gamma^l_{sp} b^s_k - \gamma^l_{ks} b^s_p) +$$

$$+\gamma^l_{kj}(\gamma^s_{ip} b^k_s - \gamma^k_{sp} b^s_i - \gamma^k_{is} b^s_p) + \gamma^l_{ik}(\gamma^s_{jp} b^k_s - \gamma^k_{sp} b^s_j - \gamma^k_{js} b^s_p). \tag{4.12}$$

If we consider (3.7) as $SV_1 = 0$ – linear operator acting on V_1, and split the operator into two parts: $S = S(\Gamma_0) + S(\Gamma_1)$, then (4.12) allows us to rewrite $S(\Gamma_0)V_1 = (\widetilde{\mathcal{L}}_{V_2}\Gamma_0)^l_{ij,p} - (\widetilde{\mathcal{L}}_{V_2}\Gamma_0)^l_{pj,i}$ as:

$$(\gamma^k_{pj}\gamma^s_{ki} - \gamma^k_{ij}\gamma^s_{kp}) b^l_s + (\gamma^l_{pk}\gamma^k_{js} - \gamma^k_{pj}\gamma^l_{ks}) b^s_i +$$

$$+(\gamma^l_{pk}\gamma^k_{si} - \gamma^l_{ik}\gamma^k_{sp}) b^s_j + (\gamma^k_{ij}\gamma^l_{ks} - \gamma^l_{ik}\gamma^k_{js}) b^s_p. \tag{4.13}$$

Recall that $n = 2$, and indexes $1 = i < p = 2$ must therefore stay fixed at $i = 1$, $p = 2$, while the remaining pair of indexes take any values. That turns (4.13) into a system of 4 expressions, which we index, as in the previous section with (j, l) on left:

$$\begin{array}{ll}
(11) & (\gamma^1_{22}\gamma^2_{11} - \gamma^1_{12}\gamma^2_{12}) b^1_1 + (\gamma^2_{11}\gamma^1_{22} - \gamma^1_{12}\gamma^2_{12}) b^2_2 + \\
 & (\gamma^1_{2k}\gamma^k_{12} - \gamma^1_{1k}\gamma^k_{22}) b^2_1 + (\gamma^k_{21}\gamma^2_{k1} - \gamma^k_{11}\gamma^2_{k2}) b^1_2 \\
(22) & (\gamma^2_{21}\gamma^1_{21} - \gamma^1_{22}\gamma^2_{11}) b^1_1 + (\gamma^1_{12}\gamma^2_{12} - \gamma^2_{11}\gamma^1_{22}) b^2_2 + \\
 & (\gamma^k_{22}\gamma^1_{k1} - \gamma^k_{12}\gamma^1_{k2}) b^2_1 + (\gamma^2_{2k}\gamma^k_{11} - \gamma^2_{1k}\gamma^k_{21}) b^1_2 \\
(12) & 2(\gamma^2_{2k}\gamma^k_{11} - \gamma^k_{21}\gamma^2_{k1}) b^1_1 + 2(\gamma^2_{21}\gamma^1_{21} - \gamma^2_{11}\gamma^1_{22}) b^2_1 \\
(21) & 2(\gamma^1_{2k}\gamma^k_{21} - \gamma^1_{1k}\gamma^k_{22}) b^2_2 + 2(\gamma^1_{22}\gamma^2_{11} - \gamma^1_{12}\gamma^2_{12}) b^1_2.
\end{array}$$

Considered by itself, this system is degenerate. Indeed, setting

$$(\gamma^1_{22}\gamma^2_{11} - \gamma^1_{12}\gamma^2_{12}) =: A, \quad (\gamma^1_{2k}\gamma^k_{12} - \gamma^1_{1k}\gamma^k_{22}) =: B, \quad (\gamma^k_{21}\gamma^2_{k1} - \gamma^k_{11}\gamma^2_{k2}) =: C,$$

the system's matrix becomes:

$$\begin{pmatrix} A & A & B & C \\ -A & -A & -B & -C \\ -2C & 0 & -2A & 0 \\ 0 & 2B & 0 & 2A \end{pmatrix}.$$

The determinant of this is identically zero.

Let us now consider the other half of (3.7), the part $S(\Gamma_1)V_1$ (we will unite the halves afterward). Here we just adapt equation (3.10) to the case $n = 2$, $i = 1$, $p = 2$:

$$\begin{array}{ll}
(11) & (c^{12}_{11} - c^{11}_{21})b^1_1 + (c^{12}_{11} - c^{11}_{21})b^2_2 + (c^{12}_{12} - c^{11}_{22})b^2_1 + (c^{21}_{21} - c^{22}_{11})b^1_2 \\
(22) & (c^{22}_{12} - c^{21}_{22})b^1_1 + (c^{22}_{12} - c^{21}_{22})b^2_2 + (c^{11}_{22} - c^{12}_{12})b^2_1 + (c^{22}_{11} - c^{21}_{21})b^1_2 \\
(12) & 2(c^{22}_{11} - c^{21}_{21})b^1_1 + [(c^{11}_{12} - c^{12}_{11}) + (c^{22}_{21} - c^{21}_{22})]b^2_1 \\
(21) & 2(c^{12}_{12} - c^{11}_{22})b^2_2 + [(c^{21}_{22} - c^{22}_{21}) + (c^{12}_{11} - c^{11}_{12})]b^1_2.
\end{array}$$

Setting

$$c^{12}_{11} - c^{11}_{21} =: a, \quad c^{12}_{12} - c^{11}_{22} =: b, \quad c^{21}_{21} - c^{22}_{11} =: c, \quad c^{22}_{12} - c^{21}_{22} =: d,$$

the above system's matrix can be written as:

$$\begin{pmatrix} a & a & b & c \\ d & d & -b & -c \\ -2c & 0 & d-a & 0 \\ 0 & 2b & 0 & a-d \end{pmatrix}. \tag{4.14}$$

It is also degenerate. The full (united) system of equations has the matrix:

$$\begin{pmatrix} a+A & a+A & b+B & c+C \\ d-A & d-A & -(b+B) & -(c+C) \\ -2(c+C) & 0 & d-A-(a+A) & 0 \\ 0 & 2(b+B) & 0 & a+A-(d-A) \end{pmatrix}.$$

It is also degenerate: in fact, it has exact same structure as (4.14). Even though not of full rank, generically the above system has rank 3, resulting in a 1-dimensional stabilizer.

Let us now consider the next, second jet of our connection. To calculate its stabilizer, we need to solve the following equation from (3.5) for V_4:

$$\mathcal{L}_{V_1}\Gamma_2 + \widetilde{\mathcal{L}}_{V_2}\Gamma_1 + \widetilde{\mathcal{L}}_{V_3}\Gamma_0 + \frac{\partial^2 V_4}{\partial x^2} = 0.$$

Its compatibility conditions are:

$$(\mathcal{L}_{V_1}\Gamma_2 + \widetilde{\mathcal{L}}_{V_2}\Gamma_1 + \widetilde{\mathcal{L}}_{V_3}\Gamma_0)^l_{ij,p} = (\mathcal{L}_{V_1}\Gamma_2 + \widetilde{\mathcal{L}}_{V_2}\Gamma_1 + \widetilde{\mathcal{L}}_{V_3}\Gamma_0)^l_{pj,i}. \quad (4.15)$$

We will use the same strategy as in the previous section to prove that in this case stabilizer is trivial. Namely we will obtain a connection 2-jet, for which the above equation will be a nondegenerate homogeneous linear system. We set $\Gamma_0 = \Gamma_1 = 0$, then (4.15) becomes:

$$(\mathcal{L}_{V_1}\Gamma_2)^l_{ij,p} = (\mathcal{L}_{V_1}\Gamma_2)^l_{pj,i}. \quad (4.16)$$

We introduce notation for coefficients of Γ_2:

$$\Gamma_2{}^l_{ij} = \sum_{s,t=1}^{2} d^l_{ijst}x^s x^t, \quad d^l_{ijst} = d^l_{jits}.$$

With these,

$$(\mathcal{L}_{V_1}\Gamma_2)^l_{ij,p} = 2d^l_{ijkt}b^k_p x^t + 2d^l_{ijkp}b^k_t x^t - 2d^k_{ijpt}b^l_k x^t + 2d^l_{kjpt}b^k_i x^t + 2d^l_{ikpt}b^k_j x^t$$

(b^l_k are still coefficients of V_1, as in (3.8), and (4.16) (with $i = 1, p = 2$) is:

$$(d^l_{kj2t} - d^l_{2jkt})b^k_1 + (d^l_{1jkt} - d^l_{kj1t})b^k_2 + (d^l_{1jk2} - d^l_{2jk1})b^k_t +$$
$$(d^k_{2j1t} - d^k_{1j2t})b^l_k + (d^l_{1k2t} - d^l_{2k1t})b^k_j = 0.$$

With the triple of indexes (j, l, t) arbitrary, we have system of 8 equations in 4 variables: the coefficients of V_1. This is the system, equations are labelled by this index triple:

$$\begin{aligned}
(111) \qquad & 2(d^1_{1112} - d^1_{1211})b^1_1 + (d^1_{1112} - d^1_{1211})b^2_2 + \\
& (d^1_{1112} - d^1_{1211})b^2_1 + (d^2_{1211} - d^2_{1112})b^1_2 = 0 \\
(221) \qquad & 2(d^2_{1212} - d^2_{2211})b^1_1 + (d^2_{1212} - d^2_{2211})b^2_2 + \\
& (d^2_{1222} - d^2_{2221} + d^1_{2211} - d^1_{1212})b^2_1 + (d^2_{1112} - d^2_{1211})b^1_2 = 0
\end{aligned}$$

$$
\begin{aligned}
&(121) && 3(d^2_{1112}-d^2_{1211})b^1_1+(d^2_{1122}-d^2_{2211}+d^1_{2111}-d^1_{1112})b^2_1=0\\
&(211) && (d^1_{1212}-d^1_{2211})b^1_1+2(d^1_{1212}-d^1_{2211})b^2_2+\\
& && (d^1_{1222}-d^1_{2221})b^2_1+(d^1_{1112}-d^1_{2111}+d^2_{2211}-d^2_{1221})b^1_2=0\\
&(112) && (d^1_{1122}-d^1_{1212})b^1_1+2(d^1_{1122}-d^1_{1212})b^2_2+\\
& && (d^1_{1222}-d^1_{2221})b^2_1+(d^1_{1112}-d^1_{2111}+d^2_{1212}-d^2_{1122})b^1_2=0\\
&(222) && (d^2_{1222}-d^2_{2221})b^1_1+2(d^2_{1222}-d^2_{2221})b^2_2+\\
& && (d^1_{2221}-d^1_{1222})b^2_1+(d^2_{1122}-d^2_{2211})b^1_2=0\\
&(122) && 2(d^2_{1122}-d^2_{1212})b^1_1+(d^2_{1122}-d^2_{1212})b^2_2+\\
& && (d^2_{1222}-d^2_{2221}+d^1_{1212}-d^1_{1122})b^2_1+(d^2_{1112}-d^2_{2111})b^1_2=0\\
&(212) && 3(d^1_{1222}-d^1_{2212})b^2_2+(d^2_{2221}-d^2_{1222}+d^1_{1122}-d^1_{2211})b^1_2=0.
\end{aligned}
$$

Setting:

$$
\begin{gathered}
a=(d^1_{1112}-d^1_{1211}),\quad e=(d^1_{1212}-d^1_{1122}),\quad g=(d^1_{1222}-d^1_{2221}),\\
h=(d^1_{1122}-d^1_{1212}),\quad b=(d^2_{1211}-d^2_{1112}),\quad c=(d^2_{1212}-d^2_{2211}),\\
d=(d^2_{1222}-d^2_{2221}),\quad f=(d^2_{1122}-d^2_{1212}),
\end{gathered}
$$

we see the system take form:

$$
\begin{pmatrix}
2a & a & a & b\\
2c & c & d+e & -b\\
-3b & & f+c-a & \\
-e & -2e & g & a-c\\
h & 2h & g & a-f\\
d & 2d & -g & f+c\\
2f & f & d-h & -b\\
 & 3g & & h-e-d
\end{pmatrix}.
$$

This is nondegenerate for a generic connection. For example, if $a=d=c=e=f=h=0$, then it is:

$$
\begin{pmatrix}
 & & & b\\
 & & & -b\\
-3b & & & \\
 & & g & \\
 & & g & \\
 & & -g & \\
 & & & -b\\
 & 3g & &
\end{pmatrix}.
$$

Now we can summarize what we know about exceptional stabilizers:

Proposition 4.1. *The stabilizer of a k-jet of a generic connections:*
for $n = 1$ is 1-dimensional (equal to G_1/G_2) for any k;
for $n = 2$ is equal to G_1/G_2 for $k = 0$, is 1-dimensional for $k = 1$, and is trivial for $k \geq 2$.

5. Back to Poincaré series: proof of Theorem 2.4

We will use Propositions 3.3 and 3.4 to find dimension of a generic orbit:

$$\dim \mathcal{O}_k(\Gamma) = \dim(T_{\mathrm{id}}(K_k/G_\Gamma)) =$$

$$\dim(\{V|V = V_1 + V_2 + \ldots + V_{k+1} + V_{k+2}\}) - n^2\delta_0^k - \delta_1^n(1-\delta_0^k) - \delta_2^n\delta_1^k,$$

where δ is a Kronecker symbol, taking care of nonzero stabilizers for various k and n; V_i is an n-component vector, each component a homogeneous polynomial of degree i in $x^1, \ldots, x^n$. So we have:

$$\dim \mathcal{O}_k = n\sum_{m=1}^{k+2}\binom{n+m-1}{n-1} - \delta_1^n - \delta_2^n\delta_1^k \quad \text{for} \quad k \geq 1,$$

$$\dim \mathcal{O}_0 = n\sum_{m=1}^{2}\binom{n+m-1}{n-1} - n^2 = n\binom{n+2-1}{n-1} = \frac{n^2(n+1)}{2}.$$

Dimension of the moduli space of connection k-jets $\mathcal{M}_k$ is:

$$\dim \mathcal{M}_k = \dim \mathcal{F}_k - \dim \mathcal{O}_k,$$

where $\mathcal{F}_k$ is the space of connection k-jets. For $k = 0$:

$$\dim \mathcal{M}_0 = \dim \mathcal{F}_0 - \dim \mathcal{O}_0 = n\frac{n(n+1)}{2} - \frac{n^2(n+1)}{2} = 0.$$

For $k \geq 1$:

$$\dim \mathcal{M}_k = \dim \mathcal{F}_k - \dim \mathcal{O}_k$$

$$= n\frac{n(n+1)}{2}\sum_{m=0}^{k}\binom{n+m-1}{n-1} - n\sum_{m=1}^{k+2}\binom{n+m-1}{n-1} + \delta_1^n + \delta_2^n\delta_1^k.$$

The Poincaré series is:

$$p_\Gamma(t) = \dim \mathcal{M}_0 + \sum_{k=1}^{\infty}(\dim \mathcal{M}_k - \dim \mathcal{M}_{k-1})t^k$$

$$= n\sum_{k=1}^{\infty}\left[\frac{n(n+1)}{2}\binom{n+k-1}{n-1} - n\binom{n+k+1}{n-1}\right]t^k + (t-t^2)\delta_2^n, \quad (5.17)$$

as required. Simplifying (5.17), we obtain the following

Fact. The Poncaré series $p_\Gamma(t)$ is a rational function. Namely,

$$p_\Gamma(t) = (t-t^2)\delta_2^n + nD_\Gamma\left(\frac{1}{1-t}\right) + \frac{(n-1)n^2(n+1)}{2}, \quad (5.18)$$

where D_Γ is a differential operator of order $n-1$:

$$D_\Gamma = \frac{n(n+1)}{2}\binom{n+t\frac{d}{dt}-1}{n-1} - n\binom{n+t\frac{d}{dt}+1}{n-1},$$

with

$$\binom{n+t\frac{d}{dt}-1}{n-1} = \frac{1}{(n-1)!}(t\frac{d}{dt}+1)\ldots(t\frac{d}{dt}+n-1),$$

$$\binom{n+t\frac{d}{dt}+1}{n-1} = \frac{1}{(n-1)!}(t\frac{d}{dt}+3)\ldots(t\frac{d}{dt}+n+1).$$

Indeed, denote

$$\varphi_m(t) = \sum_{k=0}^{\infty} k^m t^k, \quad m \in \mathbb{Z}_+,$$

then

$$\varphi_m(t) = \sum_{k=0}^{\infty} k^{m-1}kt^{k-1}t = t\left(\sum_{k=0}^{\infty} k^{m-1}t^k\right)'$$

$$= \left(t\frac{d}{dt}\right)\varphi_{m-1}(t) \text{ for } \quad m \in \mathbb{N}.$$

Thus

$$\varphi_m(t) = \left(t\frac{d}{dt}\right)^m \varphi_0(t) = \left(t\frac{d}{dt}\right)^m\left(\frac{1}{1-t}\right).$$

Hence,

$$\sum_{k=0}^{\infty}\left[\frac{n(n+1)}{2}\binom{n+k-1}{n-1} - n\binom{n+k+1}{n-1}\right]t^k$$

$$= \left[\frac{n(n+1)}{2}\binom{n+t\frac{d}{dt}-1}{n-1} - n\binom{n+t\frac{d}{dt}+1}{n-1}\right]\left(\frac{1}{1-t}\right).$$

We have:

$$a_0 = \frac{n(n+1)}{2} - n\binom{n+1}{n-1} = -\frac{(n-1)n(n+1)}{2}.$$

So

$$p_\Gamma(t) = (t-t^2)\delta_2^n + n\sum_{k=0}^{\infty}\left[\frac{n(n+1)}{2}\binom{n+k-1}{n-1} - n\binom{n+k+1}{n-1}\right]t^k$$

$$+\frac{(n-1)n^2(n+1)}{2} = (t-t^2)\delta_2^n + D_\Gamma\left(\frac{1}{1-t}\right) + \frac{(n-1)n^2(n+1)}{2}.$$

□

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