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ON CONFIGURATION SPACES OF PLANAR PENTAGONS

Abstract. Algebraic formulae for the Euler characteristic are applied to the topological study of moduli spaces of planar polygons. In particular, a complete description of moduli spaces of planar pentagons is obtained using a computer algorithm for the calculation of the Euler characteristic of algebraic varieties.

1. Introduction

Configuration spaces of polygons in Euclidean spaces naturally appear in mechanics [4], [6] and engineering (see, e.g., [15]) and suggest some curious geometric problems. In particular, the topology of such configuration spaces has recently gained considerable interest [8], [9]. For polygons with a small number of sides (it is tempting to call them "fewgons" by analogy with [13]) the description of configuration spaces becomes a visual geometric problem which can be in principle investigated using Morse theory [16], [8] (cf. [17], [7]) but this requires rather delicate topological and combinatorial considerations.

At the same time some topological invariants of such configuration spaces (e.g., the Euler characteristic) can be algorithmically computed using the so-called *signature formulae for topological invariants* [12] which rely on the well-known algebraic formula for the topological (mapping) degree [5], [10]. Since nowadays such computations can be done on computer using an algorithmic implementation of the named formula developed by A. Lecki and Z. Szafraniec [14], these invariants appear to be easily accessible for "fewgons".

The aim of the present note is to illustrate the latter approach by explaining how one can describe the possible topological types of the configuration spaces of *generic planar pentagons*. The result itself is rather elementary and can be obtained by the aforementioned Morse theory approach [16], [7], but our approach is more straightforward and this type of application of signature formulae seems to be effective in a number of similar problems so it might be of certain interest as a paradigm.

In order to make the exposition easily readable and self-contained we present basic definitions and auxiliary results concerning configuration spaces and signature formulae. The main result follows from a general formula for the Euler characteristic of an algebraic subset [3], [11] and some concrete computations making use of the aforementioned algorithm for the mapping degree. Actually, the same technique is applicable for certain configuration spaces of different types and we briefly discuss some related problems and settings at the end of the paper.

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2. Configuration spaces of polygons

Let M_n denote a polygon with n angles and vertices $v_1, \dots, v_n$ in Euclidean plane $\mathbf{R}^2$ and let l_i denote the length of the side $v_i v_{i+1}$, $i = 1, \dots, n, v_{n+1} := v_1$. By A_n we denote a "polygonal chain" with n vertices and $n-1$ sides. Thus A_n is obtained from M_n by ignoring the length of the "virtual last side" $v_n v_1$. Borrowing terminology from robotics [15], one can call it a $(n-1)$-arm (another appealing name is "nunchaku").

Informally speaking we want to consider the set of all possible positions of M_n in Euclidean space $\mathbf{R}^k$ which preserve lengths of the sides (so the sides are considered as rigid segments). In order to achieve certain normalization (in particular, to avoid dull shifts of M_n as a whole), it is reasonable to "freeze" one of the sides, say the first one, by requiring that it should coincide with the first coordinate ort $\mathbf{e}_1$ (whose endpoint $(1, \dots, 0)$ will be denoted by e). Thus we always assume that $l_1 = 1$ (another reasonable normalization is mentioned in the last section).

This apparently causes no loss of generality. In the sequel the distance between two points $p, q \in \mathbf{R}^k$ is denoted by $d(p, q)$.

Definition 1. *For each natural $k \geq 2$, introduce the kth configuration space $C(M_n, k)$ of polygon M_n as the set of all maps $F : V \to \mathbf{R}^k$ of the set of its vertices V into $\mathbf{R}^k$ such that*

1) *for each $i = 1, \dots, n$, $d(F(v_i), F(v_{i+1})) = l_i$;*
2) *$F(v_1) = \mathbf{0} (= (0, \dots, 0))$, $F(v_2) = e$.*

In the same way one defines configuration spaces $C(A_n, k)$ of an $(n-1)$-arm A_n (just omit the requirement that $F(v_n) = \mathbf{0}$). We always consider them endowed with natural (metric) topologies and wonder about their possible homeomorphy types. Of course for an $(n-1)$-arm this issue is trivial since each of its $n-2$ "free" vertices can trace a $(k-1)$-sphere when positions of preceding vertices are fixed. In other words, the configuration space of an $(n-1)$-arm is homeomorphic to the product of spheres of appropriate dimension.

Lemma 1. $C(A_n, k) \cong (S^{k-1})^{n-2}$

For polygons, the situation is not that simple and there does not seem to exist a topological description of configuration spaces for polygons with arbitrary number of vertices, so in this note we restrict ourselves to configuration spaces $C(M_5, 2)$ of *planar pentagons*. At the same time, some general properties of configuration spaces can be established without big effort. Notice that in our setting the set of all n-gons is parametrized by a non-negative vector of lengths $l = (l_2, \dots, l_n) \in \mathbf{R}_+^{n-1}$. We write $M_n(l)$ for the the corresponding polygon. It turns out that for an *open dense subset* of parameter space, $C(M_n(l), k)$ is a smooth manifold. In such cases we will say that this happens *for almost all* points of the parameter space and the corresponding polygon is called *generic*.

Proposition 1 ([8], [7]). *For almost all $l \in \mathbf{R}_+^{n-1}$, $C(M_n(l), k)$ is a smooth compact orientable submanifold of dimension $k(n-2) - n + 1$.*

This result is well-known and quite straightforward (see, e.g., [7], [8]). From the definition it follows that the configuration space can be represented as a preimage of the point $(l_2^2, \dots, l_n^2)$ under an algebraic (quadratic) mapping

$$\Phi : \mathbf{R}^{k(n-2)} \to \mathbf{R}^{n-1}.$$

The source is the parameter space for all possible positions of $(n-2)$ "movable" vertices and one has $(n-1)$ conditions expressing the constancy of the squares of side lengths (for planar pentagons this map is written down explicitly in Section 5). It is obvious that such preimages are compact. The rest follows from the Sard lemma [2] since the preimage of a regular value of Φ is a smooth orientable manifold of the indicated dimension. Similar results about the structure of configuration spaces can be obtained using a version of implicit function theorem developed in [1].

Corollary 1. *For almost all pentagons M_5, the configuration space $C(M_5, 2)$ is a smooth compact orientable surface.*

Notice that the configuration space of a polygon need not be connected. This is obviously the case for a triangle M_3 where the configuration space is the union of two points, which expresses the evident fact that there are only two possible positions of a triangle obtained from each other by reflection in the "freezed" first side $\mathbf{e}_1$. By adding a very short side at one of vertices of a triangle, say at e, one obtains a quadrangle with a disconnected configuration space. Indeed, as a result of such procedure each of the points in $C(M_3)$ becomes a circle (the added short side can be rotated around e).

Repeating this procedure it is easy to obtain examples of polygons with configuration spaces homeomorphic to a disjoint union of two tori. It turns out that this is the only possibility for a disconnected configuration space and this can only happen if a polygon has "three long sides" (see Theorem 1 of [8]). In our situation this result of [8] takes the folowing form. Fix a length vector l and denote by L the sum of all side lengths of the corresponding pentagon $M_5(l)$ (recall that we set $l_1 = 1$).

Proposition 2. *The configuration space of $C(M_5(l), 2)$ is not connected if and only if there exist three (pairwise different) indices i, j, k such that*

$$l_i + l_j > \frac{L}{2}, l_i + l_k > \frac{L}{2}, l_j + l_k > \frac{L}{2}.$$

It is quite easy to see that in such case the configuration space can never have more than two components.

Corollary 2. *If a configuration space of a planar pentagon is smooth and disconnected then it is homeomorphic to a disjoint union of two tori T^2.*

Thus to obtain a complete description for planar pentagons, it only remains to classify possible topological types of connected configuration spaces in which case everything is determined by its Euler characteristic. In fact the final result (Theorem 2) shows that the answer is always given by the value of the Euler characteristic since it is vanishing only in the case of disconnected configuration space.

Now one can follow a standard paradigm of singularity theory [2] which involves investigating the set of "non-generic" values of parameters (bifurcation diagram). In our case this is the set $B(n, k)$ of all values of l for which $C(M_n(l), k)$ is not smooth. One can await that the topological type of configuration space remains unchanged on each connected component of the complement to $B(n, k)$ and all possible changes happen when the value of parameter crosses the bifurcation diagram [2].

These points will be clarified in the next section. Notice that similar results hold for configuration spaces of linkages [17] and graphs [7], as well as for configuration spaces of a slightly different type mentioned in the last section (cf. [9]).

3. Singular configuration spaces

The condition for a collection of side lengths to be a regular value of Φ is well known.

Proposition 3 ([7], [9]). *Polygon $M_5(l)$ has a singular configuration space if and only if there exist numbers $c_i \in \{-1, 1\}, i = 2, \dots, n$, such that*

$$\sum_{i=2}^{n} c_i l_i = 1. \tag{1}$$

This means that a configuration space is singular exactly in those cases when the corresponding polygon can degenerate in a line segment, i.e., when the side lengths are such that all vertices can be put on the same line (the first coordinate axis in our case).

Thus the bifurcation diagram $B(n, k)$ coincides with the union of sixteen hyperplanes H_c defined by the above linear equations of the form (1). Moreover, the topological type of configuration space behaves exactly in such way as is predicted by singularity theory.

Proposition 4 ([17], [7], [9]). *The topological type of configuration space does not change on each component of the complement to $B(n, k)$.*

These results imply in particular that all possible topological types of non-singular configuration spaces $C(M_5, 2)$ can be listed by finding them for a finite number of concrete values of the length vector l. Indeed, from Proposition 3 it follows that there are exactly 16 hyperplanes H_c which obviously cut the space of parameters in a finite union of disjoint domains (obviously, there can be no more than 2^{16} such domains; in fact, since among these 16 hyperplanes there are 8 pairs of parallel hyperplanes, a minute thought shows that the amount of such domains cannot exceed $6561 = 3^8$). Since the topology of configuration space does not change on each of these pieces, it is apparently sufficient to take a single value of length vector in each of them and determine the topological types of configuration spaces for those concrete pentagons.

Our next aim is to show that in each concrete case this can be done using signature formulae from [12].

Remark 1. In fact, one can give a sufficiently detailed description of possible singularities of configuration spaces. Basically they have isolated singularities of cone type which appear in the process of surgery corresponding to points of the bifurcation diagram. For example, for the configuration spaces considered in [9] it is known that they are analytic spaces with only isolated singularities and their singularities are equivalent to homogeneous quadratic cones ([9], Theorem 2.6). Since the configuration spaces above can be also defined by quadratic equations one can arrive to similar conclusions using the known results about singularities of intersection of quadrics.

Remark 2. In fact important information on the topology of configuration space of a planar pentagon can be read off the values of the last two sides v_3v_4, v_4v_1 [7]. In particular, in this way one obtains a decomposition of the plane quadrant $\mathbf{R}^2_+$ of possible values of lengths of these two sides and it is sufficient to determine the topological type of configuration space in each of arising domains, which helps to reduce the number of concrete cases needed to be analyzed.

4. Explicit formula for the Euler characteristic

We proceed by recalling some results from [5], [10], [11]. Let $f_1, \dots, f_p$ be real polynomial functions on $\mathbf{R}^q$ and $X = \{f_1 = 0, \dots, f_p = 0\} \subset \mathbf{R}^q$. Assume that X is compact and denote by d the maximum of algebraic degrees of f_i. As is well known, the Euler characteristic $\chi(X)$ can be expressed through the local topological degree of certain endomorphism [3], [11]. Recall that the local topological degree $deg_p\, F$ of an endomorphism is defined at every point p which is an isolated point of the full preimage of its image [5], [10].

Let us introduce the homogenizations of polynomials f_i defined as

$$g_i(x_0, x_1, \dots, x_q) = x_0^{d+1} f_i\left(\frac{x_1}{x_0}, \dots, \frac{x_q}{x_0}\right)$$

and an auxiliary polynomial

$$H(x_0, \dots, x_q) = \sum_{i=1}^{p} g_i^2 - \sum_{j=0}^{q} x_j^{2d+4} \,. \tag{2}$$

Theorem 1 ([3], [11]). *Polynomial H has an isolated critical point at the origin of $\mathbf{R}^{q+1}$ and one has*

$$2\chi(X) = (-1)^q - deg_0\, grad\, H \,. \tag{3}$$

Recall now that the local topological degree can be effectively computed as the signature of the Gorenstein quadratic form on the local algebra [5], [10]. The general method of calculating the matrix of Gorenstein quadratic form is presented in [12]. This basically requires finding a monomial base for the local algebra of a map germ and writing down the multiplication table for this base [12]. In simple cases this can be done "by hand" (cf. [12]). In general, there exists (at least one) effective algorithm for computing the local degree using the aforementioned method and algorithms for computing Gröbner bases [14].

Thus in order to solve the problem for planar pentagons, it remains to choose one point in each component of the complement to $B(5,2)$ and compute $\chi(C(M_5(l),2))$ at these points using such an algorithm. This can be performed using a computer and one eventually comes to the desired result.

5. Topological types of configuration spaces

Let us apply preceding results in the case of a pentagon $M_5(l)$. First of all notice that its configuration space $C(M_5(l),2)$ is an algebraic set defined by explicit equations. Indeed, positions of last three "movable" vertices can be parametrized by a point P in $\mathbf{R}^6 = (\mathbf{R}^2)^3$ and the condition that it realizes the image of a map F as in Definition 1 can be apparently written as a system of four quadratic equations expressing the fact that lengths of all sides remain fixed.

Taking into account our normalization convention, these equations take the form

$$(x_1-1)^2+x_2^2=l_2^2,\ (x_3-x_1)^2+(x_4-x_2)^2=l_3^2,$$
$$(x_5-x_3)^2+(x_4-x_2)^2=l_4^2,\ x_5^2+x_6^2=l_5^2.$$

In other words, $C(M_5(l),2)$ is naturally isomorphic to the full preimage of the origin under the mapping $\Phi : \mathbf{R}^6 \to \mathbf{R}^4$ defined by the left hand sides of the above equations. Thus one can compute its Euler characteristic by formula (3) and this can be done quite effectively according to [14].

In order to get an idea of the complexity of this procedure one can use (2) in order to explicitly write down the mapping $grad\, H : \mathbf{R}^7 \to \mathbf{R}^7$ whose local degree should be computed. Then one needs to find a monomial base in the local algebra of this map germ and write down the matrix of Gorenstein quadratic form whose signature equals the local degree [12]. In our case this will be a $(5^7 \times 5^7)$-matrix so this looks quite awful and cannot be actually treated "by bare hands", but modern computers are able to do this job using the program described in [14].

Thus we have achieved a complete reduction of the problem to a finite number of computer calculations. The remaining steps are not of a big interest since everything is reduced to the choice of a reasonable representative (vector l) in each component of the complement of the bifurcation set and performing calculations for those pentagons. Omitting numerical details we present the final conclusion.

Theorem 2. *All possible topological types of non-singular configuration spaces $C(M_5, 2)$ are contained in the following list: $S^0 \times S^1 \times S^1$ (disjoint union of two tori T^2) or compact closed orientable surface M_g^2 of genus g not exceeding four.*

Taking into account Proposition 2 we see that for every concrete pentagon the topological type of the configuration space may be found by computing the Euler characteristic by formula (3). It is quite amazing to visualize the configuration space in concrete cases, e.g., to verify that the configuration space of a regular pentagon is homeomorphic to a sphere with four handles.

6. Concluding remarks

As was already mentioned, it is possible to describe the topological types of singular configuration spaces as well. In particular, the singularity types are of a very simple nature (cf. [9]). The topological changes which occur when the side length vector l crosses the bifurcation diagram, can be easily described by a standard "wall-crossing" argument [17] and they appear to be just surgeries associated with non-degenerate (Morse) critical points of the parameter l_n considered as a function on the configuration space $C(A_{n-1}, k)$ [17], [7].

Along the same lines one can also consider configuration spaces of "spatial" polygons $C_0(M_n, 3)$ described in [9]. They are obtained by factoring the set of all possible positions of M_n in $\mathbf{R}^3$ over the natural action of the group of isometries [9].

It turns out that such configuration spaces are two-dimensional in the case of a quadrangle in $\mathbf{R}^3$ and one can again apply formula (3) for determining possible topological types of $C_0(M_4, 3)$. Realizing the above scheme in this case one finds out that the Euler characteristic vanishes for all regular configuration spaces so their topological type is always that of a two-torus T^2.

Finally, it is also possible to describe a class of graphs Γ_p having an arbitrary amount of vertices p and such that their configuration spaces are nevertheless two-dimensional. This gives many examples of two-dimensional configuration spaces of high genus which are sufficiently curious from the geometric point of view (cf. [7]).

The values of genus obtained in such way may be compared with the general estimates for the possible value of the Euler characteristic of configuration spaces. From the results of [2], [12], it follows that $\chi(C(\Gamma_p, 2))$ does not exceed the so-called Petrovsky number $P(2(p-2)+1, 5)$ which is easily calculable for any concrete value of p [12]. It seems to be an open problem to obtain an exact upper estimate for the genera of configuration spaces of such graphs in terms of p.

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