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EXACT SMALL BALL CONSTANTS FOR SOME GAUSSIAN PROCESSES UNDER L^2–NORM

Abstract. We find some logarithmic and exact small deviation asymptotics for the L^2-norm of certain Gaussian processes closely connected with the Wiener process. In particular the processes obtained by centering and integrating Brownian motion and Brownian bridge are examined

1. Introduction

The problem of small deviations for norms of Gaussian processes has obtained much attention in recent years (see, for example, the reviews [20] and [18].)

Suppose $X(t), 0 \le t \le 1$, is a Gaussian process with mean zero and covariance function $\sigma(s,t) = EX(t)X(s)$ for $s,t \in [0,1]$. Let

$$||X||_2 = \left(\int_0^1 X^2(t)dt \right)^{1/2}$$

and

$$Q(X;\varepsilon) = P(||X||_2 \le \varepsilon).$$

An interesting problem is to define the behavior of $Q(X;\varepsilon)$ as $\varepsilon \longrightarrow 0$. As an example, consider the Wiener process $W(t), 0 \le t \le 1$, and the Brownian bridge $B(t), 0 \le t \le 1$. The following small deviation asymptotics, as $\varepsilon \longrightarrow 0$, were obtained a long time ago

$$P(||W||_2 \le \varepsilon) \sim 4\pi^{-1/2}\varepsilon \exp(-(1/8)\varepsilon^{-2}), \tag{1}$$

$$P(||B||_2 \le \varepsilon) \sim 2\sqrt{2}\pi^{-1/2} \exp(-(1/8)\varepsilon^{-2}) \tag{2}$$

and follow from known exact distributions of L^2-norm for W and B. However, for other Gaussian processes, such formulas are seldomly available.

Theoretically the problem of small deviation asymptotics was solved by Sytaya [24], but in an implicit way. Therefore, the efforts of many scientists starting from [4], [12] and [27] were aimed at the simplification of the expression for $Q(X;\varepsilon)$, see the references in [20] and [5].

By the Kac–Siegert formula it is well-known that

$$\int_0^1 X^2(t)dt = \sum_{n=1}^{\infty} \lambda_n \xi_n^2, \tag{3}$$

where ξ_n, $n \geq 1$, are independent standard normal r.v.'s and $\lambda_n > 0, n \geq 1$, are the eigenvalues of the integral equation

$$\lambda f(t) = \int_0^1 \sigma(s,t) f(s) ds, \qquad 0 \leq t \leq 1. \tag{4}$$

Thus, equivalently, we are led to the problem of studying the asymptotic behavior of $P(\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq \varepsilon^2)$, as $\varepsilon \to 0$. This problem is considered as solved when the eigenvalues λ_n can be found explicitly. However, they are known only for a limited number of examples, see [17], [5] and [20].

The aim of the present work is to calculate the exact small deviation asymptotics for some Gaussian processes, which are of interest in Statistics, but have not been considered in previous papers. The results were obtained in October of 2001 when the second author was visiting Rome University "La Sapienza" and were published in the preprint [1]. They were also presented at the Seminar of Prof. I. A. Ibragimov in POMI in Spring of 2002.

Some general exact small deviation results were obtained later and appeared in [6]–[8] and [21]–[22].

2. Gaussian processes under consideration

In some statistical problems it is reasonable to consider centered empirical processes and the corresponding limiting Gaussian processes. In particular, we are interested in the centered (by its mass) Wiener process

$$W^c(t) = W(t) - \int_0^1 W(u) du$$

and the centered Brownian bridge

$$B^c(t) = B(t) - \int_0^1 B(u) du.$$

The idea of such type of centering is very old and dates back at least to Watson [25], who used it for testing nonparametric hypotheses on a circle.

Another operation, which has received great attention in recent years, is the integration of Gaussian processes, see, e.g., [2], [15],[2], and [16]. Let

$$\overline{W}(t) = \int_0^t W(u)du, \quad \overline{B}(t) = \int_0^t B(u)du, \quad 0 \le t \le 1,$$

be the integrated Wiener process and the integrated Brownian bridge, respectively. Besides purely probabilistic studies of these processes, we note that $\overline{B}(t)$ appeared in [10] in the context of goodness-of-fit testing. Analogously, $\overline{W}(t)$ may be used for samples of Poisson size when the empirical process is replaced by the Kac process [13] and the limiting process is the W.

We can also combine the operations of centering and integration. In this context, Henze and Nikitin [11] (see also [16]) considered two different processes: the centered integrated Brownian bridge

$$B^0(t) = \overline{B}(t) - \int_0^1 \overline{B}(u)du$$

and the integrated centered Brownian bridge

$$B^*(t) = \int_0^t (B(s) - \int_0^1 B(u)du)ds = \overline{B}(t) - t\overline{B}(1).$$

The latter process can be viewed as the bridge of the integrated Brownian bridge $\overline{B}(t)$. These studies were also motivated by the construction of new Watson-type goodness-of-fit tests, see [11].

Quite analogously, we can consider the centered integrated Wiener process

$$W^0(t) = \overline{W}(t) - \int_0^1 \overline{W}(u)du$$

and the integrated centered Wiener process

$$W^*(t) = \int_0^t (W(s) - \int_0^1 W(u)du)ds = \overline{W}(t) - t\overline{W}(1)$$

which, apparently, have not been previously considered. Note that the operations of centering and integrating do not commute, so that B^0 differs from B^*, as well as W^0 differs from W^*.

Chen and Li [2] considered the m-fold integrated Wiener process

$$W_m(t) = \frac{1}{m!}\int_0^t (t-s)^m dW(s), \quad m \geq 0.$$

They showed that, as $\varepsilon \to 0$, one has

$$\lim_{\varepsilon \to 0} \varepsilon^{2/2m+1} \ln P(||W_m||_2 \leq \varepsilon) = -D_m,$$

where

$$D_m = \frac{1}{2}(2m+1)\Big((2m+2)\sin\frac{\pi}{2m+2}\Big)^{-\frac{2m+2}{2m+1}}.$$

For $m = 0$ (Wiener process) the right hand side is equal to $-(1/8)$ according to (1), for $m = 1$ (process $\overline{W}(t)$) it is equal to $-(3/8)$. The latter result was obtained for the first time in [15].

3. Centered Wiener process and centered Brownian bridge

The process $B^c(t) = B(t) - \int_0^1 B(u)du$ was introduced by Watson [25]. He found that the covariance function of this process is

$$\sigma_{B^c}(s,t) = s \wedge t - st + \frac{1}{2}(s^2 + t^2 - s - t) + \frac{1}{12}, \; 0 \leq s, t \leq 1,$$

and that the spectrum of the corresponding integral operator consists of the eigenvalues $\lambda_n = (2n\pi)^{-2}, n \geq 1$, having multiplicity 2. This makes the calculation of small deviations more difficult because in (3) the r.v.'s ξ_n^2 should be replaced by r.v.'s χ_2^2 . Fortunately, it was proved long ago (see, e.g., [23], p.148) that there exists the following equality in distribution

$$||B^c||_2 = \pi^{-1} \sup_{0 \leq t \leq 1} |B(t)|.$$

The random variable $\sup_{0 \leq t \leq 1} |B(t)|$ has the well-known Kolmogorov distribution function, for which there exist explicit formulas suitable for small and large values of the argument (see, e.g., [19], §18). Hence, we get, as $\varepsilon \to 0$,

$$P(||B^c||_2 \leq \varepsilon) = P(\sup_{0 \leq t \leq 1} |B(t)| \leq \varepsilon\pi) \sim \sqrt{\frac{2}{\pi}}\varepsilon^{-1} \exp\Big(-\frac{1}{8}\varepsilon^{-2}\Big).$$

To study the small deviations of the process W^c note that

$$\sigma_{W^c}(s,t) = s \wedge t + \frac{1}{2}(s^2+t^2) - s - t + \frac{1}{3}.$$

Solving the integral equation (4) with this kernel by differentiation, we easily get the boundary-value problem

$$\lambda f''(t) = -(f(t) - \int_0^1 f(u)du),$$
$$f'(0) = f'(1) = 0.$$

The set of solutions consists of the eigenvalues $\lambda_n = (n^2\pi^2)^{-1}, n \geq 1$, with eigenfunctions $f_n(t) = C\cos n\pi t, n \geq 1$. Thus the spectrum coincides with that of the Brownian bridge B. Hence we get another proof of the well-known equality in distribution, see [3]:

$$||B||_2 = ||W^c||_2.$$

It follows from (2) that, as $\varepsilon \to 0$,

$$P(||W^c||_2 \leq \varepsilon) \sim 2\sqrt{2}\pi^{-1/2}\exp(-(1/8)\varepsilon^{-2}).$$

We see that the centering by its integral does not change the exponential term for small deviation probabilities of the Wiener process and Brownian bridge, but it changes the factor preceding the exponential term.

4. Integrated Brownian bridge: exact small deviations

The covariance function of the integrated Brownian bridge $\overline{B}$ reads

$$\sigma_{\overline{B}}(s,t) = \frac{1}{2}st(s\wedge t) - \frac{1}{6}(s\wedge t)^3 - \frac{1}{4}s^2t^2, \quad 0 \leq s,t \leq 1.$$

The spectrum of the corresponding integral equation was found in [10]. The derivation of this spectrum is based on the transcendental equation

$$\tan x + \tanh x = 0, \tag{5}$$

whose solutions are denoted by $k_1 < k_2 < \dots$, and then $\lambda_n = (k_n)^{-4}, n \geq 1$.

It was noticed in [10] that, since $\tanh x \sim 1$ for large x , the solutions k_j of (5), arranged in ascending order of magnitude, satisfy the approximation $k_j \sim (j-\frac{1}{4})\pi$, for large j. Therefore, for large values of j, we have

$$\lambda_j \sim \sigma_j := \left(j-\frac{1}{4}\right)^{-4}\pi^{-4}. \tag{6}$$

It follows from Theorem 2 of [17] or from Theorem 6.2 of [20] that, as $\varepsilon \to 0$,

$$P(||\overline{B}||_2 \leq \varepsilon) = P(\sum_{j=1}^{\infty} \lambda_n \xi_n^2 \leq \varepsilon^2) \sim \Pi_{n=1}^{\infty}(\sigma_n/\lambda_n)^{1/2} P(\sum_{j=1}^{\infty} \sigma_n \xi_n^2 \leq \varepsilon^2), \tag{7}$$

provided that the condition $\sum_{n=1}^{\infty} |1-\lambda_n/\sigma_n| < +\infty$ is fulfilled. Hence, we need only to verify that

$$\sum_{j=1}^{\infty} |1-(j-\frac{1}{4})^4\pi^4 k_j^{-4}| < +\infty \quad . \tag{8}$$

Clearly $k_j \to \infty$ as $j \to \infty$ and, for large j, we have $k_j = (j-1/4)\pi+\delta_j$, with $\delta_j \to 0$, as $j \to \infty$. Plugging this into (5) we get

$$\frac{\tan\delta_j - 1}{\tan\delta_j + 1} + \frac{1-\exp(\pi/2-2j\pi-2\delta_j)}{1+\exp(\pi/2-2j\pi-2\delta_j)} = 0$$

or the equivalent equation

$$\tan\delta_j = \exp(\pi/2-2j\pi-2\delta_j),$$

from which it is evident that $\delta_j = O(\exp(-2j\pi))$, $j \to \infty$. It follows that

$$k_j = -\pi/4 + j\pi + O(\exp(-2j\pi)), \quad j \to \infty, \tag{9}$$

and therefore, the series (8) is convergent. Hence (7) is proved.

Next step is to use the technique developed in [5] for the asymptotic evaluation of the probability in the right-hand side of (7). We adopt the same notations as in [5]. Let Φ be the distribution function of the standard normal law and denote, for $t, u \geq 0$,

$$\phi(t) = (t - 1/4)^{-4}, \quad f(t) = (1 + 2t)^{-1/2},$$
$$F(t) = 2\Phi(\sqrt{t}) - 1,$$
$$I_0(u) = \int_1^\infty \ln f(u\phi(t))dt,$$
$$I_1(u) = \int_1^\infty u\phi(t)(\ln f)^{'}(u\phi(t))dt,$$
$$I_2(u) = \int_1^\infty (u\phi(t))^2(\ln f)^{''}(u\phi(t))dt,$$
$$C_\phi = \frac{1}{2}\sum_{j=1}^\infty \int_0^1 \ln \frac{\phi(j)\phi(j+1)}{\phi^2(t+j)}dt.$$

The following Theorem 1 is a concretization of Corollary 3.2 from [5]. Note that with our choice of ϕ, f and F all regularity conditions assumed in [5] are satisfied.

Theorem 1.

$$P\left(\sum_{j=1}^\infty \phi(j)\xi_j^2 \leq r\right) \sim \sqrt{\frac{\Gamma(3/2)F((u\phi(1))^{-1})}{2\pi I_2(u)}}\exp(I_0(u) - C_\phi/2 + ur), \tag{10}$$

where $u = u(r)$ is any function satisfying

$$\lim_{r\to 0} \frac{I_1(u) + ur}{\sqrt{I_2(u)}} = 0. \tag{11}$$

We begin by the asymptotic analysis of $I_s(u), s = 0, 1, 2,$ as $u \to \infty$. Changing variables and integrating by parts in the definition of $I_0(u)$ we get

$$\begin{aligned}
I_0(u) &= -\frac{1}{2}\int_{3/4}^{\infty} \ln(1+\frac{2u}{t^4})dt \\
&= \frac{3}{8}\ln(1+\frac{512u}{81}) - 4u\int_{3/4}^{\infty}\frac{dt}{2u+t^4} \\
&= \frac{3}{8}\ln(1+\frac{512u}{81}) - 4u\int_{0}^{\infty}\frac{dt}{2u+t^4} + 4u\int_{0}^{3/4}\frac{dt}{2u+t^4} \\
&= \frac{3}{8}\ln(1+\frac{512u}{81}) - J_1(u) + J_2(u).
\end{aligned}$$

From [9], formula 3.241, we derive

$$J_1(u) = (u/2)^{1/4}\pi.$$

Moreover, by the Lebesgue Dominated Convergence Theorem, as $u \to \infty$,

$$J_2(u) = 2\int_0^{3/4}\frac{dt}{1+\frac{t^4}{2u}} \to 3/2.$$

In view of all this we have, as $u \to \infty$,

$$I_0(u) \sim \frac{3}{8}\ln\left(\frac{512u}{81}\right) - (u/2)^{1/4}\pi + 3/2. \tag{12}$$

Note that

$$(\ln f(t))' = -\frac{1}{1+2t}, \quad (\ln f(t))'' = \frac{2}{(1+2t)^2}. \tag{13}$$

By taking into account formula (13) and by repeating the arguments used for the analysis of $I_0(u)$, we have

$$\begin{aligned} I_1(u) &= -u\int_{3/4}^{\infty}\frac{dt}{2u+t^4} = -u\int_0^{\infty}\frac{dt}{2u+t^4} + u\int_0^{3/4}\frac{dt}{2u+t^4} \\ &\sim -2^{-9/4}\pi u^{1/4}, \\ I_2(u) &= 2u^2\int_{3/4}^{\infty}\frac{dt}{(2u+t^4)^2} = 2u^2\int_0^{\infty}\frac{dt}{(2u+t^4)^2} - 2u^2\int_0^{3/4}\frac{dt}{(2u+t^4)^2} \\ &\sim 3\pi 2^{-17/4}u^{1/4}. \end{aligned}$$

Now, if we choose u in such a way that $u = \pi^{4/3}2^{-3}r^{-4/3}$, then $u^{1/4} = \pi^{1/3}2^{-3/4}r^{-1/3}$ and $ur = -I_1(u) + O(1)$, so that u satisfies condition (11).

In order to apply formula (10) it is necessary to calculate the constant C_ϕ; in our case we have

$$\begin{aligned} C_\phi &= \frac{1}{2}\sum_{j=1}^{\infty}\int_0^1 \ln\frac{(t+j-1/4)^8}{(j-1/4)^4(j+3/4)^4}dt \\ &= 4\sum_{j=1}^{\infty}[\int_0^1 \ln(t+j-1/4)dt - \frac{1}{2}\ln((j-1/4)(j+3/4))] \\ &= 2\sum_{j=1}^{\infty}[(2j+1/2)\ln\frac{j+3/4}{j-1/4} - 2]. \end{aligned}$$

To simplify the last sum we need some formulas from the theory of Gamma function (see, e.g., [26], Ch. 12). Consider the integral

$$I(z) = \int_0^{\infty} e^{-tz}\left\{\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right\}t^{-1}dt, \tag{14}$$

defined for any complex z with positive real part. We recall the Binet's integral representation for the logarithm of Gamma function, valid for any complex z with positive real part (see [26], §12.31)

$$\ln\Gamma(z+1) = (z+1/2)\ln z - z + 1 + I(z) - I(1). \tag{15}$$

Setting in (14) $z = j + \frac{3}{4}$ and $z = j - \frac{1}{4}$ and taking the difference of these identities we get

$$I(j-1/4) - I(j+3/4) = (j+1/4)\ln\frac{j+3/4}{j-1/4} - 1.$$

and hence

$$\begin{aligned} C_\phi &= 2\sum_{j=1}^{\infty}[(2j+1/2)\ln\frac{j+3/4}{j-1/4} - 2] \\ &= 4\sum_{j=1}^{\infty}[I(j-1/4) - I(j+3/4)] = 4I(3/4). \end{aligned}$$

Applying Binet's formula once again, we have

$$\ln\Gamma(3/4) = \frac{1}{4}\ln(3/4) + 3/4 + I(3/4) - I(1). \tag{16}$$

It is well-known that $I(1) = 1 - \frac{1}{2}\ln(2\pi)$ (see [26], section 12.31) and therefore, from (16), we have

$$I(3/4) = \ln\Gamma(3/4) - \frac{1}{4}\ln(3/4) + 3/4 - \frac{1}{2}\ln(2\pi).$$

Finally, we conclude that

$$C_\phi = 4I(3/4) = 4\ln\Gamma(3/4) - \ln(3/4) + 3 - 2\ln(2\pi)$$

and

$$\exp(-C_\phi/2) = 3^{1/2}\pi\Gamma^{-2}(3/4)\exp(-3/2).$$

Considering all these partial results we have, as $u \to \infty$, that

$$\begin{aligned} &F(1/u\phi(1)) = 2\Phi((\frac{3}{4})^2u^{-1/2}) - 1 \sim \sqrt{\frac{2}{\pi}}(\frac{3}{4})^2u^{-1/2}, \\ &\sqrt{2\pi I_2(u)} \sim 3^{1/2}2^{-13/8}\pi u^{1/8}, \\ &\exp(I_0(u) + ur) \sim (512u/81)^{3/8}\exp(3/2 - 3\pi 2^{-9/4}u^{1/4}). \end{aligned}$$

In force of formula (14) and recalling that $u = \frac{1}{8}(\pi/r)^{4/3}$, we have as $r \to 0$,

$$P\left(\sum_{j=1}^{\infty}\sigma_j\xi_j^2 \le r\right) \sim \frac{2^{11/4}}{(\sqrt{3})\Gamma^2(3/4)}\exp(-(3/8)r^{-1/3}).$$

To get the final result we must take into account the constant

$$C_{\sigma\lambda} = \Pi_{n=1}^{\infty}(\sigma_n/\lambda_n)^{1/2},$$

that should be calculated numerically. Since the numbers λ_j and σ_j, as established above, are very close, the infinite product converges very fast. Using the ten largest eigenvalues λ_j found in [10], simple calculations give the value $C_{\sigma\lambda} \approx 1.0075...$. Hence, we get from (7) the exact asymptotics

$$P(||\overline{B}||_2 \leq \varepsilon) \sim 1.0075... \frac{2^{11/4}}{(\sqrt{3})\Gamma^2(3/4)} \exp(-(3/8)\varepsilon^{-2/3}).$$

Note that the factor preceding the exponent does not depend on ε. More refined arguments in [22] and [6] prove that the whole constant in the right-hand side is in fact equal to $8/\sqrt{3\pi}$.

5. Exact small deviation asymptotics for the integrated Wiener process

Same technique as in the previous section enables us to get the exact small deviation asymptotics of $Q(\overline{W};\varepsilon)$ as $\varepsilon \to 0$. The calculations are similar to those for the integrated Brownian bridge and we will omit some details. The covariance of the integrated Wiener process is given by

$$\sigma_{\overline{W}}(s,t) = \frac{1}{2}st(s \wedge t) - \frac{1}{6}(s \wedge t)^3, \quad 0 \leq s,t \leq 1.$$

The spectrum of the corresponding integral operator can be found from the following boundary-value problem

$$\lambda f^{(IV)}(t) = f(t),$$
$$f(0) = f^{'}(0) = f^{''}(1) = f^{'''}(1) = 0.$$

The solution of this problem can be found in [14]. The spectrum consists of the eigenvalues $\lambda_n = m_n^{-4}, n \geq 1$, where $m_1 < m_2 < ...$ are the solutions of the auxiliary transcendental equation

$$\cos m \cosh m + 1 = 0, \tag{17}$$

while the eigenfunctions are $f_n(t) = (\cosh m_n + \cos m_n)(\sinh m_n t - \sin m_n t) - (\sinh m_n + \sin m_n)(\cosh m_n t - \cos m_n t), n \geq 1$.

As in the previous section, it can be proved that, with an exponential error and for large j,

$$\lambda_j \sim \tau_j := (\pi(j - 1/2))^{-4},$$

so that we can evaluate the asymptotics of $P(\sum_{j=1}^{\infty}\xi_j^2(j-1/2)^{-4}\le r)$, by taking into account the constant $C_{\tau\lambda}=\prod_{n=1}^{\infty}(\tau_n/\lambda_n)^{1/2}$.

The asymptotic calculations of the functions $I_s(u)$, $s=0,1,2$, are very similar to those of the previous section. While the asymptotics of $I_1(u)$ and $I_2(u)$ (and therefore the choice of $u=u(r)$) coincide with those of the integrated Brownian bridge, the limit expression of $I_0(u)$ in this case reads

$$I_0(u)\sim\frac{1}{4}\ln(32u)-(u/2)^{1/4}\pi+1.$$

Some differences appear also when calculating the constant C_ϕ. By repeating the arguments of the previous section we have

$$\begin{aligned}C_\phi&=\frac{1}{2}\sum_{j=1}^{\infty}\int_0^1\ln\frac{(t+j-1/2)^8}{(j-1/2)^4(j+1/2)^4}dt=4\sum_{j=1}^{\infty}(j\ln\frac{j+1/2}{j-1/2}-1)\\&=4\sum_{j=1}^{\infty}[I(j-1/2)-I(j+1/2)]=4I(1/2).\end{aligned}$$

The integral $I(1/2)=(1+\ln 1/2)/2$ is calculated in [26], Section 12.31, so that $C_\phi=2-2\ln 2$ and $\exp(-C_\phi/2)=2\exp(-1)$.

To evaluate the constant $C_{\tau\lambda}=\prod_{n=1}^{\infty}(\tau_n/\lambda_n)^{1/2}$ we must find numerically the first few roots $m_1,m_2,...$ of the equation (17) and then calculate $\lambda_j=(m_j)^{-4}, j\ge 1$. After easy calculations we get the approximate value $C_{\tau\lambda}=1.4142...$. Collecting together all these calculations we obtain the exact asymptotics, namely

$$P(||\overline{W}||_2\le\varepsilon)\sim 1.4142...\cdot(8/\sqrt{3\pi})\varepsilon^{1/3}\exp(-(3/8)\varepsilon^{-2/3}).$$

Note that the methods from [22] and [6], [7] show that in fact $C_{\tau\lambda}=\sqrt{2}$.

6. Centered integrated Brownian bridge and integrated centered Brownian bridge

We consider now the processes

$$B^0(t)=\overline{B}(t)-\int_0^1\overline{B}(u)du$$

and

$$B^*(t) = \int_0^t (B(s) - \int_0^1 B(u)du)ds = \overline{B}(t) - t\overline{B}(1)$$

introduced above. It follows from [10] that for the first process, having mean zero and covariance function

$$\sigma_{B^0}(s,t) = \frac{st \cdot s \wedge t}{2} - \frac{(s\wedge t)^3}{6} - \frac{s^2t^2}{4} - \frac{s^2+t^2}{6} - \frac{s^4+t^4}{24} + \frac{s^3+t^3}{6} + \frac{1}{45},$$

$0 \leqslant s,t \leqslant 1$, the spectrum has the form $\lambda_n = (\pi n)^{-4}, n \geq 1$.

We can apply the result of Example 2 from §18 in [19], see also Example 1 from [17], to obtain the exact small deviation asymptotics

$$P(||B^0||_2 \leq \varepsilon) \sim 2^{5/2}3^{-1/2}\pi^{-1/2}\varepsilon^{-1/3}\exp(-\frac{3}{8}\varepsilon^{-2/3}), \quad \varepsilon \to 0.$$

In case of the process B^*, possessing covariance function

$$\sigma_{B^*}(s,t) = \frac{st \cdot s \wedge t}{2} - \frac{(s\wedge t)^3}{6} - \frac{s^2t^2}{4} - \frac{st^2}{4} + \frac{st^3}{6} - \frac{ts^2}{4} + \frac{ts^3}{6} + \frac{st}{12},$$
$$0 \leqslant s,t \leqslant 1,$$

the spectrum has a more complicated structure. In [10] it is shown that the spectrum contains the following two series of eigenvalues $\lambda_n = (2\pi n)^{-4}$ and $\mu_n = (2k_n)^{-4}$, $n \geq 1$, where k_n are the solutions of the equation (5).

By the Kac–Siegert formula we have

$$||B^*||_2^2 = \sum_{n=1}^{\infty} \xi_n^2/16\pi^4n^4 + \sum_{m=1}^{\infty} \eta_m^2/16k_m^4 := V_1 + V_2 \quad , \qquad (18)$$

where $\{\eta_m\}$ is a sequence of independent standard normal variables, which is independent of $\{\xi_n\}$. By applying again the result of Li [17] and formula (6), we can replace in (18) k_m by $\pi(m-1/4)$.

We can obtain the exact small deviation asymptotics of the sum (18) of two independent random variables of the same nature using the following theorem, which was kindly indicated to us by Prof. M. Lifshits.

Theorem 2. *Let V_1, $V_2 > 0$ be two independent random variables with known behavior of small deviations, namely*

$$P(V_1 \leq r) \sim c_1 r^{a_1} \exp(-b_1 r^{-d})$$

and

$$P(V_2 \le r) \sim c_2 r^{a_2} \exp(-b_2 r^{-d}),$$

as $r \to 0$. *Then we have the following small deviation asymptotics for their sum:*

$$P(V_1 + V_2 \le r) \sim K r^{a_1+a_2-d/2} \exp(-S^{d+1} r^{-d}),$$

where

$$S = b_1^{1/(d+1)} + b_2^{1/(d+1)},$$

$$K = c_1 c_2 \sqrt{\frac{2\pi d}{d+1}} S^{d/2-1/2-a_1-a_2} b_1^{(2a_1+1)/2(d+1)} b_2^{(2a_2+1)/2(d+1)}.$$

The proof is elementary but rather laborious and we omit it.

Let us apply Theorem 2 to the sum (18). In our case we have

$$c_1 = 2^{11/6} 3^{-1/2} \pi^{-1/2},\ a_1 = -1/6, b_1 = 3 \cdot 2^{-13/3},\ d = 1/3;$$
$$c_2 = 1.0075...2^{11/4} 3^{-1/2} \Gamma^{-2}(3/4),\ a_2 = 0, b_2 = 3 \cdot 2^{-13/3}.$$

Hence, after some calculations, we get the exact asymptotics

$$P(V_1 + V_2 \le r) \sim 1.0075... \cdot 3^{-1/2} 2^{7/4} \Gamma^{-2}(3/4) r^{-1/3} \exp(-(3/8) r^{-1/3}),$$

or equivalently

$$P(||B^*||_2 \le \varepsilon) \sim 1.0075... \cdot 3^{-1/2} 2^{7/4} \Gamma^{-2}(3/4) \varepsilon^{-2/3} \exp(-(3/8) \varepsilon^{-2/3}).$$

Using the sharp value of the constant given in the end of section 5, we see that the constant in the right-hand side is in fact equal to $4/\sqrt{3\pi}$.

7. Centered integrated Wiener process and integrated centered Wiener process

We consider now the process

$$W^0(t) = \overline{W}(t) - \int_0^1 \overline{W}(u) du$$

introduced above. Its covariance function is

$$\sigma_{W^0}(s,t) = \frac{1}{2}(s \wedge t) \cdot st - \frac{1}{6}(s \wedge t)^3 - \frac{s^2+t^2}{4} + \frac{s^3+t^3}{6} - \frac{s^4+t^4}{24} + \frac{1}{20},$$

$0 \le s, t \le 1$.

By differentiating the integral equation (4) we get the boundary-value problem

$$\lambda f^{(IV)}(t) = f(t) - \int_0^1 f(u)du,$$
$$f'(0) = f''(1) = f'''(0) = f'''(1) = 0.$$

Put $p(t) = f(t) - \int_0^1 f(s)ds$, then we have for p the similar boundary-value problem

$$\lambda p^{(IV)}(t) = p(t),$$
$$p'(0) = p''(1) = p'''(0) = p'''(1) = 0.$$

This problem has the following solutions (see case (1,3,2,3) in the equation 4.3 in [14]): $p_n(t) = C(\cos k_n \cosh nt + \cosh k_n \cos nt)$ and $\lambda_n = k_n^{-4}, n \geq 1$, where as above k_n are the solutions of (5). It is easy to prove that $f_n(t) = p_n(t), n = 1, 2...$, see [11]. Hence we observe that

$$||W^0||_2 = ||\overline{W}||_2$$

in distribution and the exact small deviation asymptotics of $||W^0||_2$ has the same form as in Section 6.

In the case of the integrated centered Wiener process

$$W^*(t) = \int_0^t (W(s) - \int_0^1 W(u)du)ds = \overline{W}(t) - t\overline{W}(1),$$

the covariance function has the form

$$\sigma_{W^*}(s,t) = \frac{1}{2}(s \wedge t)st - \frac{1}{6}(s \wedge t)^3 + \frac{s^3t + st^3}{6} - \frac{s^2t + st^2}{2} + \frac{st}{3},$$

$0 \leq s, t \leq 1$.

The integral equation can be reduced to the boundary-value problem

$$\lambda f^{(IV)}(t) = f(t),$$
$$f(0) = f(1) = f''(0) = f''(1) = 0,$$

whose solutions (according to [14]) are $\lambda_n = (n\pi)^{-4}, n \geq 1$, and $f_n(t) = C \sin n\pi t, n \geq 0$. We have already met such spectrum in our paper, so that

$$||W^*||_2 = ||B^0||_2$$

in distribution and we can apply the result obtained above. Hence,

$$P(||W^*||_2 \le \varepsilon) \sim 2^{5/2} 3^{-1/2} \pi^{-1/2} \varepsilon^{-1/3} \exp(-\frac{3}{8}\varepsilon^{-2/3}),\ \varepsilon \to 0.$$

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