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APPLICATIONS OF THE WULFF CONSTRUCTION TO THE NUMBER THEORY

Abstract. We apply the geometric construction of solutions of some variational problems of combinatorics to estimate the number of partitions and of plane partitions of an integer.

1. Statement of results

In this note we will explain how certain classical statements from the number theory can be rederived using recent results of the asymptotic combinatorics. Specifically, we will derive the asymptotic behavior of the number of partitions and plain partitions of an integer N.

We recall that a partition p of an integer N is an array of non-negative integers $n_1 \geq n_2 \geq ... \geq n_k \geq ...$, such that $\sum_{i=1}^{\infty} n_i = N$. One associates with p a well-known geometric object called Young diagram, $Y(p)$, with N cells. The set of all Young diagrams with N cells will be denoted by $\mathcal{Y}_N$.

Similarly, a plane partition S of an integer N is a two-dimensional array of non-negative integers n_{ij}, such that for any i we have $n_{i1} \geq n_{i2} \geq ... \geq n_{ik} \geq ...$, for any j we have $n_{1j} \geq n_{2j} \geq ... \geq n_{kj} \geq ...$, while again $\sum_{i,j=1}^{\infty} n_{ij} = N$. The corresponding geometric picture is called a *skyscraper*, or a *mausoleum*, or a *3D Young diagram*. The set of all skyscrapers with N cells (= the set of all plane partitions of N) will be denoted by $\mathcal{S}_N$.

Our goal is to explain that as $N \to \infty$

$$\ln |\mathcal{Y}_N| \sim \pi \left(\frac{2}{3}\right)^{1/2} N^{1/2}, \tag{1}$$

and

$$\ln |\mathcal{S}_N| \sim 3 \left(\frac{\zeta(3)}{4}\right)^{1/3} N^{2/3}. \tag{2}$$

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In fact, much more precise information about the behavior of these functions is known. For $|\mathcal{Y}_N|$ it is the famous result of Hardy-Ramanujan-Rademacher, see [1]. The corresponding asymptotic behavior of $|\mathcal{S}_N|$ was found by Wright in [10].

To get the relations (1) and (2), we are using, correspondingly, the results of [9] and [2], where the shape of the typical Young diagram, resp. skyscraper, were found. (Here "typical" means with respect to the uniform distribution on the sets $\mathcal{Y}_N$ and $\mathcal{S}_N$.) These shapes are solutions of certain variational maximizing problems described in details in the next section. Briefly speaking, these problems consist in finding a hypersurface G_η in a certain class that maximizes the surface integral $\mathcal{V}_\eta(G) = \int_G \eta(\mathbf{n}_x)\, ds_x$, where $\mathbf{n}_x$ is the normal to G at x, and η is a corresponding given function.

For the case of Young diagrams the function $\eta_{\mathcal{Y}}$ is defined on a unit quarter-circle $\Delta^1 = S^1 \cap \mathbb{R}^2_+$, and is given by

$$\eta_{\mathcal{Y}}(\mathbf{n}) = -\left(n_1 \ln \frac{n_1}{n_1+n_2} + n_2 \ln \frac{n_2}{n_1+n_2}\right).$$

The typical Young diagram with N cells, scaled by the factor $N^{-1/2}$, goes along the curve $\mathcal{C}$:

$$\exp\left\{-\frac{\pi}{\sqrt{6}}x\right\} + \exp\left\{-\frac{\pi}{\sqrt{6}}y\right\} = 1.$$

This result was obtained by Vershik, see [9] or [8].

In the skyscraper case the function $\eta_{\mathcal{S}}$ is defined on the set $\Delta^2 = S^2 \cap \mathbb{R}^3_+$ of unit vectors $\mathbf{n}$ and is given by

$$\eta_{\mathcal{S}}(\mathbf{n}) = \frac{|\mathbf{n}|_1}{\pi} \sum_{i=1}^{3} L\left(\pi \frac{n_i}{|\mathbf{n}|_1}\right),$$

where $|\mathbf{n}|_1 = n_1 + n_2 + n_3$, and L is the Lobachevsky function defined for $x \in [0, \pi]$ by

$$L(x) = -\int_0^x \ln(2\sin t)\, dt.$$

The surface $G_{\eta_{\mathcal{S}}}$ in $\mathbb{R}^3$ that describes the shape of a typical skyscraper with N cells scaled by the factor $N^{-1/3}$ to the unit volume is given by

$$G_{\eta_S} = \left(\frac{\zeta(3)}{4}\right)^{-1/3} \{(f(A,B,C)-\ln A, f(A,B,C)-\ln B, f(A,B,C)-\ln C)\},$$

where $A, B, C > 0$, $A + B + C = 1$, and

$$f(A,B,C) = \frac{1}{4\pi^2} \int\limits_{[0,2\pi]} \int\limits_{[0,2\pi]} \ln\left(A + Be^{iu} + Ce^{iv}\right) du dv,$$

see [2].

The geometric construction, which gives the solution to problems described above, was found in [4]. In fact, it is very close to the Wulff construction [11]. Both constructions are described below, together with the relation between them. Due to the relation (11), the evaluation of the integrals $\mathcal{V}_\eta(G_\eta)$ becomes very easy. It turns out that

$$\mathcal{V}_{\eta_\mathcal{Y}}(G_{\eta_\mathcal{Y}}) = \pi\left(\frac{2}{3}\right)^{1/2},$$

while

$$\mathcal{V}_{\eta_S}(G_{\eta_S}) = 3\left(\frac{\zeta(3)}{4}\right)^{1/3}.$$

We know from [9] that most of the Young diagrams with N cells go along the curve $N^{1/2}\mathcal{C}$. On the other hand, if G is a graph of a decaying function $g(x)$ with $\int_0^\infty g(x)\,dx = N$, then the functional $\mathcal{V}_{\eta_\mathcal{Y}}(G)$ has the following meaning: "the number of the diagrams with N cells going along the curve G is of the order $\exp\{\mathcal{V}_{\eta_\mathcal{Y}}(G)\}$". Therefore, the value (3) multiplied by $N^{1/2}$ is precisely the exponent of (1). Correspondingly, most of the skyscrapers with N cells go along the surface $N^{1/3}G_{\eta_S}$, see [2]. Therefore the value $\mathcal{V}_{\eta_S}\left(N^{1/3}G_{\eta_S}\right) = N^{2/3}\,\mathcal{V}_{\eta_S}(G_{\eta_S})$ has to be the exponent (2). For more details the reader should consult the papers [9], [3], [2] and the review paper [5].

2. The maximizing problem

Let S^d denote the d-dimensional unit sphere in $\mathbb{R}^{d+1}$. Introduce the subset $\Delta^d = S^d \cap \mathbb{R}_+^{d+1}$ of "positive" unit vectors lying in the positive octant. Let a function η on Δ^d be given, which is supposed to be continuous, nonnegative: $\eta(\cdot) \geq 0$, and vanishing on $\partial\Delta^d$. We suppose for simplicity that the function η vanishes also in some neighborhood of $\partial\Delta^d$. (The general result which is needed for the formulas in the previous section is then obtained by letting this neighborhood shrink to zero). Let now $G \subset \mathbb{R}^{d+1}$ be an embedded hypersurface, possibly with a boundary. We assume that for almost every $x \in G$ the normal vector $\mathbf{n}_x$ to G is defined, and moreover

$$\mathbf{n}_x \in \Delta^d \quad \text{for a.e.} \quad x \in G. \tag{5}$$

Then we can define the functional

$$\mathcal{V}_\eta(G) = \int_G \eta(\mathbf{n}_x)\, ds_x,$$

here ds is the usual volume d-form induced from the Riemannian metric on $\mathbb{R}^{d+1}$ by the embedding $G \subset \mathbb{R}^{d+1}$.

Denote by $Q_N \subset \mathbb{R}_+^{d+1}$ the cube consisting of points $x \in \mathbb{R}^{d+1}$ with $0 \leq x_i \leq N$, $i = 1, 2, ..., d+1$. Denote by O the point $(0, 0, ..., 0) \in Q_N$, and let $A = (N, N, ..., N) \in Q_N$ be the opposite vertex.

Let the number $V \in \left(0, N^{d+1}\right)$ be given. We introduce the family D_V^N of hypersurfaces $G \subset Q_N$ as follows:
i) G splits the cube Q_N into two parts, with the points O and A belonging to different parts; denote them by $Q_N(G, O)$ and $Q_N(G, A)$;
ii) the property (5) holds;
iii) for the volume vol(G, N), defined as the ($(d+1)$-dimensional) volume vol $(Q_N(G, O))$ of the body $Q_N(G, O)$, we have

$$\mathrm{vol}(G, N) = V.$$

We want to solve the following variational problem: find the *upper bound* of $\mathcal{V}_\eta$ over D_V^N :

$$v_\eta = \sup_{G \in D_V^N} \mathcal{V}_\eta(G), \tag{6}$$

as well as the maximizing surface(s) $V_\eta \in D_V^N$, such that $\mathcal{V}_\eta(V_\eta) = v_\eta$, if the maximizer does exist.

It turns out that there exists a geometric construction which provides a solution to the variational problem (6). It was found in [5].

Let

$$K_\eta^> = \left\{\mathbf{x} \in \mathbb{R}_+^{d+1} : \forall \mathbf{n} \in \Delta^d \ (\mathbf{x}, \mathbf{n}) \geq \eta(\mathbf{n})\right\}, \tag{7}$$

$$G_\eta = \partial\left(K_\eta^>\right). \tag{8}$$

Note that $K_\eta^>$ is a convex set, so the hypersurface G_η has normals at almost every point. Moreover, these normals fall into Δ^d. Define the dilatation parameter $\lambda(V,N)$ as the unique solution of the equation

$$\text{vol}\left(Q_N \cap \left(\mathbb{R}_+^{d+1} \setminus \lambda(V,N) K_\eta^>\right)\right) = V. \tag{9}$$

Theorem 1. *The unique solution to the variational problem* (6) *is the surface*

$$G_{\eta,N,V} = Q_N \cap \lambda(V,N) G_\eta, \tag{10}$$

provided N is large enough.

Remark 2. The analogue of the variational problem (6) obtained by removing the constraint (5) is ill posed.

Note that the set $K_\eta^>$ is a convex region in $\mathbb{R}^{d+1}$. If at some boundary point the region $K_\eta^>$ has a unique support plane, then this plane is of the form

$$L_\eta(\mathbf{n}) = \left\{\mathbf{x} \in \mathbb{R}^{d+1} : (\mathbf{x},\mathbf{n}) = \eta(\mathbf{n})\right\}$$

for the corresponding $\mathbf{n}$. Therefore for N large we have

$$\text{vol}(G_\eta) = \frac{\mathcal{V}_\eta(G_\eta)}{d+1}. \tag{11}$$

The proof of the above theorem relies on the following construction which associates with the variational problem (6) a certain Wulff variational problem. Once the correspondence is established, the proof follows easily from known properties of the Wulff problem.

3. The corresponding Wulff problem

First we formulate the general Wulff problem. We restrict ourselves to the symmetric case, in order to make the relation with the above maximizing problem more transparent. Let a real function τ on S^d be given. We suppose that the function is continuous, positive: $\tau(\cdot) \geq const > 0$, and symmetric with respect to the reflections in coordinate planes: $\tau(\mathbf{n}) \equiv \tau(n_1, n_2, ..., n_{d+1}) = \tau(\pm n_1, \pm n_2, ..., \pm n_{d+1})$ for any choice of signs. Then for every hypersurface M^d (possibly with a boundary) embedded in $\mathbb{R}^{d+1}$ we can define the **Wulff functional**

$$\mathcal{W}_\tau\left(M^d\right) = \int_{M^d} \tau(\mathbf{n}_x)\, ds_x. \tag{12}$$

Here $x \in M^d$ is a point on the manifold M^d, and the vector $\mathbf{n}_x$ is the unit vector parallel to the normal to M^d at x. We suppose that M^d is smooth enough.

Suppose additionally that the hypersurface M^d lies in fact in $\mathbb{R}_+^{d+1}$, and separates the origin O from infinity in $\mathbb{R}_+^{d+1}$. Let $N^{d+1} \subset \mathbb{R}_+^{d+1}$ be the union of all finite components of $\mathbb{R}_+^{d+1} \setminus M^d$; we denote the volume $\left|N^{d+1}\right|$ of N^{d+1} by vol $\left(M^d\right)$, and call it *the volume inside* M^d. We denote by D' the collection of all such hypersurfaces M^d in $\mathbb{R}_+^{d+1}$ with finite volumes. By $D'_\Lambda \subset D'$ we denote the collection of all these hypersurfaces M^d for which the volume vol $\left(M^d\right)$ inside M^d equals Λ.

The *Wulff problem* consists in finding the lower bound of $\mathcal{W}_\tau$ over D'_Λ :

$$w_\tau = \inf_{M \in D'_\Lambda} \mathcal{W}_\tau (M), \tag{13}$$

as well as the minimizing surface(s) $W_\tau^{(\Lambda)}$, such that $\mathcal{W}_\tau \left(W_\tau^{(\Lambda)}\right) = w_\tau$, if it exists.

The answer is given by the following **Wulff construction**. Let

$$K_\tau^{\leq} = \left\{\mathbf{x} \in \mathbb{R}^{d+1} : \forall \mathbf{n}\ (\mathbf{x}, \mathbf{n}) \leq \tau(\mathbf{n})\right\}, \quad \text{and} \quad W_\tau = \mathbb{R}_+^{d+1} \cap \partial K_\tau^{\leq}. \tag{14}$$

Note that the set $K_\tau^{\leq}$ is a symmetric convex bounded region in $\mathbb{R}^{d+1}$. If at some boundary point the region $K_\tau^{\leq}$ has a unique support plane, then this plane is of the form

$$L_\tau (\mathbf{n}) = \left\{\mathbf{x} \in \mathbb{R}^{d+1} : (\mathbf{x}, \mathbf{n}) = \tau(\mathbf{n})\right\} \tag{15}$$

for the corresponding $\mathbf{n}$. Therefore

$$\text{vol}\,(W_\tau) = \frac{1}{d+1} \mathcal{W}_\tau (W_\tau). \tag{16}$$

Theorem 3. *The variational problem* (13) *has a unique solution*

$$W_\tau^{(\Lambda)} = \left(\frac{\Lambda(d+1)}{\mathcal{W}_\tau (W_\tau)}\right)^{1/(d+1)} W_\tau,$$

which is a scaled version of the **Wulff shape** W_τ, *see* (14).

Remark 4. The standard use of the name "Wulff shape" refers to the full surface $\partial K_\tau^{\leq}$ itself. In our symmetric setting the surface $\partial K_\tau^{\leq}$ is a union of W_τ and its multiple reflections in coordinate planes.

Remark 5. The variational problem (13) might have no solutions if the function τ was allowed to vanish.

The paper [7] contains a simple proof that $\mathcal{W}_\tau\left(W_\tau^{(\Lambda)}\right) \leq \mathcal{W}_\tau(M)$ for every $M \in D'_\Lambda$. The uniqueness of the minimizing surface is proven in [6].

Now we are going to construct the Wulff problem corresponding to the problem (6) (and its solution (10)). For this we have to specify the function $\tau_\eta(\mathbf{n})$ and the value Λ of the volume. First we put

$$\Lambda = N^d - V.$$

We define $\tau(\mathbf{n})$ under assumption that the normalization constant $\lambda(V, N)$ from the equation (9) equals to 1. This is done to simplify the notation. In this case for $\mathbf{n} \in \Delta^d$ we put

$$\tau_\eta(\mathbf{n}) = \mathrm{dist}\,(A, L_\eta(\mathbf{n})),$$

see (15). For the remaining $\mathbf{n}$-s the function τ is defined by symmetry.

Note that

$$\tau_\eta(\mathbf{n}) + \eta(\mathbf{n}) = \left(\mathbf{n}, \overrightarrow{OA}\right).$$

Therefore for G in D_V^N we have:

$$\mathcal{V}_\eta(G) + \mathcal{W}_{\tau_\eta}(G) = \int\limits_G \left(\mathbf{n}_x, \overrightarrow{OA}\right) ds_x.$$

Let Π be the hyperplane orthogonal to the vector $\overrightarrow{OA}$, and P denote the orthogonal projection on Π. Then the last integral is nothing else but the d-dimensional area $S(P(G))$ of the set $P(G)$, so

$$\mathcal{V}_\eta(G) + \mathcal{W}_{\tau_\eta}(G) = S(P(G)), \tag{17}$$

and therefore

$$\sup_{G \in D_V^N} \mathcal{V}_\eta(G) \leq \sup_{G \in D_V^N} S(P(G)) - \inf_{G \in D_V^N} \mathcal{W}_{\tau_\eta}(G) \tag{18}$$

Denote by G_0 the surface made from d faces of the cube Q_N containing the vertex O. Then clearly $S(P(G_0)) \geq S(P(G))$ for every $G \in D_V^N$. Since we suppose that the function η vanishes in some neighborhood of

$\partial\Delta^d$, we have that $S\left(P\left(G_\eta\right)\right) = S\left(P\left(G_0\right)\right)$, provided N is large enough. Also, by the very definition of the function τ_η and in view of Theorem 3,

$$\inf_{G\in D_V^N} \mathcal{W}_{\tau_\eta}\left(G\right) = \mathcal{W}_{\tau_\eta}\left(G_\eta\right),$$

so (17) and (18) imply that indeed

$$\sup_{G\in D_V^N} \mathcal{V}_\eta\left(G\right) = \mathcal{V}_\eta\left(G_\eta\right).$$

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