Записки научных
семинаров ПОМИ
Том 299, 2003 г.

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A REMARK ON THE sl_2 APPROXIMATION OF THE KONTSEVICH INTEGRAL OF THE UNKNOT

Abstract. The Kontsevich integral of a knot K is a sum $I(K) = 1 + \sum_{n=1}^{\infty} h^n \sum_{D\in A_n} a_D D$ over all chord diagrams with suitable coefficients. Here A_n is the space of chord diagrams with n chords. A simple explicit formula for coefficients a_D is not known even for the unknot. Let $E_1, E_2, \dots$ be elements of $A = \bigoplus_n A_n$. Say that a sum $I'(K) = 1 + \sum_{n=1}^{\infty} h^n E_n$ is an sl_2 approximation of the Kontsevich integral if the values of the sl_2 weight system W_{sl_2} on both sums are equal, $W_{sl_2}(I(K)) = W_{sl_2}(I'(K))$.

For any natural n fix points $a_1, \dots, a_{2n}$ on a circle. For any permutation $\sigma \in S_{2n}$ of $2n$ elements define the chord diagram $D(\sigma)$ with n chords as the diagram with chords formed by pairs $a_{\sigma(2i-1)}$ and $a_{\sigma(2i)}$, $i = 1, \dots, n$. We show that

$$1 + \sum_{n=1}^{\infty} \frac{h^{2n}}{2^n (2n)!(2n+1)!} \sum_{\sigma\in S_{2n}} D(\sigma)$$

is an sl_2 approximation of the Kontsevich integral of the unknot.

§1. sl_2 approximations

The Kontsevich integral of a knot K is a sum $I(K) = 1 + \sum_{n=1}^{\infty} h^n \sum_{D\in A_n} a_D D$ over all chord diagrams with suitable coefficients [4]. Here A_n is the space of chord diagrams with n chords, h is formal parameter. A simple explicit formula for coefficients a_D is not known even for the unknot. Let $E_1, E_2, \dots$ be elements of $A = \bigoplus_n A_n$. Say that a sum $I'(K) = 1 + \sum_{n=1}^{\infty} h^n E_n$ is an sl_2 approximation of the Kontsevich integral if the values of the sl_2 weight system W_{sl_2} on both sums are equal, $W_{sl_2}(I(K)) = W_{sl_2}(I'(K))$.

For any natural n fix points $a_1, \dots, a_{2n}$ on a circle. For any permutation $\sigma \in S_{2n}$ of $2n$ elements define the chord diagram $D(\sigma)$ with n chords as the diagram with chords formed by pairs $a_{\sigma(2i-1)}$ and $a_{\sigma(2i)}$, $i = 1, \dots, n$. Introduce an element $\Sigma_n = \sum_{\sigma\in S_{2n}} D(\sigma)$ in A_n.

Supported in part by NSF grant DMS-9801582.

Theorem 1.1. *The sum*

$$1+\sum_{n=1}^{\infty}\frac{h^{2n}}{2^n(2n)!(2n+1)!}\Sigma_n$$

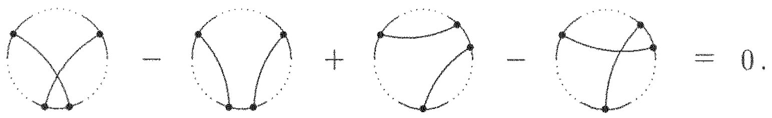
is an sl_2 approximation of the Kontsevich integral of the unknot.

The theorem is proved in Sec. 6.

We thank S. Chmutov for useful discussions.

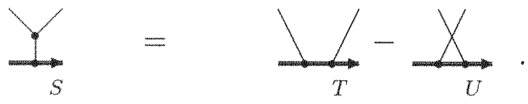
§2. Three algebras

The Kontsevich integral takes values in the graded completion of the chord diagram algebra $A=\bigoplus_n A_n$. The $\mathbb{Q}$ vector space A is generated by the usual chord diagrams modulo the four-term relation



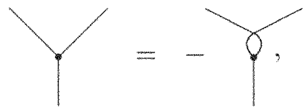
The product of chord diagrams is their connected sum, see [1].

We also consider the algebra $\mathcal{T}$ of trivalent diagrams. A trivalent diagram is a connected graph with only trivalent vertices and a distinguished oriented circle, such that at each vertex, which does not lie on the circle, one of two possible cyclic orderings of the three edges meeting at this vertex is chosen. The $\mathbb{Q}$ vector space $\mathcal{T}$ is generated by trivalent diagrams modulo the STU relation,



These three trivalent diagrams are identical outside the corresponding fragment on the picture. Pieces of the circle are pictured by thick lines. The product of trivalent diagrams is defined as their connected sum with respect to the distinguished circles.

The following relations follow from the STU relation,

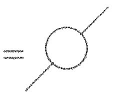
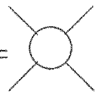
(AS): 

(IHX): 

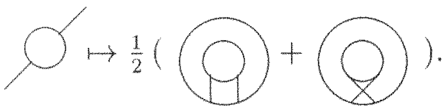
Applying the STU relation one can express a given trivalent diagram as a linear combination of chord diagrams. This gives a natural mapping $\mathcal{T} \to A$ which is an isomorphism of algebras, see [1].

The third algebra is the algebra $\mathcal{U}$ of uni-trivalent diagrams. A uni-trivalent diagram is a graph whose vertices either univalent or trivalent, at each trivalent vertex a cyclic ordering of the three edges is chosen. The vertices of valency 1 of a uni-trivalent diagram are called "legs" of the diagram. We consider the $\mathbb{Q}$ vector space $\mathcal{U}$ generated by uni-trivalent graphs modulo the AS and IHX relations. The product in the algebra $\mathcal{U}$ is the disjoint union $\sqcup$. There is a natural isomorphism $\mathcal{U} \to \mathcal{T}$ as vector spaces, but not as algebras. The isomorphism maps every uni-trivalent diagram to the average of all possible ways of placing its univalent vertices along the circle, see [1].

Introduce uni-trivalent graphs w_{2n}, the "wheels with $2n$ legs,"

$w_2 =$  , $w_4 =$  ,

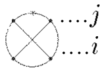
Under the isomorphism $\mathcal{U} \to \mathcal{T}$, for instance, we have



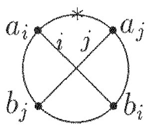
§3. The sl_2-weight system

A weight system on A (resp. on $\mathcal{T}, \mathcal{U}$) with values in a vector space is a linear homomorphism of A (resp. of $\mathcal{T}$, $\mathcal{U}$) to the vector space. The composition of the Kontsevich integral of a knot with the linear homomorphism defines a knot invariant of the knot. Here we recall the construction of the weight system associated to a Lie algebra $\mathfrak{g}$ with an ad-invariant nondegenerate bilinear form Φ.

Let $a_1, \dots, a_m$ and $b_1, \dots, b_m$ be two dual bases of $\mathfrak{g}$: $\Phi(a_i, b_j) = \delta_{i,j}$. Fix a chord diagram D and a base point on its circle which is different from the endpoints of the chords. Label each chord by a number i such that $1 \le i \le m$. Attach to one endpoint of the chord labelled by i the element a_i and to another endpoint the element b_i. For example, for the chord diagram

$D =$ 

with the base point $*$ and labels i and j we have

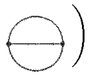


Walk around the circle starting from the base point in the direction of the orientation of the circle and write in one word the elements associated to the endpoints. The constructed word is an element of the universal enveloping algebra $U(\mathfrak{g})$. Define $W(D)$ to be the sum of such words where the sum is over all labels of the chords. In the example,

$$W(D) = \sum_{i,j} a_i b_j b_i a_j \in U(\mathfrak{g}).$$

In our pictures we always assume that the circle is oriented counterclockwise.

The element $W(D)$ does not depend on the base point, does not depend on the choice of dual bases in $\mathfrak{g}$, belongs to the center $Z(\mathfrak{g})$ of the universal enveloping algebra, satisfies the four-term relation. The mapping $W : A \to Z(\mathfrak{g})$ is an algebra homomorphism, $W(D_1 \cdot D_2) = W(D_1) \cdot W(D_2)$, see [4].

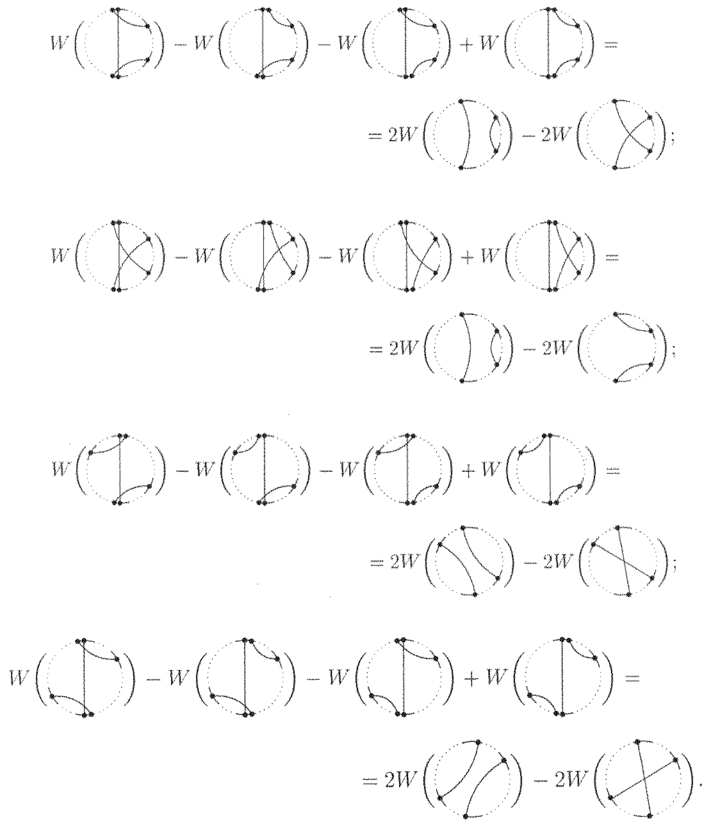
Example. $W\Big($  $\Big)$ is the quadratic Casimir element of $Z(\mathfrak{g})$ associated to the chosen invariant form.

For $\mathfrak{g} = sl_2$, we have $Z(sl_2) \cong \mathbb{C}[c]$, and for a chord diagram D with n chords,

$$W_{sl_2}(D) = c^n + \lambda_1 c^{n-1} + \lambda_2 c^{n-2} + \ldots + \lambda_{n-1} c.$$

We choose Tr as an ad-invariant form on sl_2 where Tr is the trace of matrices in the standard two dimensional representation of sl_2. A recurrent formula for W_{sl_2} is constructed in [3].

Theorem 3.1. *Let $W = W_{sl_2}$ be the weight system associated to sl_2 and the ad-invariant form Tr. Then*



This theorem allows one to compute $W(D)$ since the two chord diagrams of the right hand side have one chord less than the diagrams of the left hand side, and the last three diagrams of the left hand side are simpler than the first one since they have less intersections between their chords.

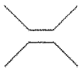
The theorem indicates six-term elements of the kernel of the sl_2 weight system. The subspace I of the algebra A generated by the six-term elements forms an ideal. The quotient algebra A/I is generated by two elements  and . The ideal generated by the six term elements and the element

 $+2$  $-$  $\cdot$ 

is the kernel of the sl_2 weight system.

The linear isomorphisms $\mathcal{U} \to \mathcal{T}$ and $\mathcal{T} \to A$ induce weight systems $W_{sl_2} : \mathcal{T} \to Z(sl_2)$, $W_{sl_2} : \mathcal{U} \to Z(sl_2)$.

Theorem 3.2 [3]. *The weight system W_{sl_2} satisfies the following three term relation*

$W_{sl_2}($  $) = 2W_{sl_2}($  $) - 2W_{sl_2}($  $)$

for any uni-trivalent diagrams differed only by the pictured fragments.

Corollary 3.3 [3].

$W_{sl_2}\Big($  $\Big) = 4W_{sl_2}\Big($ $\Big).$

§4. Bernoulli numbers and Bernoulli polynomials

The modified Bernoulli numbers are defined by the series

$$\sum_{n=0}^{\infty} b_{2n} x^{2n} = \frac{1}{2} \ln \frac{e^{x/2} - e^{-x/2}}{x/2}.$$

The Bernoulli polynomials $B_n(x)$ are defined by the series

$$\frac{ze^{zx}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}.$$

The polynomial $B_n(x)$ has degree n. Its top coefficient equals 1.

Following [5] introduce the shifted Bernoulli polynomials $q_n(x)$ by the condition

$$q_n(\frac{x^2-1}{2}) = \frac{2}{(2n+1)!}\,\frac{B_{2n+1}(\frac{1+x}{2})}{x}.$$

The polynomial $q_n(x)$ has degree n. Its top coefficient equals $\frac{1}{2^n(2n+1)!}$.

Theorem 4.1. *For any natural n, we have*

$$W_{sl_2}(w_{2n}) = 2^{2n+1}\,(2n+1)!\,q_n(c),$$

$$\sum_{\sigma\in S_{2n}} W_{sl_2}(D(\sigma)) = 2^n\,(2n)!\,(2n+1)!\,q_n(c).$$

The theorem is proved in Sec. 6.

§5. The Kontsevich integral of the unknot

A formula for the logarithm of the Kontsevich integral I of the unknot in terms of wheels is given in [2, 6].

Theorem 5.1.

$$\begin{aligned} I &= \exp\,\Big(\sum_{n=1}^{\infty} b_{2n}h^{2n}w_{2n}\Big) \\ &= 1 + \Big(\sum_{n=1}^{\infty} b_{2n}h^{2n}w_{2n}\Big) + \frac{1}{2}\Big(\sum_{n=1}^{\infty} b_{2n}h^{2n}w_{2n}\Big)^2 + \dots . \end{aligned}$$

The value of the sl_2 weight system on the Kontsevich integral of the unknot is calculated in [5].

Theorem 5.2.

$$W_{sl_2}(I) = \sum_{n=0}^{\infty} q_n(c)h^{2n}.$$

§6. Proofs

Lemma 6.1. *For $n_1+n_2+\ldots+n_k = n$, we have*

$$W_{sl_2}(w_{2n_1}\sqcup w_{2n_2}\sqcup\ldots\sqcup w_{2n_k}) = \frac{2^{n_1+n_2+\cdots+n_k+k}}{(2n)!}W_{sl_2}(\Sigma_n).$$

Proof of the lemma. We prove the lemma for $k = 1$, general case is similar. The three term relation applied to a vertex of w_{2n} gives

$$W_{sl_2}(w_{2n}) = 2W_{sl_2}(\{\ |\ \} \sqcup w_{2n-2}) - 2W_{sl_2}(t_{2n-2})$$

where $\{\ |\ \}$ is the uni-trivalent graph with one edge and two univalent vertices and $t_{2n-2} = \underbrace{\overline{\,|\,|\ldots}}_{2n-2\text{ legs}}$ Application of the three term relation to the first two vertices of t_{2n-2} gives

$$W_{sl_2}(t_{2n-2}) = 2\,W_{sl_2}(\{\,|\,\} \sqcup t_{2n-4}) \ - \ 2\,W_{sl_2}(\{|\} \sqcup t_{2n-4}) = 0.$$

Then Corollary 3.3 implies $W_{sl_2}(w_{2n}) = 2^{n+1}\,W_{sl_2}(\sqcup_n\{\,|\,\})$, where $\sqcup_n\{\,|\,\}$ is the diagram with n edges and $2n$ univalent vertices. Glueing legs of $\sqcup_n\{\,|\,\}$ to the circle in all possible ways and dividing the sum of the resulting chord diagrams by $(2n)!$ gives the lemma for $k = 1$,

$$W_{sl_2}(w_{2n}) = \frac{2^{n+1}}{(2n)!} \sum_{\sigma \in S_{2n}} W_{sl_2}(D_n(\sigma)). \qquad (1)$$

$\square$

Proof of Theorems 1 and 4.1. Theorems 5.1, 5.2, and Lemma 6.1 imply that $W_{sl_2}(\Sigma_n) = \mathrm{const}\, q_n(c)$. Both sides are polynomials in c of degree n. Comparing the coefficients of c^n one gets Theorem 1.1 and the second equality of Theorem 4.1. Equation (1) implies the first equality of Theorem 4.1. $\square$

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Поступило 19 ноября 2001 г.

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