Записки научных
семинаров ПОМИ
Том 295, 2003 г.

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INITIAL BOUNDARY VALUE PROBLEMS FOR LINEAR VISCOELASTIC FLOWS GENERATED BY INTEGRODIFFERENTIAL EQUATIONS

Dedicated to Olga Aleksandrovna Ladyzhenskaya on her jubilee

1. Introduction

In the paper we consider motions of linear viscoelastic fluids with rheological equation of integral type. Let the deviator of the stress tensor σ and the rate of strain tensor D be connected by the linear integral equation

$$\sigma(x,t) = \int_0^t K(t-\tau)D(x,\tau)d\tau. \tag{1}$$

These equations were investigated in the paper by M. Renardy [3], in the monographs of N. W. Tschoegl [9] and of J. M. Golden and G. A. C. Graham [10] and [42]. Among the recent publications devoted to this equations we may point out papers of S. Shaw, J. Whiteman [11], A. Doubova, E. Fernandez-Cara, M. Gonzalez-Burgos [12], and M. Yakupov [13].

The equations of motion of linear viscoelastic fluids are approximations of equations of nonlinear viscoelastic flows for small displacement gradients. The theoretical possibility of applications of such equations was motivated in the monograph of R. B. Bird, R. C. Armstrong, and O. Hassanger [16]. In practice, these equations are used fore mere than 100 years. In monographs of G. J. Creus [15], J. W. Goodwin and R. W. Hughes [17] and F. R. Schwarzl [18], experimental data confirming these linear viscoelastic models and restrictions on of its application are presented.

Initial boundary value problems for equations of motion of linear viscoelastic fluids were investigated in the papers by A. P. Oskolkov [5, 6, 7]. However, A. P. Oskolkov considered only rheological equations of some special types expressed in the form of differential equation.

The motion of incompressible media is described by the system of equations

$$\frac{\partial}{\partial t}v + v_k\frac{\partial}{\partial x_k}v + \operatorname{grad} p = \operatorname{div}\sigma + f,$$
$$\operatorname{div} v = 0. \tag{2}$$

If we impose the relation between the deviator of the stress tensor σ and the rate of strain tensor $D = \frac{1}{2}(\nabla v + (\nabla v)^T)$, then the type of a fluid is stated. This relation is called the rheological equation. First we consider the rheological equation of the most general type

$$\sigma = \mathbf{K}D. \tag{3}$$

We assume that the operator $\mathbf{K}$ is positive definite in the sense that:

$$\int\limits_{Q_T}((\mathbf{K}D) : D) \geq 0 \tag{4}$$

for any function v which is a solution of initial boundary value problem. This requirement makes a physical sense because the corresponding quadratic form is equal to

$$-(\operatorname{div}\sigma, v)_{2,\Omega} = \int\limits_{\Omega}\sigma : \nabla v = \int\limits_{\Omega}\sigma : D, \tag{5}$$

what means that the dissipation of the energy is positive (see G. Astarita, G. Marucci [4], Chap. 2–3). This condition is fulfilled for any non-Newtonian (and Newtonian) fluid in the case when the boundary integral is equal zero. Under this hypothesis we obtain the estimate of the solution of initial boundary value problem.

Then, we impose an additional condition of positive definiteness of the relation

$$-(\operatorname{rot}\operatorname{div}(\mathbf{K}D), \operatorname{rot} v)_{2,Q_T} \geq 0 \tag{6}$$

This relation is a corollary of (4) if the operator $\mathbf{K}$ commutates with the operators $\frac{\partial}{\partial x_k}$, $k = 1, 2$, and if the boundary integral vanishes. Moreover, this equation is fulfilled if the operator $\mathbf{K}$ has the form (1). Under this hypothesis we prove the existence of a weak solution of initial boundary value problem with periodic boundary conditions and solvability of the Cauchy problem in the two-dimensional case. We use the methods developed by O. A. Ladyzhenskaya for the Navier-Stokes equations.

2. Formulation of the problem

Let us consider the following system of equations

$$\begin{gathered}\frac{\partial v}{\partial t}+v_k\frac{\partial v}{\partial x_k}-\operatorname{div}\mathbf{K}D+\operatorname{grad}p(x,t)=f(x,t)\\ \operatorname{div}v(x,t)=0\end{gathered}\tag{7}$$

The operator $\mathbf{K}$ is positive definite, i.e., the operator $\mathbf{K}$ satisfies the condition (4).

At first, we consider an initial boundary value problem with periodic boundary conditions. Let $\Omega_R=[-R,R]\times[-R,R]$ be a square in $\mathbf{R}^2$. System (7) is considered in the cylinder $Q_T=\Omega_R\times[0,T]$, T is a finite number or $T=\infty$. Suppose that the functions v and p satisfy periodic boundary conditions on the sides of the square. We formulate them for the function v :

$$\begin{gathered}v\big|_{x_1=-R}=v\big|_{x_1=R},\ v\big|_{x_2=-R}=v\big|_{x_2=R},\\ v_{x_1}\big|_{x_1=-R}=v_{x_1}\big|_{x_1=R},\ v_{x_2}\big|_{x_2=-R}=v_{x_2}\big|_{x_2=R}.\end{gathered}\tag{8}$$

Moreover, the function v satisfies the initial condition

$$v\big|_{t=0}=v_0(x),\ x\in\Omega_R.\tag{9}$$

With the help of inequality (4) we obtain the first a-priori estimate which is true for any non-Newtonian (and Newtonian) fluid.

To get a-priori estimates we suppose that all the integrals in the following identities are finite.

Multiply the the first equation of the system (7) by v and integrate over Ω. After integrating by parts we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{2,\Omega}+\int\limits_\Omega((\mathbf{K}D):D)=\int\limits_\Omega fv\tag{10}$$

Integration with respect to t gives

$$\frac{1}{2}(\|v(t)\|^2_{2,\Omega}-\|v(0)\|^2_{2,\Omega})+\int\limits_{Q_T}((\mathbf{K}D):D)=\int\limits_{Q_T}fv\tag{11}$$

By the nonnegativity of the term $\int_{Q_T}((\mathbf{K}D) : D)$ the following estimate holds for the norm of v:

$$\|v(t)\|^2_{2,\Omega} \le \|v(0)\|^2_{2,\Omega} + 2\int\limits_0^t \|f\|_{2,\Omega}\|v\|_{2,\Omega},$$

$$\|v(t)\|^2_{2,\Omega} \le \|v(0)\|^2_{2,\Omega} + \sup_{t\in[0,T]} \|v\|_{2,\Omega} \int\limits_0^t \|f\|_{2,\Omega}. \tag{12}$$

After maximization the left-hand side, we obtain the estimate

$$\sup_{t\in[0,T]} \|v\|_{2,\Omega} = \int\limits_0^t \|f\|_{2,\Omega} + \sqrt{(\int\limits_0^t \|f\|_{2,\Omega})^2 + \|v(0)\|^2_{2,\Omega}} \tag{13}$$

We prove existence of at least one weak solution from the space $\mathbf{L}_\infty(0, T; \hat{\mathbf{J}}^1_2(\Omega))$, which satisfies the integral identity

$$\int\limits_{Q_T} (-v\phi_t - v_k v \phi_{x_k} + (\mathbf{K}D)\nabla\phi) dx dt$$
$$+ \int\limits_{\Omega_R} v\phi\big|_{t=T} dx - \int\limits_{\Omega_R} v_0\phi\big|_{t=0} dx = \int\limits_{Q_T} f\phi \tag{14}$$

for any solenoidal periodic ϕ such that $\phi_x, \phi_{xt} \in \mathbf{L}_2(Q_T)$. All the integrals in (14) are finite if the operator $\mathbf{K}$ is continuous in the space $L_2(Q_T)$.

We project system (7) onto the space of divergent-free vector fields and annihilate the term $\operatorname{grad} p$, find a weak solution of obtained boundary value problem.

3. A priori estimates of the solutions

Now we assume, that the operator $\mathbf{K}$ possesses the following property:

$$-(\operatorname{rot}\operatorname{div}(\mathbf{K}D), \operatorname{rot} v)_{2,Q_T} \ge 0 \tag{15}$$

The operation rot is understood as the scalar function in the space $\mathbf{R}^2$

$$\operatorname{rot} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

This condition is fulfilled for any linear viscoelastic fluid.

We estimate the space derivative of the function v.

Consider the stream function $v = (-\psi_{x_2}, \psi_{x_1})$. To obtain the following relation we apply the operator rot to both parts of the first equation of system (7). Then it holds:

$$\Delta\psi_t - \operatorname{rot}\operatorname{div}(\mathbf{K}D) = \psi_{x_1}\Delta\psi_{x_2} - \psi_{x_2}\Delta\psi_{x_1} + \varphi, \tag{16}$$

where $\varphi(x,t) = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}$. Moreover, we suppose that the function ψ satisfies periodic boundary conditions. Let all the first- and the second-order space derivatives of the function ψ be periodic functions in Ω_R.

The initial condition takes the form

$$\psi\big|_{t=0} = b(x), \tag{17}$$

where $b_{x_1} = -v_{02}, b_{x_2} = v_{01}$.($v_i$ is the i-th coordinate of the function v.)

If we multiply equation (16) by $\Delta\psi$ in the space $\mathbf{L}_2(\Omega_R)$ and integrate by parts, we arrive at the identity

$$\frac{1}{2}\frac{d}{dt}\|\Delta\psi_t\|^2_{2,\Omega} - (\operatorname{rot}\operatorname{div}(\mathbf{K}D), \operatorname{rot} v)_{2,\Omega} = (\varphi, \Delta\psi)_{2,\Omega} \tag{18}$$

This relation is similar to relation (10). Thus we find an estimate similar to estimate (13) for the function $\Delta\psi$, namely

$$\sup_{t\in[0,T]} \|\Delta\psi\|_2 \le \|\varphi\|_{2,1} + \sqrt{\|\varphi\|^2_{2,1} + \|\Delta\psi(0)\|^2_{2,\Omega}}. \tag{19}$$

Since $\|v_x\|_2 \le \|\Delta\psi\|_2$ (19) implies a further estimate

$$\sup_{t\in[0,T]} \|v_x\|_2 \le c(\|\varphi\|_{2,1} + \sqrt{\|\varphi\|^2_{2,1} + \|\Delta\psi(0)\|^2_2}). \tag{20}$$

4. Existence of the solution

To prove the existence of a weak solution we apply Galerkin's method with the system of eigenfunctions of the Stokes problem

$$\begin{gathered} -\Delta v = \lambda v, \operatorname{div} v = 0, \\ v \in \hat{\mathbf{J}}^1_2(\Omega) \cap \mathbf{W}^2_2(\Omega) = \hat{\mathbf{J}}^2_2(\Omega) \end{gathered} \tag{21}$$

as basis functions. By [1], the functions $\{\varphi^l\}_{l=0}^\infty$ form an orthonormal system in the space $\hat{\mathbf{J}}^0_2(\Omega)$, and so it is the case in the spaces $\hat{\mathbf{J}}^1_2(\Omega)$ and $\hat{\mathbf{J}}^2_2(\Omega)$.

The Galerkin approximations are defined as

$$v^n(x,t) = \sum_{l=0}^{n} c_{ln}(t)\varphi^l(x).$$

where the coefficients $c_{ln}(t)$ are determined from the system

$$\frac{d}{dt}(v^n, \varphi^l) + (\mathbf{K}v^n_x, \varphi^l_x) + (v^n_k v^n_{x_k}, \varphi^l) = (f, \varphi^l) \tag{22}$$

with initial conditions

$$c_{ln}(0) = (v_0, \varphi^l), l = 0, \dots, n.$$

The approximations v^n satisfy the same identities

$$\frac{1}{2}\frac{d}{dt}\|v^n\|^2_{2,\Omega} + \int_\Omega ((\mathbf{K}D(v^n)) : D(v^n)) = \int_\Omega f v^n$$

as the exact solutions. Thus, estimates (13), (20) are true for them. And a majorant is independent of n. This permits us to pass to the limit and prove the next theorem:

Theorem 4.1. *Let the functions be given as follows:*

$$f \in \mathbf{L}_{2,1}(Q_T),\ f_x \in \mathbf{L}_{2,1}(Q_T),\ v_0 \in \hat{\mathbf{J}}^1_2(\Omega_R)$$

Then, the initial boundary value problem (7), (8), (9) *admits a generalized solution*

$$v \in \mathbf{L}_\infty(0, T; \hat{\mathbf{J}}^1_2(\Omega_R)), \tag{23}$$

which is global with respect to t. The solution v satisfies estimates (13), (20).

The corollary of this theorem consists of the existence of a solution (23) for the systems of equations of general linear viscoelastic liquids with rheological equation of the type (1). All the conditions (4), (6) are fulfilled for these equations.

5. The Cauchy problem

From theorem 4.1 it follows that the Cauchy problem is also solvable.

Theorem 5.1. *Let*

$$f \in \mathbf{L}_{2,1}(\mathbf{R}^2 \times [0,T]),\ f_x \in \mathbf{L}_{2,1}(\mathbf{R}^2 \times [0,T]),\ v_0 \in \mathbf{J}_2^1(\mathbf{R}^2),$$

where $\mathbf{J}_2^1(\mathbf{R}^2)$ *is a space of solenoidal vector fields with finite norm* $\|u\|_2+\|u_x\|_2$. *Then there exists a generalized solution of system* (7) *which satisfies the initial Cauchy condition*

$$v(x,0) = v_0(x),\ x \in \mathbf{R}^2.$$

This solution belongs to $\mathbf{L}_\infty(0,T;\mathbf{J}_2^1(\mathbf{R}^2))$ *and satisfies estimates* (13), (20).

This theorem is also true for general linear viscoelastic liquids.

Proof. The theorem can be proved in the following way. We choose the sequence of extending domains $\Omega_{R_n}, R_n \to \infty$, and construct the solution v_n of periodic boundary value problem in every domain. The norms $\|v^n\|_{2,\infty}, \|v_x^n\|_{2,\infty}$ of all solutions v^n are bounded by a majorant, which is independent of the domain. If we use compactly supported test functions then we can pass to the limit in the identities (14) for functions v^n.

6. Equations of motion of the linear viscoelastic Maxwell fluids

The motion of the linear viscoelastic Maxwell fluid is determined by the rheological equation

$$\sigma + \sum_{s=1}^{L} \lambda_l \frac{\partial^l}{\partial t^l}\sigma = 2\nu(1 + \sum_{m=1}^{M} \kappa_m \nu^{-1} \frac{\partial^m}{\partial t^m})D, \tag{24}$$

together with $M = L-1, L = 1,2,\ldots$. The coefficients λ_l and κ_m are nonnegative and are called relaxation and retardation times correspondingly. We can represent the deviator of the stress tensor σ in the form [5]

$$\sigma = \sum_{s=1}^{L} \beta_s^{(m)} \int_0^t e^{-\alpha_s(t-\tau)} D(x,\tau)d\tau$$

where

$$\beta_s^{(m)} = P_m(\alpha_s)[Q'(\alpha_s)]^{-1},$$

and the polynomial P_m is given by

$$P_m(p) = \nu + \sum_{l=1}^{L-1} \kappa_l p^l.$$

The flow of the Maxwell fluid is described by the system (7) with the operator $\mathbf{K}_m$

$$\mathbf{K}_m = \sum_{s=1}^{L} \beta_s^{(m)} e^{-\alpha_s t}$$

The operator $\mathbf{K}_m$ is continuous in th space $L_2(Q_T)$ and possesses properties (4), (6) and therefore the existence theorem is also valid for the Maxwell fluids.

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9 января 2003 г.