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REMARKS ON REGULARITY OF WEAK SOLUTIONS TO THE NAVIER–-STOKES EQUATIONS NEAR THE BOUNDARY

Abstract. We give a simple proof of the so-called ε-regularity of suitable weak solutions to the Navier–Stokes equations near the boundary.

Dedicated to Olga Alexandrovna Ladyzhenskaya

1. Introduction

The partial regularity of suitable weak solutions to the Navier–Stokes near the boundary was studied in the paper [6], where it was proved an analogue of the Caffarelli-Kohn-Nirenberg theorem, see [1], for points belonging to the lateral surface of the space-time cylinder. In this note, we give a new simple proof of the so-called ε-regularity which is one of the main steps to achieve the final statement of the theory of partial regularity for the Navier–Stokes system.

So, let us consider the three-dimensional Navier–Stokes equations

$$\partial_t v + \operatorname{div}(v \otimes v) - \Delta v = f - \nabla p, \quad \operatorname{div} v = 0$$

in a domain $Q_T \equiv \Omega \times]0, T[$ of the space $\mathbb{R}^4$. We assume, in addition, that

$$v = 0 \qquad \text{on} \qquad \Gamma \times [0, T], \qquad \Gamma \subset \partial\Omega.$$

Our aim is to investigate differentiability properties of v in a neighborhood of a point $z_0 = (x_0, t_0) \in \Gamma \times]0, T]$ under minimal assumptions on starting properties of v and p. Now, it is well understood that these minimal assumptions can be expressed in the form that the pair v and p must be a suitable weak solution to the Navier–Stokes equations. For details, we refer the reader to the paper [6].

Definition 1.1. *Let Ω be a domain in $\mathbb{R}^3$, Γ be an open subset of the set $\partial\Omega$, and T be a positive parameter. Suppose that a function f satisfies the condition*

$$f \in L_2(Q_T; \mathbb{R}^3). \tag{1.1}$$

We say that a pair of functions v and p is a suitable weak solution to the Navier–Stokes equations in Q_T near the boundary $\Gamma\times]0,T]$ if the following four conditions hold. The functions v and p have the properties

$$\left.\begin{aligned} &v \in L_{2,\infty}(Q_T) \cap W_2^{1,0}(Q_T), \quad p \in L_{\frac{3}{2}}(Q_T), \\ &\nabla^2 v \in L_{\frac{9}{8},\frac{3}{2}}(Q_T), \quad \nabla p \in L_{\frac{9}{8},\frac{3}{2}}(Q_T). \end{aligned}\right\} \tag{1.2}$$

They meet the Navier–Stokes equations

$$\left.\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) - \Delta v = f - \nabla p, \\ \operatorname{div} v = 0 \end{aligned}\right\} \tag{1.3}$$

a.e. in Q_T. The function v satisfies the boundary condition

$$v = 0 \qquad on \qquad \Gamma\times]0,T]. \tag{1.4}$$

The functions v and p satisfy the local energy inequality

$$\left.\begin{aligned} &\int\limits_{\Omega} |v(x,t)|^2 \phi(x,t)\,dx + 2 \int\limits_{Q_t \equiv \Omega\times]0,t[} |\nabla v|^2 \phi\,dx\,dt' \le \\ &\le \int\limits_{Q_t} \left\{ |v|^2(\partial_t \phi + \Delta \phi) + (|v|^2 + 2p)\, v \cdot \nabla \phi + 2f \cdot v\phi \right\} dx\,dt' \end{aligned}\right\} \tag{1.5}$$

for a. a. $t \in]0,T]$ and for all non-negative functions $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$, vanishing in a neighborhood of the set $\Omega \times \{t=0\} \cup (\partial\Omega \setminus \Gamma) \times [0,T]$.

To formulate our results, we introduce the following notion:

$$B(x_0,R) \equiv \{x \in \mathbb{R}^3 \parallel |x - x_0| < R\}, \quad x' = (x_1, x_2) \in \mathbb{R}^2,$$
$$B^+(x_0,R) \equiv \{x \in B(x_0,R) \parallel x = (x', x_3), \quad x_3 > x_{03}\},$$
$$B(\theta) \equiv B(0,\theta), \quad B \equiv B(1), \quad B^+(\theta) \equiv B^+(0,\theta), \quad B^+ \equiv B^+(1);$$
$$\Gamma^+(x_0,R) \equiv \{x \in B(x_0,R) \parallel x = (x', x_{03})\},$$
$$\Gamma^+(\theta) \equiv \Gamma^+(0,\theta), \quad \Gamma^+ \equiv \Gamma^+(1);$$
$$Q(z_0,R) \equiv B(x_0,R)\times]t_0 - R^2, t_0[, \quad z_0 = (x_0,t_0),$$
$$Q^+(z_0,R) \equiv B^+(x_0,R)\times]t_0 - R^2, t_0[,$$
$$Q(\theta) \equiv Q(0,\theta), \quad Q \equiv Q(1), \quad Q^+(\theta) \equiv Q^+(0,\theta), \quad Q^+ \equiv Q^+(1);$$
$$(v)_\omega \equiv \frac{1}{|\omega|} \int\limits_{\omega} v\,dz, \qquad [p]_\Omega(t) \equiv \frac{1}{|\Omega|} \int\limits_{\Omega} p(x,t)\,dx.$$

Just for simplicity we assume that

$$f = 0.$$

Theorem 1.2. *Given $\tau \in]0,1[$, there are positive numbers $\varepsilon_0 = \varepsilon_0(\tau)$ and $c_0 = c_0(\tau)$ with the following property. Let a pair v and p be an arbitrary weak solution to the Navier–Stokes equations in $Q^+(z_0, R)$ near $\Gamma^+(z_0, R)\times]t_0 - R^2, t_0]$. Suppose that v and p satisfy the additional condition*

$$U(z_0, R; v, p) \equiv \frac{1}{R^2} \int\limits_{Q^+(z_0,R)} \left(|v|^3 + |p|^{\frac{3}{2}}\right) dz < \varepsilon_0. \tag{1.6}$$

Then, the function v is Hölder continuous in the closure of the set $Q^+(z_0, \tau R)$ and the estimate

$$\sup_{z \in Q^+(z_0,\tau R)} |v(z)| \le \frac{c_0}{R} \tag{1.7}$$

is valid.

2. Main Lemma

In this section, we are going to prove the following key statement.

Lemma 2.1. *Given $\theta \in]0, 1/2[$, there exists a positive constant ε, depending on θ only, with the following property. For any suitable weak solution v and p to the Navier–Stokes equations in Q^+ near $\Gamma^+ \times]-1, 0]$, satisfying the additional condition*

$$Y_1^+(v, p) < \varepsilon, \tag{2.1}$$

the following estimate is valid:

$$Y_\theta^+(v, p) \le c\theta^{\frac{1}{2}} Y_1^+(v, p). \tag{2.2}$$

Here, c is an absolute positive constant.

Proof. Assume that the statement of the lemma is false. This means that a number $\theta \in]0, 1/2[$ and a sequence v^k and p^k of a suitable weak solutions in Q^+ near $\Gamma^+ \times]-1, 0]$ exist such that

$$Y_1(v^k, p^k) = \varepsilon_k \to 0 \tag{2.3}$$

as $k \to +\infty$, and

$$Y_\theta(v^k, p^k) > c\theta^{\frac{1}{2}} \varepsilon_k. \tag{2.4}$$

The constant c will be chosen later in order to get a contradiction. We introduce new functions

$$u^k = v^k/\varepsilon_k, \qquad q^k = (p^k - [p^k]_{Q^+(\theta)})/\varepsilon_k.$$

By (2.3) and (2.4), they satisfy the relations

$$Y_1(u^k, q^k) = 1, \tag{2.5}$$

$$Y_\theta(u^k, q^k) > c\theta^{\frac{1}{2}}, \tag{2.6}$$

the system, see (1.3),

$$\left.\begin{aligned} \partial_t u^k + \varepsilon_k \operatorname{div}(u^k \otimes u^k) - \Delta u^k = -\nabla q^k, \\ \operatorname{div} u^k = 0 \end{aligned}\right\} \qquad \text{in } Q^+, \tag{2.7}$$

and the boundary condition, see (1.4),

$$u^k = 0 \qquad \text{on} \qquad \Gamma^+ \times]-1, 0]. \tag{2.8}$$

In addition, functions u^k and q^k have the following differentiability properties, see (1.2),

$$\left.\begin{aligned} &u^k \in L_{2,\infty}(Q^+) \cap W_2^{1,0}(Q^+), \quad q^k \in L_{\frac{3}{2}}(Q^+), \\ &\nabla^2 u^k \in L_{\frac{9}{8},\frac{3}{2}}(Q^+), \quad \nabla q^k \in L_{\frac{9}{8},\frac{3}{2}}(Q^+), \end{aligned}\right\} \tag{2.9}$$

and, by (1.5), the local energy inequality

$$\left.\begin{aligned} &\int\limits_{B^+} |u^k(x,t)|^2 \phi(x,t)\, dx + 2 \int\limits_{B^+ \times]-1,t[} |\nabla u^k|^2 \phi \, dx\, dt' \le \\ &\le \int\limits_{B^+ \times]-1,t[} \left\{ |u^k|^2 (\partial_t \phi + \Delta \phi) + (|u^k|^2 \varepsilon_k + 2q^k)\, u^k \cdot \nabla \phi \right\} dx\, dt' \end{aligned}\right\} \tag{2.10}$$

holds for a. a. $t \in]-1, 0]$ and for all non-negative functions $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$, vanishing in a neighborhood of the set $B^+ \times \{t = -1\} \cup (\partial B^+ \setminus \Gamma^+) \times [-1, 0]$.

Choosing a cut-off function ϕ in an appropriate way, we easily deduce from (2.10) the estimate

$$\|u^k\|_{2,\infty,Q^+(\tau)} + \|\nabla u^k\|_{2,Q^+(\tau)} \le c_1(\tau),$$

which is valid for any $\tau \in]0, 1[$. But then Hölder inequality and the known multiplicative inequality together with (2.11) imply two bounds

$$\|u^k\|_{\frac{10}{3},Q^+(\tau)} \le c_2(\tau)$$

and

$$\|\mathrm{div}\ u^k\otimes u^k\|_{\frac{9}{8},\frac{3}{2},Q^+(\tau)}\le c_2(\tau).$$

Both holds for any $\tau\in]0,1[$.

Now, we are in a position to apply local estimates of solutions to the Stokes equations near the boundary, see [5], Proposition1. As a result, we have

$$\begin{aligned}&\|\partial_t u^k\|_{\frac{9}{8},\frac{3}{2},Q(\tau_1)}+\|\nabla^2 u^k\|_{\frac{9}{8},\frac{3}{2},Q(\tau_1)}+\|\nabla\, q^k\|_{\frac{9}{8},\frac{3}{2},Q(\tau_1)}\\ &\quad\le c_3(\tau_1,\tau)\Big(\|u^k\|_{\frac{9}{8},\frac{3}{2},Q(\tau)}+\|\nabla\, u^k\|_{\frac{9}{8},\frac{3}{2},Q(\tau)}\\ &+\|q^k\|_{\frac{9}{8},\frac{3}{2},Q(\tau)}+\|\mathrm{div}\ u^k\otimes u^k\|_{\frac{9}{8},\frac{3}{2},Q(\tau)}\Big)\le c_3'(\tau_1,\tau).\end{aligned}\tag{2.14}$$

The latter estimate is valid for any $0<\tau_1<\tau<1$.

Without loss of generality, we claim that, by standard compactness arguments,

$$\begin{cases} u^k\rightharpoonup u & \text{in}\quad L_3(Q)\\ q^k\rightharpoonup q & \text{in}\quad L_{\frac{3}{2}}(Q)\\ u^k\to u & \text{in}\quad \mathbb{L}_3(Q^+(5/6)).\end{cases}\tag{2.15}$$

Moreover, for these limit functions, the following relations are valid:

$$Y_1(u,q)\le 1,\qquad [p]_{B^+}=0\qquad \text{for } t\in]-1,0[,\tag{2.16}$$

$$\left.\begin{aligned}\|u\|_{2,\infty,Q(\tau)}+\|\nabla\, u\|_{2,Q(\tau)}&\le c_1(\tau)\\ \|u\|_{\frac{10}{3},q(\tau)}&\le c_2(\tau)\\ \|\mathrm{div}\ u\otimes u\|_{\frac{9}{8},\frac{3}{2},Q(\tau)}&\le c_2(\tau)\\ \|\partial_u\|_{\frac{9}{8},\frac{3}{2},Q(\tau_1)}+\|\nabla^2 u\|_{\frac{9}{8},\frac{3}{2},Q(\tau_1)}+\|\nabla\, q\|_{\frac{9}{8},\frac{3}{2},Q(\tau_1)}&\le c_3'(\tau_1,\tau).\end{aligned}\right\}\tag{2.17}$$

Here, τ_1 and τ are arbitrary numbers satisfying $0<\tau_1<\tau<1$. It remains to note that the pair u and q solves the following boundary value problem:

$$\left.\begin{aligned}\partial_t u-\Delta\, u&=-\nabla\, q\\ \mathrm{div}\ u&=0\end{aligned}\right\}\qquad \text{in } Q^+\tag{2.18}$$

and

$$u=0\qquad\text{on}\qquad \Gamma^+\times]-1,0].\tag{2.19}$$

From the theory of regularity up to the boundary for the Stokes system, it follows that the function u is Hölder continuous in $\overline{Q}(\tau)$ and

$$\sup_{z\in\overline{Q}(\tau)}|u(z)|\le c_4\tau^{\frac{1}{2}}\tag{2.20}$$

for any $0 < \tau < \tau_0 < 1$. In (2.20), c_4 depends on τ_0 only.

Now, we pass to the limit in (2.6) and find with the help of (2.15) and (2.20) the following bound

$$c\theta^{\frac{1}{2}} \le c_4\theta^{\frac{1}{2}} + \theta \limsup_{k \to +\infty} Y_\theta^{+2}(q^k). \tag{2.21}$$

Let us fix a domain $\widetilde{B}$ with sufficiently smooth boundary so that

$$B^+(4/5) \subset \widetilde{B} \subset B^+(5/6)$$

and consider the following initial boundary value problem

$$\left.\begin{aligned} \partial_t u^{1k} - \Delta\, u^{1k} + \nabla\, q^{1k} &= -\varepsilon_k \operatorname{div} u^k \otimes u^k \\ \operatorname{div} u^{1k} &= 0 \end{aligned}\right\} \quad \text{in } \widetilde{Q} = \widetilde{B} \times]-(5/6)^2, 0[,$$

$$[q^{1k}]_{\widetilde{B}}(s) = 0, \qquad s \in]-(5/6)^2, 0[,$$

$$u^{1k} = 0 \qquad \text{on} \qquad \partial\widetilde{B} \times [-(5/6)^2, 0],$$

$$u^{1k} = 0 \qquad \text{on} \qquad \widetilde{B} \times \{s = -(5/6)^2\}.$$

It is well known that, for the solution to this problem, we have the coercive estimate

$$\|\nabla\, q^{1k}\|_{\frac{9}{8},\frac{3}{2},\widetilde{Q}} + \|q^{1k}\|_{\frac{9}{8},\frac{3}{2},\widetilde{Q}} \le c_5\,\varepsilon_k \|\operatorname{div} u^k \otimes u^k\|_{\frac{9}{8},\frac{3}{2},\widetilde{Q}}.$$

By the imbedding theorem and by (2.17), we find

$$\|q^{1k}\|_{\frac{3}{2},Q(\frac{4}{5})} \le c_5'\varepsilon_k. \tag{2.22}$$

If we introduce functions

$$u^{2k} = u^k - u^{1k}, \qquad q^{2k} = q^k - q^{1k},$$

we see that they satisfy the relations

$$\begin{aligned} \|\nabla^2 u^{2k}\|_{\frac{9}{8},\frac{3}{2},Q(\frac{4}{5})} + \|\nabla\, u^{2k}\|_{\frac{9}{8},\frac{3}{2},Q(\frac{4}{5})} + \|\nabla\, q^{2k}\|_{\frac{9}{8},\frac{3}{2},Q(\frac{4}{5})} \\ + \|q^{2k}\|_{\frac{9}{8},\frac{3}{2},Q(\frac{4}{5})} \le c_6, \end{aligned} \tag{2.23}$$

$$\left.\begin{aligned} \partial_t u^{2k} - \Delta\, u^{2k} &= -\nabla\, q^{2k} \\ \operatorname{div} u^{2k} &= 0 \end{aligned}\right\} \qquad \text{in } Q^+(4/5), \tag{2.24}$$

$$u^{2k} = 0 \qquad \text{on} \qquad \Gamma^+ \times]-1, 0]. \tag{2.25}$$

By Proposition 2, see [5], for solutions to problem (2.24), (2.25), subjected to inequality (2.23), the following estimate is valid:

$$\|\nabla q^{2k}\|_{18,\frac{3}{2},Q^+(\frac{3}{4})} \le c_6'. \tag{2.26}$$

Next, using (2.22), we can deduce from (2.21)

$$c\theta^{\frac{1}{2}} \le c_4\theta^{\frac{1}{2}} + \theta \limsup_{k\to+\infty} Y_\theta^{+2}(q^{2k}).$$

In turn, Poincaré's inequality implies

$$c\theta^{\frac{1}{2}} \le c_4\theta^{\frac{1}{2}} + c_7\theta \limsup_{k\to+\infty} \Big(\frac{\theta^{\frac{3}{2}}}{\theta^5} \int\limits_{-\theta^2}^{0} ds \int\limits_{B^+(\theta)} |\nabla q^{2k}|^{\frac{3}{2}} dy\Big)^{\frac{2}{3}}.$$

Now, (2.26) together with Hölder inequality with respect to the spatial variables leads to the estimate

$$c\theta^{\frac{1}{2}} \le c_4\theta^{\frac{1}{2}} + c_7'\theta \limsup_{k\to+\infty} \Big(\frac{1}{\theta^{\frac{7}{2}}} \int\limits_{-\theta^2}^{0} ds \Big(\int\limits_{B^+(\theta)} |\nabla q^{2k}|^{18} dy\Big)^{\frac{1}{12}} |B^+(\theta)|^{\frac{11}{12}}\Big)^{\frac{2}{3}}$$

$$\le c_4\theta^{\frac{1}{2}} + c_7''\theta^{\frac{1}{2}} = c_8\theta^{\frac{1}{2}},$$

where c_8 is a known absolute constant. In order to get a contradiction, it is sufficient to let $c = 2c_8$. Lemma 2.1 is proved. •

Lemma 2.1 can be iterated in the following way.

Lemma 2.2. *Let $\theta \in]0, 1/2[$ satisfy the condition*

$$c\theta^{\frac{1}{6}} < 1. \tag{2.28}$$

Assume that

$$Y_1^+(v, p) < \varepsilon. \tag{2.29}$$

Then, for any $k = 1, 2, ...,$

$$Y_{\theta^{k-1}}^+(v, p) < \varepsilon,$$
$$Y_{\theta^k}^+(v, p) \le \theta^{\frac{k}{3}} Y_1^+(v, p). \tag{2.30}$$

Proof. We argue by induction on k. For $k = 1$, this is nothing but the statement of Lemma 2.1.

Assume now that statements (2.30) are valid for $s = 1, 2, ..., k \geq 2$. Our goal is to prove that they are valid for $s = k + 1$ as well.

We make natural scaling

$$v^k(y,s) = \theta^k v(\theta^k y, \theta^{2k} s), \qquad p^k(y,s) = \theta^{2k} v(\theta^k y, \theta^{2k} s)$$

for $(y,s) \in Q^+$. It is easy to check that v^k and p^k form suitable weak solution in Q^+. Since

$$Y_1^+(v^k, p^k) = \theta^k Y_{\theta^k}^+(v,p) < \varepsilon,$$

we can derive from Lemma 2.1 and from (2.28)

$$Y_\theta^+(v^k, p^k) \leq c\theta^{\frac{1}{2}} Y_1^+(v^k, p^k) < \theta^{\frac{1}{3}} Y_1^+(v^k, p^k)$$

and therefore

$$Y_{\theta^{k+1}}^+(v,p) \leq \theta^{\frac{1}{3}} Y_{\theta^k}^+(v,p).$$

The latter immediately implies statements (2.30) for $s = k + 1$. Lemma 2.2 is proved. •

A direct consequence of Lemma 2.2 and scaling

$$v^R(y,s) = Rv(x_0 + Ry, t_0 + R^2 s), \qquad p^R(y,s) = R^2 p(x_0 + Ry, t_0 + R^2 s)$$

is the following statement.

Lemma 2.3. *Let θ and ε be as in Lemma 2.2. Let v and p be an arbitrary suitable weak solution to the Navier–Stokes equations in $Q^+(z_0, R)$ near $\Gamma(x_0, R)\times]t_0 - R^2, t_0]$. Assume that v and p satisfy the additional condition*

$$RY^+(z_0, R; v, p) < \varepsilon. \tag{2.31}$$

Then, for $k = 1, 2, ...$, we have

$$Y^+(z_0, \theta^k R; v, p) \leq \theta^{\frac{k}{3}} Y^+(z_0, R; v, p). \tag{2.32}$$

3. Interior Regularity

In this section, we are going to formulate results on the interior (in the parabolic sense) regularity. The detail proof in the style of Section 2 can be found in the forthcoming paper [2], see [3] for an alternative approach.

To formulate results for the interior case, we need the following notation

$$Y(z_0, R; v, p) = Y^1(z_0, R; v) + Y^2(z_0, R; p),$$

$$Y^1(z_0,R;v)=\Big(\frac{1}{|Q(R)|}\int\limits_{Q(z_0,R)}|v-(v)_{z_0,R}|^3\,dz\Big)^{\frac{1}{3}},$$

$$Y^2(z_0,R;p)=R\Big(\frac{1}{|Q(R)|}\int\limits_{Q(z_0,R)}|p-[p]_{z_0,R}|^{\frac{3}{2}}\,dz\Big)^{\frac{2}{3}},$$

$$(v)_{z_0,R}=(v)_{Q(z_0,R)},\qquad [p]_{x_0,R}=[p]_{B(x_0,R)},$$

$$Y^1_\theta(v)=Y^1(0,\theta;v),\qquad Y^2_\theta(p)=Y^2(0,\theta;p),$$

$$Y_\theta(v,p)=Y(0,\theta;v,p),\qquad (v)_{,\theta}=(v)_{0,\theta},\qquad [p]_{,\theta}=[p]_{0,\theta}.$$

Lemma 3.1. *Given numbers $\theta\in]0,1/2[$ and $M>3$, there exist two positive constants $\varepsilon_1(\theta,M)$ and $c_1(M)$ such that, for any suitable weak solution v and p to the Navier–Stokes equations in Q, satisfying the additional conditions*

$$|(v)_{,\theta}|<M,\qquad Y_1(v,p)<\varepsilon_1, \tag{3.1}$$

the following estimate is valid:

$$Y_\theta(v,p)\le c_1\theta^{\frac{2}{3}}Y_1(v,p). \tag{3.2}$$

Lemma 3.2. *Given numbers $M>3$ and $\beta\in]0,2/3[$, we choose $\theta\in]0,1/2[$ so that*

$$c_1(M)\theta^{\frac{2-3\beta}{6}}<1. \tag{3.3}$$

Let $\overline{\varepsilon}_1(\theta,M)=\min\{\varepsilon_1(\theta,M),\theta^5M/2\}$. If

$$|(v)_{,1}|<M,\qquad Y_1(v,p)<\overline{\varepsilon}_1, \tag{3.4}$$

then, for any $k=1,2,\dots$,

$$\theta^{k-1}|(v)_{,\theta^{k-1}}|<M,\qquad Y_{\theta^{k-1}}(v,p)<\overline{\varepsilon}_1\le\varepsilon_1,$$

$$Y_{\theta^k}(v,p)\le\theta^{\frac{2+3\beta}{6}}Y_{\theta^{k-1}}(v,p). \tag{3.5}$$

Lemma 3.3. *Let M, β, θ, and $\overline{\varepsilon}_1$ be as in Lemma 3.2. Let a pair v and p be an arbitrary suitable weak solution to the Navier–Stokes equations in the parabolic cylinder $Q(z_0,R)$, satisfying the additional conditions*

$$R|(v)_{z_0,R}|<M,\qquad RY(z_0,R;v,p)<\overline{\varepsilon}_1. \tag{3.6}$$

Then, for any $k=1,2,\dots$, we have

$$Y(z_0,\theta^kR;v,p)\le\theta^{\frac{2+3\beta}{6}k}Y(z_0,R;v,p). \tag{3.7}$$

4. Proof of Theorem 1.2

Obviously, by natural scaling, it is sufficient to prove Theorem 1.2 for $z_0 = 0$ and $R = 1$. For simplicity, we consider the case $\tau = 1/2$. The general case differs from particular one just by technical details.

From Lemma 2.3 and 3.3, we can easily deduce the following statements.

Lemma 4.1. *There are two absolute positive constants ε_+ and c_+ such that, for any suitable weak solution v and p to the Navier–Stokes equations in $Q^+(z_0, R)$ near $\Gamma(x_0, R)\times]t_0 - R^2, t_0]$, satisfying the condition*

$$RY^+(z_0, R; v, p) < \varepsilon_+,$$

the following estimate is valid:

$$Y^+(z_0, \rho; v, p) \le c_+ \Big(\frac{\rho}{R}\Big)^{\frac{1}{3}} Y^+(z_0, R; v, p), \qquad \forall \rho \in]0, R].$$

Lemma 4.2. *There are two absolute positive constants $\varepsilon_\times$ and $c_\times$ such that, for any suitable weak solution v and p to the Navier–Stokes equations in $Q(z_0, R)$, satisfying the condition*

$$R\widetilde{Y}(z_0, R; v, p) < \varepsilon_\times,$$

where

$$\widetilde{Y}(z_0, R; v, p) \equiv \Big(\frac{1}{|Q(R)|} \int\limits_{Q(z_0,R)} |v|^3\, dz\Big)^{\frac{1}{3}} + \Big(\frac{1}{|Q(R)|} \int\limits_{Q(z_0,R)} |p - [p]_{x_0,R}|^{\frac{3}{2}}\, dz\Big)^{\frac{2}{3}},$$

the following estimate is valid:

$$Y(z_0, \rho; v, p) \le c_\times \Big(\frac{\rho}{R}\Big)^{\frac{1}{3}} Y(z_0, R; v, p), \qquad \forall \rho \in]0, R].$$

To prove Lemma 4.2, we may take $M = 2002$ and use two simple inequalities:

$$|(v)_{z_0,R}| \le \widetilde{Y}(z_0, R; v, p), \qquad Y(z_0, R; v, p) \le 2\widetilde{Y}(z_0, R; v, p).$$

In turn, Lemma 4.1 and 4.2 imply the following result.

Proposition 4.3. *There are two absolute positive constants $\overline{\varepsilon}$ and $\overline{c}$ such that, for any suitable weak v and p to the Navier–Stokes equations in Q^+ near $\Gamma^+\times]-1,0]$, satisfying the additional condition*

$$RV^+(z_0,R;v,p)<\overline{\varepsilon},\qquad z_0\in\overline{Q^+}(2/3),\quad 0<R\le 1/4,$$

we have the estimate

$$V(z_0,\rho;v,p)\le\overline{c}\Big(\frac{\rho}{R}\Big)^{\frac{1}{3}}V^+(z_0,R:v,p),$$

which is valid for all $0<\rho\le R$.

To formulate Proposition 4.3, we used the following notation. For $z_0\in\overline{Q^+}$ and $0<R\le 1$, satisfying in addition

$$B(x_0,R)\subset B,\qquad t_0>R^2-1,$$

we let

$$V^+(z_0,R;v,p)=\Big(\frac{1}{|\omega(z_0,R)|}\int\limits_{\omega(z_0,R)}|v|^3\,dz\Big)^{\frac{1}{3}}$$

$$+\Big(\frac{1}{|\omega(z_0,R)|}\int\limits_{\omega(z_0,R)}|p-[p]_{\Omega(x_0,R)}|^{\frac{3}{2}}\,dz\Big)^{\frac{2}{3}},$$

$$V(z_0,R;v,p)=\Big(\frac{1}{|\omega(z_0,R)|}\int\limits_{\omega(z_0,R)}|v-(v)_{\omega(z_0,R)}|^3\,dz\Big)^{\frac{1}{3}}$$

$$+\Big(\frac{1}{|\omega(z_0,R)|}\int\limits_{\omega(z_0,R)}|p-[p]_{\Omega(x_0,R)}|^{\frac{3}{2}}\,dz\Big)^{\frac{2}{3}},$$

where $\Omega(x_0,R)=B(x_0,R)\cap B^+$ and $\omega(x_0,R)=\Omega(x_0,R)\times]t_0-R^2,t_0[$.

The proof of Proposition 4.3 is quite standard, see [6], Section 5, or [7], Section 6, and we omit it.

Proof of Theorem 1.2. First we let

$$A=U(0,1;v,p).$$

Obviously,

$$\frac{1}{4}V^+(z_0,\frac{1}{4};v,p)\le C(A^{\frac{1}{3}}+A^{\frac{2}{3}})\le C(\varepsilon_0^{\frac{1}{3}}+\varepsilon_0^{\frac{2}{3}})<\overline{\varepsilon}$$

for sufficiently small ε_0. Here, C is an absolute positive constant. Then,

$$V(z_0, \rho; v, p) \le \overline{c}(4\rho)^{\frac{1}{3}} V^+(z_0, \frac{1}{4}; v, p) \le \overline{c}_1 \rho^{\frac{1}{3}} \overline{\varepsilon}$$

for all $z_0 \in \overline{Q^+}$ and for all $0 < \rho \le \frac{1}{4}$. By Campanato's condition, v is Hölder continuous at least in $\overline{Q^+}(1/2)$. Theorem 4.3 is proved. •

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