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PATCHWORKING SINGULARITIES A_μ AND D_μ AND MEANDERS OF THEIR SMOOTHING

Abstract. Let an algebraic curve f have a singular point of type A_μ or D_μ. Let $\tilde{f}$ be the curve obtained as a result of smoothing the singular point of the curve f. In this paper we study the local maximal meanders appearing under M-smoothing in a neighborhood of the singular point. A local maximal meander means that the number of real points of the intersection of the curve $\tilde{f}$ with a coordinate axis in the neighborhood is maximal and the points belong to one of the components of $\tilde{f}$; and an M-smoothing means that the number of components of the curve $\tilde{f}$, which appear in the neighborhood under the smoothing, is also maximal.

In Sec. 1, we give definitions, notations, and references to the results and methods which we use in this paper. In Sec. 2, we prove the topological classification of $(m,2)$-meanders in D^2 of types I, II, III (see the definitions in the section) and calculate the numbers of equivalence classes in the classification. In Secs. 3 and 4, we prove that all topological $(m,1)$- and $(m,2)$-meanders are realizable by algebraic $(m,2)$-meanders, the models of which can be described as the charts of smoothing singularities A_μ and D_μ. These charts of smoothing are maximal in the sense that, together with an $(m,1)$- or $(m,2)$-meander, the constructed chart has the maximal number of ovals. We show that the arrangment of ovals of the charts must satisfy an extension of the Gudkov–Rokhlin congruence [7].

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§1. Preliminaries

A *projective algebraic curve* is a real homogeneous polynomial f considered up to a nonzero constant factor. The degree d of the polynomial f is called the *degree* of the curve. The set

$$\mathbb{R}f = \{(x:y:z) \in \mathbb{R}P^2 \mid f = 0\}$$

its is called the *set of real points* of the curve f. In particular, $\mathbb{R}x$, $\mathbb{R}y$, and $\mathbb{R}z$ are coordinate axes.

Let

$$f(x,y,z) = \sum_{\omega_1+\omega_2+\omega_3=d} f_\omega x^{\omega_1} y^{\omega_2} z^{\omega_3}, \ \omega = (\omega_1, \omega_2, \omega_3)$$

be a real homogeneous polynomial of degree d and

$$f(x,y,1) = \sum_{\omega_1+\omega_2 \le d} f_\omega x^{\omega_1} y^{\omega_2}$$

its restriction to the affine chart $\mathbb{R}^2 = \mathbb{R}P^2 \setminus \{z=0\}$. The convex hull of the set $\{(\omega_1,\omega_2) \in \mathbb{R}^2 \mid f_\omega \neq 0\}$ is called the *Newton polygon* of the polynomial $f(x,y,1)$ and denoted by $N(f)$.

For a set $\Gamma \subset \mathbb{R}^2$ and a polynomial

$$f = \sum_{(\omega_1,\omega_2)\in N(f)} f_\omega x^{\omega_1} y^{\omega_2},$$

the polynomial

$$f^\Gamma = \sum_{(\omega_1,\omega_2)\in \Gamma\cap N(f)} f_\omega x^{\omega_1} y^{\omega_2}$$

is called the Γ-*truncation* of f.

O. Ya. Viro [8] discovered a method of glueing charts of completely nondegenerate polynomials (the Viro patchwork method). In [8] or [9], one can find the description of the method: all notations, definitions, constructions, and the Viro patchwork theorem (Theorem 1.4.A in [9]) that constitute the method. Here we use the Viro patchwork method without clarification. We denote the chart (Δ, v) of a polynomial f by $\mathbb{V}(f)$.

Let f be a peripherally nondegenerate polynomial without x and y factors having the chart $\mathbb{V}(f)$. Let the polynomial f have a singular point at the origin O. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_l$ be edges of the Newton diagram $\Gamma = \bigcup_{i=1}^{l} \Gamma_i$ of the singular point. The join $O * \Gamma$ is called the *Newton polygon of the singular point.* Let $f_1, f_2, \dots, f_s$ be peripherally nondegenerate polynomials having the charts $\mathbb{V}(f_i) = (\Delta_i, v_i)$, $i = 1, \dots, s$, such that the polynomials $f, f_1, f_2, \dots, f_s$ satisfy the Viro patchwork theorem. Let the polynomial $\tilde{f}(t)$ be the result of the patchworking of the polynomials $f, f_1, f_2, \dots, f_s$ by means of a concave function ν. The union

$$\mathbb{V} = \bigcup_{i=1}^{s} \mathbb{V}(f_i) = \Big(\bigcup_{i=1}^{s} \Delta_i, \bigcup_{i=1}^{s} v_i \Big)$$

is called the *chart of smoothing* of the singular point of the polynomial f if $\bigcup_{i=1}^{s} \Delta_i = O * \Gamma$. The boundary $\partial(O * \Gamma)$ of the Newton polygon of the singular point consists of edges $\Gamma_1, \Gamma_2, \dots, \Gamma_l$ and two more edges Γ_x and Γ_y lying on the axes $\mathbb{R}y$ and $\mathbb{R}x$, respectively. We define the Γ-truncations on these edges as

$$\tilde{f}(t)^{\Gamma_1}, \dots, \tilde{f}(t)^{\Gamma_l}, \tilde{f}(t)^{\Gamma_x}, \tilde{f}(t)^{\Gamma_y}.$$

Thus, the Γ-truncations on the boundary are well defined both for charts of polynomials and smoothings.

Note that the chart $\mathbb{V}$ of a smoothing can be nonconvex. If it is convex then one can consider this chart as the chart of a polynomial. We call a chart without the specification "of smoothing" or "of polynomial" when it is applicable to both of them.

Two charts $\mathbb{V} = (\Delta, v)$ and $\mathbb{V}' = (\Delta, v')$ are *equivalent* if the topological pairs (Δ, v) and (Δ, v') are homeomorphic.

Let $\mathbb{V}(f) = (\Delta, v)$ be a chart. There are two kinds of connected components of the set v: the components homeomorphic to the standard circle S^1 (such components are called *ovals*) and the components homeomorphic to the standard segment $I = [0, 1]$ (such components are called *arcs*).

Let k be the number of integer points with positive coordinates belonging to the interior of the set Δ.

Let $\Gamma_1, \Gamma_2, \dots, \Gamma_l$ be the edges of the set $\{(x, y) \in \Delta \mid x \geq 0, y \geq 0\}$. Let r be the number of real roots of the quasihomogeneous polynomials, which are Γ-truncations on the boundary of this set. Using the generalized Harnack inequality [4], one can evaluate the number of ovals and arcs of the set v in terms of k and r as follows. This result is well known; but we did not find any references for it.

Theorem 1.1. *The number of connected components of the set v is at most $k + r$, the number of ovals is at most k, and the number of arcs is equal to r.*

If the number of connected components of the set v is equal to $k + r$, then the chart is called the chart of an *M-smoothing.*

Let

$$r_{-+}, r_{+-}, r_{--}, s_1, s_2, s_3 : \mathbb{R}P^2 \to \mathbb{R}P^2$$

be maps defined by the formulas

$$\begin{aligned} r_{-+}(x:y:z) &= ((-x):y:z), & r_{+-}(x:y:z) &= (x:(-y):z), \\ r_{--}(x:y:z) &= (x:y:(-z)), & s_1(x:y:z) &= (x:z:y), \\ s_2(x:y:z) &= (z:y:x), & s_3(x:y:z) &= (y:x:z). \end{aligned}$$

Let hy and ha be the birational transformations defined by the formulas

$$hy(x:y:z) = (x^2:yz:xz) \quad \text{and} \quad ha(x:y:z) = (xz:y^2:xy),$$

respectively; note that $hy^{-1}(x:y:z) = (xz:xy:z^2)$ and $ha^{-1} = ha$.

Representatives of the singularities A_μ and D_μ series are called A_μ- and D_μ-*germs*.

§2. Meanders

2.1. Topological meanders. Let X be a two-dimensional differentiable manifold with the boundary ∂X, possibly $\partial X = \emptyset$. A subset $A \subset X$ that is diffeomorphic to a circle, $A \cap \partial X = \emptyset$, is called an *oval*. A subset $B \subset X$ that is diffeomorphic to a closed interval whose terminal points are distinct, say x_1 and x_2, and $B \cap \partial X = \{x_1, x_2\}$, is called an *arc*. Let Y_1 be an oval or arc and Y be an oval or arc and have a transversal intersection at m points, lying in $X \setminus \partial X$, where $m \geq 1$. The union $M = Y_1 \cup Y$ is called an m-*meander* in X. Two meanders M_1 and M_2 are *equivalent* if the topological pairs (X, M_1) and (X, M_2) are homeomorphic.

If X is a two-dimentional algebraic variety and Y_1 and Y are connected components of the nonsingular algebraic curves f_1 and f_2, respectively, then the meander $M = Y_1 \cup Y$ is algebraic. The Harnack [5, 6, 2] and Viro [8, 9] methods of construction of algebraic curves provide various examples of algebraic meanders in the projective plane $\mathbb{R}P^2$.

2.2. Topological (m,n)-meanders in D^2. Let $X = D^2$ be the standard disk $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$;

$$Y_1 = D^1 = \{(x,0) \in \mathbb{R}^2 \mid -1 \leq x \leq 1\};$$

and Y be an oval or an arc such that $M = D^1 \cup Y$ is an m-meander. The problem to find the number of equivalence classes of m-meanders for each $m \geq 1$, where Y is an arc, was stated by V. I. Arnold in [1], and, as we know, has not been proved up to now.

Consider an m-meander $M = D^1 \cup Y$ in D^2. The set $Y \setminus (D^1 \cup S^1)$, where

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

consists of m connected components, if Y is an oval; and of $m+1$ connected components, if Y is an arc. Each of these components is homeomorphic to an open segment. Denote the components of $D^2 \setminus D^1$ as

$$D^2_\pm = \{(x, y) \in D^2 \mid \pm y > 0\},$$

where $\pm$ means $+$ or $-$, respectively. Denote $Y \cap D^1 = \{a_1, \dots, a_m\}$, where $-1 < a_1 < \dots < a_m < 1$.

There are three types of m-meanders. If Y is an oval, then the meander has *type* I (in this case, m is even). If Y is an arc and both terminal points belong to one of components of the set $S^1 \setminus D^1$, then the meander has *type* II (in this case, m is even). If Y is an arc, and each component of the set $S^1 \setminus D^1$ contains one of the terminal points, then the meander has *type* III (in this case, m is odd).

1) Let M be an m-meander of type I. Let A_i be one of the m components of the set $Y \setminus D^1$, the points $(a_k, 0)$ and $(a_l, 0)$ be terminal points of A_i, and $A_i \subset D^2_\pm$. We put each A_i into 1-to-1 correspondence with the semicircle $B_i = C_i \cap D^2_\pm$, where

$$C_i = \{(x, y) \in D^2 \mid (x - \frac{a_k + a_l}{2})^2 + y^2 = \frac{(a_k - a_l)^2}{4}\}.$$

It is obvious that $B_1, \dots B_m$ are mutually disjoint. It is clear that the meanders $M = D^1 \cup Y$ and $M_1 = D^1 \cup B_1 \cup \dots \cup B_m$ are equivalent in D^2. The meander M_1 is the *standard form* of M. Let

$$L_i = D^2 \cap \{(x, y) \in \mathbb{R}^2 \mid\ y = \frac{1}{2}(a_{i-1} + a_i)\},$$

where $i = 1, \dots, m-1$. Let n_i be the number of points in the set $L_i \cap M_1$. The number

$$n = \max\{n_1, \dots, n_{m-1}\}$$

is called the *meandrous number* of the meander $M = D^1 \cup Y$.

2) Let M be an m-meander of type II or III. In this case the set $Y \setminus (D^1 \cup S^1)$ consists of $m+1$ segments: the $m-1$ segments $A_1, \dots, A_{m-1}$ have both terminal points lying on D^1, and two segments A_m and A_{m+1}, each of which has one terminal point on D^1 and the other one on S^1. For the segments $A_1, \dots, A_{m-1}$, we define the disjoint semicircles $B_1, \dots, B_{m-1}$ in the same manner as in the previous item 1). Let

$a_k, a_l \in D^1$ be terminal points of the segments A_m and A_{m+1} lying on D^1, and, without loss of generality, $a_k < a_l$. Let ε_1 and ε_2 be sufficiently small numbers, $0 < \varepsilon_1 < \varepsilon_2$. Denote

$$C_{m,1} = \{(x,y) \in \mathbb{R}^2 \mid \big[x - \frac{1}{2}(a_k - 1 - \varepsilon_1)\big]^2 + y^2 = \frac{1}{4}(a_k + 1 + \varepsilon_1)^2\},$$

$$C_{m,2} = \{(x,y) \in \mathbb{R}^2 \mid \big[x - \frac{1}{2}(a_k + 1 + \varepsilon_2)\big]^2 + y^2 = \frac{1}{4}(a_k - 1 - \varepsilon_2)^2\},$$

$$C_{m+1,1} = \{(x,y) \in \mathbb{R}^2 \mid \big[x - \frac{1}{2}(a_l - 1 - \varepsilon_2)\big]^2 + y^2 = \frac{1}{4}(a_l + 1 + \varepsilon_2)^2\},$$

$$C_{m+1,2} = \{(x,y) \in \mathbb{R}^2 \mid \big[x - \frac{1}{2}(a_l + 1 + \varepsilon_1)\big]^2 + y^2 = \frac{1}{4}(a_l - 1 - \varepsilon_1)^2\}.$$

Now we can describe ε_1 and ε_2 more precisely. If $(x_{m,1}(\varepsilon_1), \pm y_{m,1}(\varepsilon_1))$, $(x_{m,2}(\varepsilon_2), \pm y_{m,2}(\varepsilon_2))$, $(x_{m+1,1}(\varepsilon_2), \pm y_{m+1,1}(\varepsilon_2))$, and $(x_{m+1,2}(\varepsilon_1), \pm y_{m+1,2}(\varepsilon_1))$ are the points of intersection of the standard circle S^1 with the circles $C_{m,1}$, $C_{m,2}$, $C_{m+1,1}$, and $C_{m+1,2}$, respectively, then ε_1 and ε_2 must be chosen such that the inequality $0 < \varepsilon_1 < \varepsilon_2$ implies both

$$-1 < x_{m,1}(\varepsilon_1) < x_{m+1,1}(\varepsilon_2) < a_1$$

and

$$a_m < x_{m,2}(\varepsilon_2) < x_{m+1,2}(\varepsilon_1) < 1.$$

If $A_m \subset D^2_\pm$ (resp., $A_{m+1} \subset D^2_\pm$), then denote $B_{m,i} = D^2_\pm \cap C_{m,i}$ (resp., $B_{m+1,i} = D^2_\pm \cap C_{m+1,i}$), where $i = 1, 2$.

If the meander M is of type II, then there are three ("quasistandard") meanders

$$M_1 = D^1 \cup B_1 \cup \dots \cup B_{m-1} \cup B_{m,1} \cup B_{m+1,1},$$

$$M_2 = D^1 \cup B_1 \cup \dots \cup B_{m-1} \cup B_{m,1} \cup B_{m+1,2},$$

$$M_3 = D^1 \cup B_1 \cup \dots \cup B_{m-1} \cup B_{m,2} \cup B_{m+1,2},$$

each of which has disjoint B's and each of which is equivalent to the initial meander $M = D^1 \cup Y$.

If the meander M is of type III, then there are four ("quasistandard") meanders

$$M_1 = D^1 \cup B_1 \cup \dots \cup B_{m-1} \cup B_{m,1} \cup B_{m+1,1},$$

$$M_2 = D^1 \cup B_1 \cup \dots \cup B_{m-1} \cup B_{m,1} \cup B_{m+1,2},$$

$$M_3 = D^1 \cup B_1 \cup \dots \cup B_{m-1} \cup B_{m,2} \cup B_{m+1,2},$$

$$M_4 = D^1 \cup B_1 \cup \dots \cup B_{m-1} \cup B_{m,2} \cup B_{m+1,1},$$

each of which has disjoint B's and each of which is equivalent to the initial meander $M = D^1 \cup Y$. Let

$$L_j = D^2 \cap \{(x,y) \in \mathbb{R}^2 |\ y = \frac{1}{2}(a_{j-1} + a_j)\},$$

where $j = 0, 1, \dots, m, m+1$, $a_0 = -1$, and $a_{m+1} = 1$. Let n_{ij} be the number of points in the set $M_i \cap L_j$. Denote $n_i = \max\{n_{i0}, n_{i1}, \dots, n_{i,m+1}\}$. The number $n = \min\{n_1, \dots, n_p\}$ (where $p = 3$, if m is even; and $p = 4$, if m is odd) is the *meandrous number* of the meander $M = D^1 \cup Y$; and if n_i is the minimum, then the meander M_i is the *standard form* of M.

An m-meander with meandrous number n is called a *topological* (m, n)-*meander*.

If $m = 1, 2, 3$, then $n = 1$. If $m \geq 4$, then $n \leq m - 2$. If $m \geq 4$ is even, then $n \equiv m \pmod 2$.

It is obvious that there are no $(m, 1)$-meanders of type I. The classification of topological $(m, 1)$-meanders of type II (m is even) and III (m is odd) is trivial: for each $m \in \mathbb{N}$, there is only one equivalence class of such meanders, whose representative, for example, is $D^1 \cup \mathbb{R}f$, where

$$f(x, y, 1) = y - (x - x_1) \dots (x - x_m)$$

and $-1 < x_1 < \dots < x_m < 1$. If $m \geq 3$, then the classification of $(m, m-2)$-meanders is simple; one can check that for each $m \in \mathbb{N}$ and for each type I, II (m is even), and III (m is odd), there is only one equivalence class of such meanders.

Condition 2.1. *Consider an (m, n)-meander M in the standard form. Let $a_1 < \dots < a_m$ be points of intersection of Y and D^1. If M is of type* I, *then the lines $x = x_0$, where $-1 < x_0 < a_1$ or $a_m < x_0 < 1$, does not intersect Y. If M is of type* II *or* III, *then either for $-1 < x_0 < a_1$ or for $< a_m < x_0 < 1$ the family of lines $x = x_0$ does not intersect Y. We can subject the pair (D^2, M), if it is necessary, to transformation $(x, y) \mapsto (-x, y)$, and reenumerate the points $a_1 < \dots < a_m$ such that the family of lines $x = x_0$, where $-1 < x_0 < a_1$, does not intersect Y. In the remainder of the paper, without loss of generality, we consider only $(2, n)$-meanders in the standard form that satisfy this condition.*

2.3. Topological $(m, 2)$-meanders. In this section, we give a classification of $(m, 2)$-meanders (the topological classification of pairs (D^2, M)). Note that if M is an $(m, 2)$-meander of type I, then $m \geq 2$; if of type II, then $m \geq 4$; and if of type III, then $m \geq 5$.

Let $M = D^1 \cup Y$ be an $(m, 2)$-meander, and $-1 < a_1 < \cdots < a_m < 1$ be the points of intersection of D^1 and Y. Let $L_k = x_k$, where $k = 1, 2, \ldots, [\frac{m-1}{2}] + 1$, be a finite family of lines with $a_{2k-1} < x_k < a_{2k}$; and $[a]$ means the greatest integer in a. If m is odd, then we define $a_{m+1} = 1$.

The fragments $N_1, N_2, \ldots, N_{[\frac{m-1}{2}]}$ of the pair (D^2, M), defined by the formulas $N_k = (P_k, M \cap P_k)$, where

$$P_k = D^2 \cap ((a_{2k-1}, a_{2k+1}) \times \mathbb{R}x),$$

are diffeomorphic to the pair

$$N = \big((-\pi, \pi) \times [-3, 3],\ \mathbb{R}f_1 \cup \mathbb{R}f_2 \cup ((-\pi, \pi) \times 0)\big)\big),$$

where $f_1 = y - \cos x$, $f_2 = y - 2$, and $-\pi < x < \pi$. If the diffeomorphism $N_k \to N$ preserves the orientation, then we assign to the fragment N_k the index $i_k = 0$, otherwise we assign the index $i_k = 1$. If the pair (D^2, M) contains the fragments $N_1, N_2, \ldots, N_{[\frac{m-1}{2}]}$, then the *topological* $(m, 2)$-meander is said to have the *code*

$$i = (i_1, i_2, \ldots, i_{[\frac{m-1}{2}]}).$$

Every $(m, 2)$-meander of type I has exactly one code $i = (i_1, i_2, \ldots, i_{\frac{m-2}{2}})$, and if m is even, then every $(\frac{m-2}{2})$-tuple $(i_1, i_2, \ldots, i_{\frac{m-2}{2}})$ of zeros and ones defines an $(m, 2)$-meander of type I. Every $(m, 2)$-meander of type II (of type III) has exactly one code $i = (i_1, i_2, \ldots, i_{[\frac{m-1}{2}]})$, and every $[\frac{m-1}{2}]$-tuple $(i_1, i_2, \ldots, i_{[\frac{m-2}{2}]})$ of zeros and ones, except $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$, defines an $(m, 2)$-meander of type II (of type III).

Let the operation bar be defined in the set of indices $\{0, 1\}$ by the rule $\bar{0} = 1$ and $\bar{1} = 0$. This operation induces a reflection of each fragment N_k with respect to the axis $\mathbb{R}y$.

The maps id_{D^2}, r_{-+}, r_{+-}, $r_{--} : D^2 \to D^2$, defined by the formulas in Sec. 1, are homeomorphisms, and thus, the meanders M, $r_{-+}(M)$, $r_{+-}(M)$, and $r_{--}(M)$ are equivalent. These maps induce the maps

$$\mathrm{id}_C,\ \rho_{-+},\ \rho_{+-},\ \rho_{--} : C \to C$$

in the set

$$C = \{i = (i_1, i_2, \ldots, i_{[\frac{m-1}{2}]}) \mid i_k = 0 \text{ or } 1\}$$

of codes of $(m,2)$-meanders, defined by the formulas:

$$\mathrm{id}_C(i) = i,\ \rho_{-+}(i) = (i_{[\frac{m-1}{2}]}, \dots, i_2, i_1),$$

$$\rho_{+-}(i) = \bar{i} = (\bar{i}_1, \bar{i}_2, \dots, \bar{i}_s),\ \rho_{--}(i) = (\rho_{-+} \circ \rho_{+-})(i).$$

The set

$$G = \{\mathrm{id}_{D^2}, r_{-+}, r_{+-}, r_{--}\}$$

is a group with the operation of composition of the maps, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem 2.2. (1) *If* $m \equiv 0 \pmod 4$, *then there are* $2^{\frac{m}{4}-2} + 2^{\frac{m}{2}-3}$ *equivalence classes of* $(m,2)$*-meanders of type* I. *If* $m = 2$, *then there is one equivalence class of* $(2,2)$*-meanders of type* I. *If* $m \equiv 2 \pmod 4$ *with* $m \geq 6$, *then there are* $2^{\frac{m-2}{4}-1} + 2^{\frac{m-2}{2}-2}$ *equivalence classes of* $(m,2)$*-meanders of type* I.

(2) *There are* $2^{\frac{m-2}{2}} - 1$ *equivalence classes of* $(m,2)$*-meanders of type* II.

(3) *There are* $2^{\frac{m-3}{2}} - 1$ *equivalence classes of* $(m,2)$*-meanders of type* III.

Proof. 1) The set

$$C = \{(i_1, i_2, \dots, i_{\frac{m-2}{2}}) \mid i_k = 0 \text{ or } 1\}$$

of codes of $(m,2)$-meanders of type I contains $2^{\frac{m-2}{2}}$ elements (codes). The group G defines a natural action $G \times C \to C$ on the set C by the formula $(g,i) = g(i)$, where $g \in G$. The number $\mathrm{orb}\,(G)$ of orbits of codes under this action equals the number of equivalence classes of $(m,2)$-meanders of type I. There are two kinds of such orbits: the orbits consisting of two codes and the orbits consisting of four codes. On the other hand, there are two kinds of orbits consisting of two codes: orbits with the property

$$\rho_{-+}(i) = i \tag{1}$$

and orbits with the property

$$\rho_{+-}(i) = i. \tag{2}$$

If σ is the number of codes satisfying (1) and τ is the number of codes satisfying (2), then the number of codes, which constitute the orbits consisting of four codes, equals to $2^{\frac{m-2}{2}} - \sigma - \tau$. Thus, the total number of orbits of the action of G is

$$\mathrm{orb}\,(g) = \frac{\sigma + \tau}{2} + \frac{2^{\frac{m-2}{2}} - \sigma - \tau}{4} = \frac{\sigma + \tau + 2^{\frac{m-2}{2}}}{4}. \tag{3}$$

If $m = 4k$, where $k \in \mathbb{N}$, then $\sigma = 2^k$ and $\tau = 0$. If $m = 2$, then $C = \emptyset$; and in this trivial case, there is one equivalence class of $(2,2)$-meanders of type I. If $m = 4k+2$, where $k \geq 2$, then $\sigma = 2^k$ and $\tau = 2^k$. Substitution of this data into (3) completes the proof.

The proof of items 2) and 3) is trivial. $\square$

Remark 2.3. Let $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. Let $\mathrm{ar}_T(m)$ be the number of equivalence classes of topological m-meanders $M = D^1 \cup Y$ in D^2 of type T. Let $\mathrm{ar}_T(m,n)$ be the number of equivalence classes of topological (m,n)-meanders. It is easy to see that for $k \geq 1$ we have

$$\mathrm{ar}_{\mathrm{I}}(2k-1) = \mathrm{ar}_{\mathrm{II}}(2k-1) = \mathrm{ar}_{\mathrm{III}}(2k) = 0, \quad \mathrm{ar}_{\mathrm{II}}(6,2) = 3,$$

and if $m \geq 3$, then $\mathrm{ar}_T(m) = \sum_{n=1}^{m-2} \mathrm{ar}_T(m,n)$. We already know that

$$\mathrm{ar}_{\mathrm{I}}(2k,1) = \mathrm{ar}_{\mathrm{II}}(2k,1) = \mathrm{ar}_{\mathrm{III}}(2k-1,1) = 1,$$

and the numbers $\mathrm{ar}_T(m,2)$ are provided by Theorem 2.2. We announce the following formulas: if $k \geq 2$, then

$$\mathrm{ar}_{\mathrm{I}}(2k,2k-2) = \mathrm{ar}_{\mathrm{II}}(2k,2k-2) = \mathrm{ar}_{\mathrm{III}}(2k-1,2k-3) = 1;$$

if $k \geq 4$, then

$$\mathrm{ar}_{\mathrm{I}}(2k,2k-4) = 4k-8, \quad \mathrm{ar}_{\mathrm{II}}(2k,2k-4) = 4k-7,$$

and

$$\mathrm{ar}_{\mathrm{III}}(2k-1,2k-5) = 2k-4.$$

2.4. Algebraic (a,b)-meanders. Let $\mathbb{V} = (\Delta, v)$ be a chart. Denote $I_x = \Delta \cap \mathbb{R}y$. Let w be a connected component of v. If $I_x \cap w \neq \emptyset$, then the set $I_x \cup w$ is called a *local algebraic meander* (or briefly *a local meander*) of the chart $\mathbb{V}$.

If f is a polynomial without x and y factors having a singular point at the origin, then one of the terminal points of the Newton diagram of the singular point belongs to $\mathbb{R}y$ and the other one belongs to $\mathbb{R}x$; denote them as $(a,0)$ and $(0,b)$. If f is peripherally nondegenerate, then a and b are the multiplicities of the intersection at the singular point of the curve f with axes $\mathbb{R}y$ and $\mathbb{R}x$, respectively. The numbers a and b provide a restriction on the topology of local meanders. If $I_x \cup w_1, I_x \cup w_2, \ldots, I_x \cup w_l$ are local meanders provided by a chart $\mathbb{V} = (\Delta, v)$, then by the Bezout theorem, the number of points of the set $I_x \cap (w_1 \cup w_2 \cdots \cup w_l)$ is at most a.

Let $\tilde{f}$ be a curve, which is obtained as a result of patchworking the curve f by the chart of smoothing $\mathbb{V} = (\Delta, v)$. Let F be a closed neighborhood of the singular point of the curve f such that the topological pairs (Δ, v) and $(F, F \cap \mathbb{R}\tilde{f})$ are homeomorphic. By the Bezout theorem, every set $F \cap \mathbb{R}(x - x_0)$ with $(x_0, 0) \in F$ intersects the set $F \cap \mathbb{R}\tilde{f}$ at no more than b points. The numbers a and b are called the *Bezout numbers* of the local meander.

If a local meander $I_x \cup w$ has Bezout numbers a and b, the set w consists of one connected component, and the number of points of the set $I_x \cap w$ is equal to a, then the meander is called a *local maximal algebraic* (a, b)*-meander* or briefly an *algebraic* (a, b)*-meander*.

Every homeomorphism $h : (\Delta,\ I_x) \to (D^2,\ D^1)$ transfers an algebraic (a, b)-meander $I_x \cup w$ onto the corresponding topological (m, n)-meander $D^1 \cup Y$ with $m = a$ and $n \le b$. If the homeomorphism h transfers an algebraic (a, b)-meander $I_x \cup w$ onto the corresponding topological (m, n)-meander $D^1 \cup Y$ of type I or II or III, then the algebraic (a, b)-meander $I_x \cup w$ is said to be of *type* I or II or III, respectively.

Two meanders $I_x \cup w_1$ and $I_x \cup w_2$ are *equivalent* if the topological pairs $(\Delta,\ I_x \cup w_1)$ and $(\Delta,\ I_x \cup w_2)$ are homeomorphic.

Two charts $\mathbb{V} = (\Delta,\ v)$ and $\mathbb{V}' = (\Delta,\ v')$ are *meanderingly equivalent* if the topological pairs $(\Delta, I_x \cup v)$ and $(\Delta, I_x \cup v')$ are homeomorphic.

§3. Meanders of patchworking A_μ-germs

We consider germs of the singularity A_μ, whose Newton diagram consists of one segment $\Gamma_\mu = [(\mu + 1, 0), (0, 2)]$, where $\mu = 1, 2, \dots$. If $h = 0$ is an equation of a curve with such a singular point, then the standard equation of its A_μ-germ is defined by the Γ_μ-truncation: $h^{\Gamma_\mu} = 0$. From the real point of view, there are three kinds of germs of such a singularity: A^-_{2k-1}, A^+_{2k-1}, and A_{2k}, where $k \ge 1$.

The A^-_{2k-1}- and A^+_{2k-1}-germs have the standard equation

$$h^{\Gamma_{2k-1}} = y^2 + \alpha x^k y + \beta x^{2k} = 0,$$

where $\beta \ne 0$. An A^-_{2k-1}-germ (resp., A^+_{2k-1}-germ) is defined by the condition $\alpha^2 - 4\beta > 0$ (resp., $\alpha^2 - 4\beta < 0$). An A^-_{2k-1}-germ (resp., A^+_{2k-1}-germ) has two real (resp., two complex conjugate) branches. One can check that the necessary condition, which provides the maximal local meanders arising from germs of A^-_{2k-1}, is "both roots of the quasihomogeneous polynomial $y^2 + \alpha x^k y + \beta x^{2k}$ are of the same sign;" this means that $\beta > 0$.

An A_{2k}-germ has the standard equation

$$h^{\Gamma_{2k}} = y^2 + \gamma x^{2k+1} = 0,$$

where $\gamma \neq 0$. Below we consider the germs having the standard equations.

Lemma 3.1. *There exists a polynomial $c(x, y, z)$, which has the chart depicted in Fig. 1.1, whose Γ_0- and Γ_3-truncations are*

$$c^{\Gamma_0} = y^2 - \psi x \quad \text{and} \quad c^{\Gamma_3} = y^2 + \alpha x^k y + \beta x^{2k},$$

where $\alpha^2 - 4\beta > 0$, $\beta > 0$, and ψ is sufficiently small positive number.

Proof. We use the small parameter method in two steps. First, it is easy to check that the hyperbola

$$c^*(x, y, 1) = (y + \kappa)^2 + \alpha(x - x_0)(y + \kappa) + \beta(x - x_0)^2 + \varepsilon(y + \kappa)$$

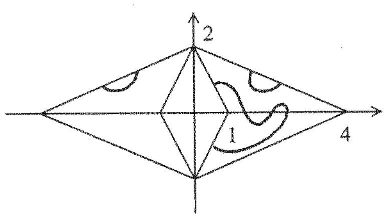
with $\alpha^2 - 4\beta > 0$, $\beta > 0$, $x_0 > 0$, with sufficiently small $\varepsilon > 0$, and

$$0 < \kappa < \frac{\alpha x_0 + \varepsilon - \sqrt{(\alpha x_0 + \varepsilon)^2 - 4\beta x_0^2}}{2}$$

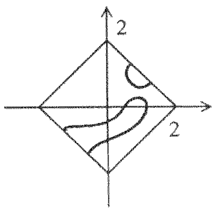
has the chart depicted in Fig. 1.2. And second, for sufficiently small ψ, $0 < \psi \ll \varepsilon$, the polynomial

$$c(x, y, 1) = (c^* \circ hy)(x, y, 1) - \psi x$$

satisfies the requirements of the lemma. $\square$



1. $\mathbb{V}(c^* \circ hy + \psi x)$



2. $\mathbb{V}(c^*)$

Fig. 1.

Denote

$$a(x,y,1)=(y-\frac{1}{2})(y-2)-x,\ b(x,y,1)=(y-\frac{1}{2}x)(y-2x)-x,$$

$$a_1(x,y,1)=(y-\frac{1}{2})(y-2)+\gamma x,\ b_1(x,y,1)=y^2+\alpha xy+\beta x^2-x,$$

and

$$c_1(x,y,1)=(y-\frac{1}{2})(y-2)-\psi x.$$

Note that $b=a\circ s_2$.

Theorem 3.2. (1) *Every A^-_{2k-1}-germ with $\alpha^2-4\beta>0$ and $\beta>0$ admits patchworking with the charts*

$$\begin{array}{llclcl}
\mathbb{V}(f_{2k}^{i_0,i_1,\dots,i_{k-1}}) & = & & & & \mathbb{V}(a\circ r_{+-}^{i_0}) \\
 & \cup & \mathbb{V}(b\circ r_{+-}^{i_1}) & \cup & & \mathbb{V}(a\circ hy\circ r_{+-}^{i_1}) \\
 & \cup & \mathbb{V}(b\circ hy\circ r_{+-}^{i_2}) & \cup & & \mathbb{V}(a\circ hy^2\circ r_{+-}^{i_2})\cup \\
 & \dots & \dots\dots\dots & \dots & & \dots\dots\dots \\
 & \cup & \mathbb{V}(b\circ hy^{k-2}\circ r_{+-}^{i_{k-1}}) & \cup & & \mathbb{V}(a\circ hy^{k-1}\circ r_{+-}^{i_{k-1}}) \\
 & \cup & \mathbb{V}(b_1\circ hy^{k-1}) & & &
\end{array} \tag{4}$$

and with the charts

$$\begin{array}{llclcl}
\mathbb{V}(f_{2k}^{i_0,i_1,\dots,i_{k-2},1}) & = & & & & \mathbb{V}(a\circ r_{+-}^{i_0}) \\
 & \cup & \mathbb{V}(b\circ r_{+-}^{i_1}) & \cup & & \mathbb{V}(a\circ hy\circ r_{+-}^{i_1}) \\
 & \cup & \mathbb{V}(b\circ hy\circ r_{+-}^{i_2}) & \cup & & \mathbb{V}(a\circ hy^2\circ r_{+-}^{i_2})\cup \\
 & \dots & \dots\dots\dots & \dots & & \dots\dots\dots \\
 & \cup & \mathbb{V}(b\circ hy^{k-3}\circ r_{+-}^{i_{k-2}}) & \cup & & \mathbb{V}(c_1\circ hy^{k-2}\circ r_{+-}^{i_{k-2}}) \\
 & \cup & \mathbb{V}(c\circ hy^{k-2}), & & &
\end{array} \tag{5}$$

where $i_t\in\{0,1\}$.

(2) *Every A^+_{2k-1}-germ with $\alpha^2-4\beta<0$ admits patchworking with the charts defined by formula (4).*

(3) *Every A_{2k}-germ admits patchworking with the charts*

$$\begin{array}{llclcl}
\mathbb{V}(f_{2k+1}^{i_0,i_1,\dots,i_k}) & = & & & & \mathbb{V}(a\circ r_{+-}^{i_0}) \\
 & \cup & \mathbb{V}(b\circ r_{+-}^{i_1}) & \cup & & \mathbb{V}(a\circ hy\circ r_{+-}^{i_1}) \\
 & \cup & \mathbb{V}(b\circ hy\circ r_{+-}^{i_2}) & \cup & & \mathbb{V}(a\circ hy^2\circ r_{+-}^{i_2})\cup \\
 & & \dots\dots\dots & & & \dots\dots\dots \\
 & \cup & \mathbb{V}(b\circ hy^{k-2}\circ r_{+-}^{i_{k-1}}) & \cup & & \mathbb{V}(a\circ hy^{k-1}\circ r_{+-}^{i_{k-1}}) \\
 & \cup & \mathbb{V}(b\circ hy^{k-1}\circ r_{+-}^{i_k}) & \cup & & \mathbb{V}(a_1\circ hy^k\circ r_{+-}^{i_k}),
\end{array} \tag{6}$$

where $i_t \in \{0, 1\}$.

(4) *The charts* (4) *for both* $\alpha^2 - 4\beta > 0$ *and* $\alpha^2 - 4\beta < 0$, (5), *and* (6) *represent local maximal meanders and are the charts of* M*-smoothing.*

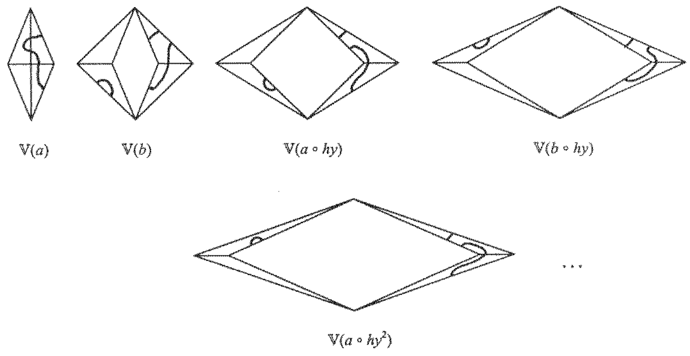


Fig. 2.

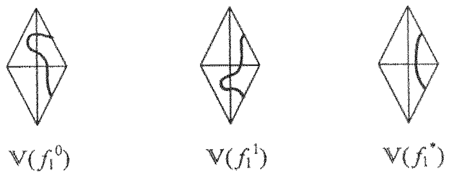


Fig. 3. $d = 1$.

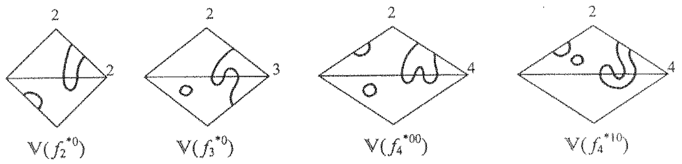


Fig. 4. $d = 2, 3, 4$.

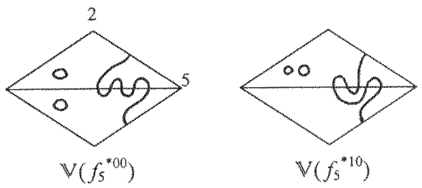


Fig. 5. $d = 5$.

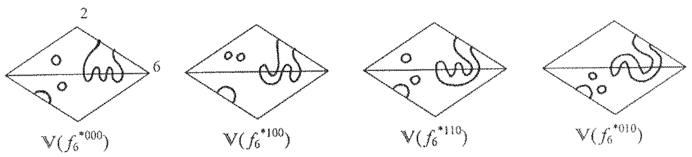


Fig. 6. $d = 6$.

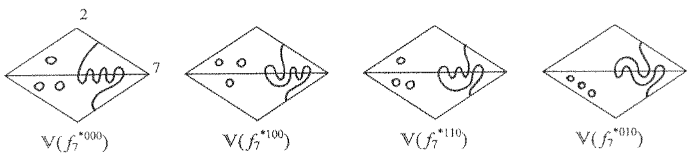


Fig. 7. $d = 7$.

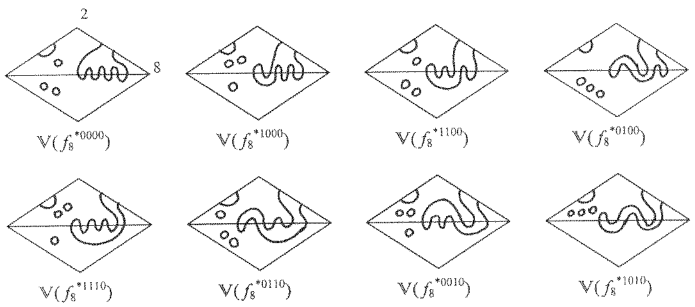


Fig. 8. $d = 8$.

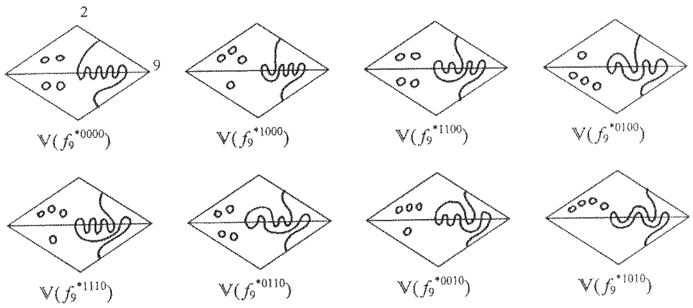


Fig. 9. $d = 9$.

Proof. Compose the sequence of polynomials

$$a,\ b,\ a \circ hy,\ b \circ hy,\ a \circ hy^2,\ b \circ hy^2,\ \ldots \tag{7}$$

with the common term $a \circ s_2^n \circ hy^{[\frac{n}{2}]}$. The first five charts of the polynomials of the sequence are depicted in Fig. 2. The polynomials of this sequence and the polynomials, which constitute the formulas (4), (5), and (6), satisfy the Viro patchwork theorem. Patchworking them, we get the required charts of M-smoothing. □

Example 3.3. In Figs. 3–9, we show the charts $\mathbb{V}(f_d^{i_0,i_1,\ldots,i_s})$ of M-smoothing with local maximal meanders of type II and III. Namely, for $d = 2, 4, 6, 8$ (type II), the charts are provided by the formula (4) and satisfy the condition $\alpha^2 - 4\beta > 0$ (and $\beta > 0$); and for $d = 1, 3, 5, 7, 9$ (type III), the charts are provided by the formula (6). The charts of smoothing of the A_{2k-1}^--germ defined by the formula (5), the A_{2k-1}^+-germs defined by formula (4) with $\alpha^2 - 4\beta < 0$, and the A_{2k}-germ defined by the formula (6) can be drawn similarly. Denote

$$f_1^*(x, y, z) = y^2 - x + 1.$$

To reduce the number of pictures in Figs. 4–9, we use the chart $\mathbb{V}(f_1^*)$ (see Fig. 3) instead of the charts $\mathbb{V}(a \circ r_{+-}^{i_1}) = \mathbb{V}(f_1^{i_1})$.

One can note that each index i_t, where $t = 1, 2, \ldots$, completely defines a fragment of the meander of the chart

$$\mathbb{V}(b \circ hy^{t-1} \circ r_{+-}^{i_t}) \cup \mathbb{V}(\tilde{a} \circ hy^t \circ r_{+-}^{i_t}),$$

where $\tilde{a} = a, a_1, c_1$. Each such a fragment consists of a piece of the meander and an oval (see Fig. 10). The chart $\mathbb{V}(c \circ hy^{k-2})$ also has such a fragment with index $i_{k-1} = 1$ (see (5)), but does not have a corresponding oval. One can see that the charts $\mathbb{V}(f_{2k}^{i_0, i_1, \dots, i_{k-1}})$ and $\mathbb{V}(f_{2k+1}^{i_0, i_1, \dots, i_k})$ can be restored by means of $(i_1, \dots, i_{k-1})$ and $(i_1, \dots, i_k)$, respectively.

The $(k-1)$- and k-tuples $(i_1, \dots, i_{k-1})$ and $(i_1, \dots, i_k)$ are called *codes of local maximal meanders* of the charts of the M-smoothing $A_{2k}^{\mp}$- and A_{2k}-germs, respectively. The codes of algebraic $(m, 2)$-meanders coincide with the codes of corresponding topological $(m, 2)$-meanders. Thus, we have:

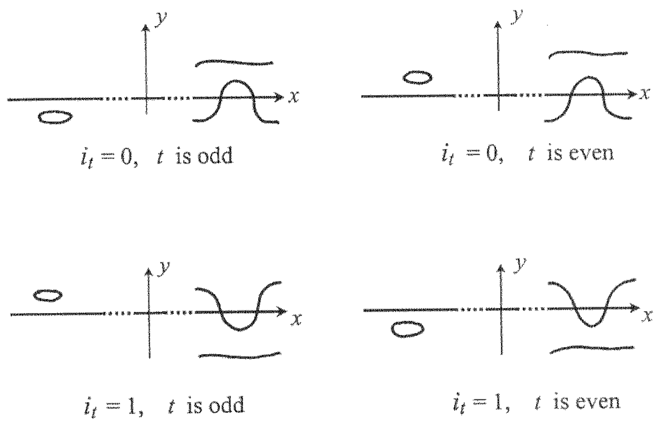


Fig. 10.

Remark 3.4. 1) Formula (4) for Theorem 3.2.1) provides all 2^{k-1} equivalence classes of algebraic $(2k, 2)$-meanders of type II, namely, representatives of algebraic $(2k, 2)$-meanders with codes $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$) realize the class of topological $(2k, 1)$-meanders (which is unique), and representatives of the remaining $2^{k-1} - 1$ classes realize all equivalence classes of topological $(2k, 2)$-meanders provided by Theorem 2.2.2).

2) Formula (5) for Theorem 3.2.1) provides all equivalence classes of algebraic $(2k, 2)$-meanders of type I, namely, if $k \geq 2$ is even, then the formula provides representatives of algebraic $(2k, 2)$-meanders that realize $2^{\frac{k}{2}-2} + 2^{k-3}$ equivalence classes of topological $(2k, 2)$-meanders of type I; and if $k \geq 3$ is odd, then the formula provides representatives of algebraic $(2k, 2)$-meanders that realize $2^{\frac{k-1}{2}-1} + 2^{k-3}$ equivalence classes of

topological $(2k,2)$-meanders of type I; where in both cases the toplogical $(2k,2)$-meanders are provided by Theorem 2.2.1).

3) Formula (4) for Theorem 3.2.2) provides all equivalence classes of algebraic $(2k,2)$-meanders of type I described in the previous item 2).

4) Formula (6) for Theorem 3.2.3) provides all 2^{k-1} equivalence classes of algebraic $(2k+1,2)$-meanders, namely, representatives of algebraic $(2k+1,2)$-meanders with codes $(0,0,\dots,0)$ and $(1,1,\dots,1)$) realize one class of topological $(2k+1,1)$-meanders (which is unique), and representatives of the remaining $2^{k-1}-1$ classes realize all topological classes of topological $(2k+1,2)$-meanders of type III provided by Theorem 2.2.3).

Corollary 3.5. *Every topological $(m,1)$- and $(m,2)$-meander is realizable by an algebraic $(m,2)$-meander.*

Remark 3.6. Consider the case of smoothing A^-_{2k-1}- and A_{2k}-germs provided by formulas (4) and (5), respectively. One can handle in a similar way the cases of smoothing A^-_{2k-1}- and A_{2k}-germs provided by formulas (5) and (4), respectively.

Let $p(i_t)$ and $n(i_t)$ be the number of ovals lying above and below the axis $\mathbb{R}y$, respectively, in the fragment with index i_t. Let $p(i_1,\dots,i_s)$ and $n(i_1,\dots,i_s)$ be the numbers of ovals of the curve $yf_d^{i_0,i_1,\dots,i_s}=0$ lying above and below the axis $\mathbb{R}y$, respectively, which has the chart with the code $(i_1,\dots,i_s)$, where

$$s=\left[\frac{d-1}{2}\right]=\frac{2d-3-(-1)^d}{4}$$

and $d=2k$ or $2k+1$. It is clear that

$$p(i_t)+n(i_t)=1;\quad p(i_1,\dots,i_s)=\sum_{t=1}^{s}p(i_t);\quad n(i_1,\dots,i_s)=\sum_{t=1}^{s}n(i_t);$$

and one can check that Theorem 1.1 implies

$$p(i_1,\dots,i_s)+n(i_1,\dots,i_s)=s.$$

One can see (Fig. 10) that

$$p(i_t)=\frac{1-(-1)^{t+1}}{2}+(-1)^{t+1}i_t$$

and can calculate that

$$p(i_1,\dots,i_s)=\frac{2s-1+(-1)^s}{4}+\sum_{t=1}^{s}(-1)^{t+1}i_t \tag{8}$$

and

$$n(i_1, \dots, i_s) = \frac{2s+1-(-1)^s}{4} - \sum_{t=1}^{s} (-1)^{t+1} i_t. \tag{9}$$

From (8) and (9) it follows that the curves $f_d^{i_0, i_1, \dots, i_s} = 0$, where $d \geq 2$, satisfy the condition

$$p(i_1, \dots, i_s) - n(i_1, \dots, i_s) = \frac{(-1)^s - 1}{2} + 2\sum_{t=1}^{s} (-1)^{t+1} i_t.$$

Let $g_d^{i_0, i_1, \dots, i_s} = 0$ be a curve which has

1) the same Newton polygon as the curve $f_d^{i_0, i_1, \dots, i_s} = 0$,
2) a meander equivalent to the meander of $f_d^{i_0, i_1, \dots, i_s} = 0$, and
3) the same maximal number s of ovals.

According to [7], the curve $yg_d^{i_0, i_1, \dots, i_s} = 0$ for odd d and the curve $yzg_d^{i_0, i_1, \dots, i_s} = 0$ for even d must satisfy the congruence

$$p(i_1, \dots, i_s) - n(i_1, \dots, i_s) \equiv \equiv \frac{(-1)^s - 1}{2} + 2\sum_{t=1}^{s} (-1)^{t+1} i_t \pmod 8. \tag{10}$$

which is an extension of the Gudkov–Rokhlin congruence, and where $s = [\frac{d-1}{2}]$. From (10) it follows that for $d = 2, 3, \dots, 8$, we obtain the classification of M-smoothing with local maximal meanders of A_1-, A_2-, ... , A_7-germs, respectively, and with the standard equations (see Figs. 4–8). (To complete such a classification for $d = 9$, one must prove or disprove the existence of an M-smoothing, whose chart has a local maximal meander, which is equvalent to the meander of the chart $\mathbb{V}(f_9^{*1010})$, and has $p(1010) = 0$ and $n(1010) = 4$, see Fig. 9).

Conjecture 3.7. *Every chart of M-smoothing an A_μ-germ with local maximal meander is meanderingly equivalent to one of the charts of M-smoothing provided by Theorem* 3.2.

Remark 3.8. In Theorem 3.2 we applied the symmetry r_{+-} to the polynomials of the sequence (7). If in addition we apply, in an appropriate manner, symmetries r_{-+} and r_{--}, then we can obtain local meanders which are not maximal.

§4. Meanders of patchworking D_μ-germs

We consider germs of the singularity D_μ, whose Newton diagram consists of two segments $\Delta = [(0,3),(1,2)]$ and $\Gamma_\mu = [(1,2),(\mu-1,0)]$, where $\mu = 4, 5, \dots$. If $h = 0$ is an equation of a completely nondegenerate curve with such a singular point, then the standard equation of the D_μ-germ is defined by the Γ_μ-truncation: $h^{\Gamma_\mu} = 0$. Let $h^\Delta = xy^2 + \varepsilon y^3$, where, without loss of generality, we choose $\varepsilon < 0$. From the real point of view, there are three kinds of germs of such a singularity: D_{2k}^-, D_{2k}^+, and D_{2k+1}, where $k \geq 2$.

The D_{2k}^-- and D_{2k}^+-germs have the standard equation $h^{\Gamma_{2k}} = xy^2 + \alpha_1 x^k y + \beta_1 x^{2k-1} = 0$, where $\beta_1 \neq 0$, and satisfy the conditions $\alpha_1^2 - 4\beta_1 > 0$ and $\alpha_1^2 - 4\beta_1 < 0$, respectively. A D_{2k}^--germ has three real branches. A D_{2k}^+-germ has one real and two complex conjugate branches.

A D_{2k+1}-germ has the standard equation

$$h^{\Gamma_{2k+1}} = xy^2 + \beta x^{2k} = 0,$$

where $\beta \neq 0$, and has two real branches.

To get local maximal meanders of M-smoothing, we reduce a D_μ-germ to a $A_{\mu-2}$-germ and then apply Theorem 3.2.

Denote

$$p_1(x,y,1) = \zeta + x + \alpha_1 y + \beta_1 y^2,$$

$$q(x,y,1) = \zeta + x + \alpha y + \beta y^2,$$

and

$$r(x,y,1) = \zeta y^2 + xy^2 + \varepsilon y^3.$$

Lemma 4.1. (1) *Every $D_{2k}^\mp$-germ with the standard equation admits patchworking with the chart of smoothing*

$$\mathbb{V}(p_1 \circ ha \circ s_3 \circ ha \circ s_3 \circ hy^{k-1}) \cup \mathbb{V}(r).$$

(1.1) *For a D_{2k}^--germ with even k, $\beta > 0$, and $\zeta < 0$, it is depicted in Fig.* 11.1.

(1.2) *For a D_{2k}^--germ with odd k, $\beta > 0$, and $\zeta < 0$, it is depicted in Fig.* 11.2.

(1.3) *For a D_{2k}^--germ with $\beta < 0$ and $\zeta > 0$, it is depicted in Fig.* 11.3.

(1.4) *For a D_{2k}^+-germ with $\zeta < 0$, it is depicted in Fig.* 11.4.

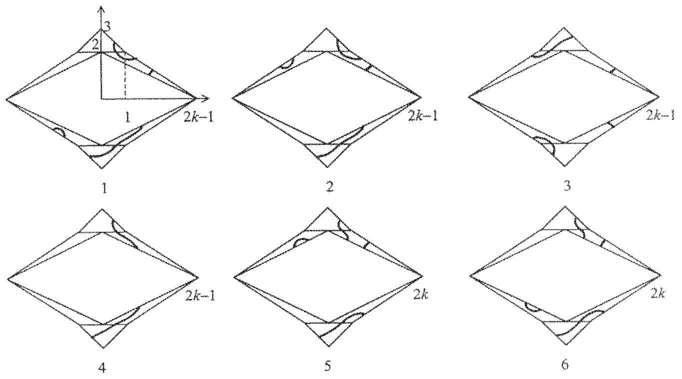


Fig. 11.

(2) *Every D_{2k+1}-germ of a singularity with the standard equation admits patchworking with the chart of smoothing*

$$\mathbb{V}(q \circ ha \circ hy^k) \cup \mathbb{V}(r).$$

(2.1) *For D_{2k+1}-germ with odd k and $\zeta < 0$, it is depicted in Fig. 11.5.*

(2.2) *For D_{2k+1}-germ with even k and $\zeta < 0$, it is depicted in Fig. 11.6.*

Proof. It is easy to check that the polynomials $h^{\Gamma_{2k}}$, $h^{\Gamma_{2k+1}}$, h^{Δ}, p_1, q, and r satisfy the conditions of the Viro patchwork theorem; and thus, we get the required charts. $\square$

Theorem 4.2. (1) *Every $D^{\mp}_{2k}$-germ with the standard equation admits patchworking with the chart*

$$\mathbb{V}^{i_0, i_1, \dots, i_{k-1}}_{2k-1} = \mathbb{V}(f^{i_0, i_1, \dots, i_{k-1}}_{2k-1}) \cup$$
$$\cup \mathbb{V}(p_1 \circ ha \circ s_3 \circ ha \circ s_3 \circ hy^{k-1}) \cup \mathbb{V}(r), \tag{11}$$

where the chart $\mathbb{V}(f^{i_0, i_1, \dots, i_{k-1}}_{2k-1})$ is provided by Theorem 3.2.2).

(2) *Every D_{2k+1}-germ with the standard equation admits patchworking with the chart*

$$\mathbb{V}^{i_0, i_1, \dots, i_{k-1}, k}_{2k} = \mathbb{V}(f^{i_0, i_1, \dots, i_{k-1}}_{2k}) \cup \mathbb{V}(q \circ ha \circ hy^k) \cup \mathbb{V}(r), \tag{12}$$

where the chart $\mathbb{V}(f^{i_0, i_1, \dots, i_{k-1}}_{2k})$ is provided by Theorem 3.2.1).

(3) *The charts* $\mathbb{V}_{2k-1}^{i_0, i_1, \dots, i_{k-1}}$ *and* $\mathbb{V}_{2k}^{i_0, i_1, \dots, i_{k-1}, k}$ *represent local maximal meanders and are the charts of* M*-smoothing.*

Proof. First, we apply Lemma 4.1 and get the chart

$$\mathbb{V}(p_1 \circ ha \circ s_3 \circ ha \circ s_3 \circ hy^{k-1}) \cup \mathbb{V}(r) \cup \mathbb{V}(h)$$

in case 1) and the chart

$$\mathbb{V}(q \circ ha \circ hy^k) \cup \mathbb{V}(r) \cup \mathbb{V}(h)$$

in case 2). Second, we patchwork in the appropriate manner the previous charts and the charts provided by Theorem 3.2 and get the required charts of smoothing. □

The $(k-1)$-tuples $(i_1, \dots, i_{k-1})$ and $(i_1, \dots, i_{k-1}, k)$ are called *codes* of local meanders of the charts $\mathbb{V}_{2k-1}^{i_0, i_1, \dots, i_{k-1}}$ and $\mathbb{V}_{2k}^{i_0, i_1, \dots, i_{k-1}, (i_k)}$ of smoothing $D_{2k}^{\mp}$- and D_{2k+1}-germs, respectively.

Remark 4.3. 1) Let $p(i_1, \dots, i_{k-1})$ and $n(i_1, \dots, i_{k-1})$ be the numbers of ovals in the chart $\mathbb{V}_{2k-1}^{i_0, i_1, \dots, i_{k-1}}$ lying above and below the axis $\mathbb{R}y$, respectively. The fragment $\mathbb{V}(f_{2k-1}^{i_0, i_1, \dots, i_{k-1}})$ of the chart $\mathbb{V}_{2k-1}^{i_0, i_1, \dots, i_{k-1}}$ (see formula (11)) completely describes both the local meander and the arrangement of the ovals of the chart. One can check that Theorem 1.1 implies $p(i_1, \dots, i_{k-1}) + n(i_1, \dots, i_{k-1}) = k-1$. Using formulas (8) and (9), one can find that

$$p(i_1, \dots, i_{k-1}) = \frac{2k-3-(-1)^k}{4} + \sum_{t=1}^{k-1} (-1)^{t+1} i_t \tag{13}$$

and

$$n(i_1, \dots, i_{k-1}) = \frac{2k-1+(-1)^k}{4} - \sum_{t=1}^{k-1} (-1)^{t+1} i_t. \tag{14}$$

2) Let $p(i_1, \dots, i_{k-1}, k)$ and $n(i_1, \dots, i_{k-1}, k)$ be the numbers of ovals in the chart $\mathbb{V}_{2k}^{i_0, i_1, \dots, i_{k-1}, k}$ lying above and below the axis $\mathbb{R}y$, respectively. The fragment $(i_0, i_1, \dots, i_{k-1})$ completely describes the local meander of the chart and describes the arrangement of the ovals except one. This exceptional oval appears as a result of patchworking of charts $\mathbb{V}(f_{2k}^{i_0, i_1, \dots, i_{k-1}})$ and $\mathbb{V}(q \circ ha \circ hy^k)$ (see formula (12)); and the arrangement of the oval (above or below the axis $\mathbb{R}y$) can be described by the formulas

$$p(k) = \frac{1-(-1)^k}{2} \quad \text{and} \quad p(k) + n(k) = 1. \tag{15}$$

One can check that Theorem 1.1 implies $p(i_1, \dots, i_{k-1}, k) + n(i_1, \dots, i_{k-1}, k) = k$. Using formulas (8), (9) and (15) one can find that

$$p(i_1, \dots, i_{k-1}, k) = \frac{2k - 1 - 3(-1)^k}{4} + \sum_{t=1}^{k-1}(-1)^{t+1} i_t \tag{16}$$

and

$$n(i_1, \dots, i_{k-1}, k) = \frac{2k + 1 + 3(-1)^k}{4} - \sum_{t=1}^{k-1}(-1)^{t+1} i_t. \tag{17}$$

We already know that the pair of polynomials

$$f_{2k-1}^{i_0, i_1, \dots, i_{k-1}} \quad \text{and} \quad p_1 \circ ha \circ s_3 \circ ha \circ s_3 \circ hy^{k-1}$$

(resp., $f_{2k}^{i_0, i_1, \dots, i_{k-1}}$ and $q \circ ha \circ hy^k$) satisfies the conditions of the Viro patchwork theorem; thus, there exists a polynomial $g_{2k-1}^{i_0, i_1, \dots, i_{k-1}}$ (resp., $g_{2k}^{i_0, i_1, \dots, i_{k-1}, k}$) having the chart

$$\mathbb{V}(g_{2k-1}^{i_0, i_1, \dots, i_{k-1}}) = \mathbb{V}(f_{2k-1}^{i_0, i_1, \dots, i_{k-1}}) \cup \mathbb{V}(p_1 \circ ha \circ s_3 \circ ha \circ s_3 \circ hy^{k-1})$$

$$\big(\text{ resp., } \ \mathbb{V}(g_{2k}^{i_0, i_1, \dots, i_{k-1}, k}) = \mathbb{V}(f_{2k}^{i_0, i_1, \dots, i_{k-1}}) \cup \mathbb{V}(q \circ ha \circ hy^k)\big).$$

We can easily see that the numbers $p = p(i_1, \dots, i_{k-1})$ and $n = n(i_1, \dots, i_{k-1})$
(resp., $p = p(i_1, \dots, i_{k-1}, k)$ and $n = n(i_1, \dots, i_{k-1}, k)$) of ovals in the chart $\mathbb{V}(g_{2k-1}^{i_0, i_1, \dots, i_{k-1}})$ (resp., $\mathbb{V}(g_{2k}^{i_0, i_1, \dots, i_{k-1}, k})$) lying above and below the axis $\mathbb{R}y$ can be found by formulas (13) and (14) (resp., (16) and (17)) and satisfy the equation

$$p - n = -\frac{1 + (-1)^k}{2} + 2\sum_{t=1}^{k-1}(-1)^{t+1} i_t$$

$$\Big(\text{resp., } \ p - n = -\frac{1 + 3(-1)^k}{4} + \sum_{t=1}^{k-1}(-1)^{t+1} i_t\Big).$$

Let $\tilde{g}_{2k-1}^{i_0, i_1, \dots, i_{k-1}} = 0$ (resp., $\tilde{g}_{2k}^{i_0, i_1, \dots, i_{k-1}, k} = 0$) be a curve which has

1) the same Newton polygon as the curve $g_{2k-1}^{i_0, i_1, \dots, i_{k-1}} = 0$ (resp., $g_{2k}^{i_0, i_1, \dots, i_{k-1}, k} = 0$),

2) a meander equivalent to the meander of $g_{2k-1}^{i_0, i_1, \dots, i_{k-1}} = 0$ (resp., $g_{2k}^{i_0, i_1, \dots, i_{k-1}, k} = 0$), and

3) the same maximal number $k-1$ (resp., k) of ovals.

According to [7] the curve $y\tilde{g}_{2k-1}^{i_0,\,i_1,\,\dots,i_{k-1}} = 0$ (resp., $yz\tilde{g}_{2k}^{i_0,\,i_1,\,\dots,i_{k-1},k} = 0$) must satisfy the congruence

$$p - n = -\frac{1+(-1)^k}{2} + 2\sum_{t=1}^{k-1}(-1)^{t+1} i_t \pmod 8 \qquad (18)$$

$$\Big(\text{resp.,}\ \ p - n = -\frac{1+3(-1)^k}{4} + \sum_{t=1}^{k-1}(-1)^{t+1} i_t \pmod 8\Big) \qquad (19)$$

which is an extension of the Gudkov–Rokhlin congruence. From (18) (resp., (19)) it follows that, for $p+n \le 4$, which means that for $k = 2,4$ (resp., $k = 3,5$) we obtain the classification of M-smoothing D_4- and D_6–germs (resp., D_5- and D_7-germs) and the classification of [singular M-]curves $\tilde{g}_{2k-1}^{i_0,\,i_1,\,\dots,i_{k-1}} = 0$ (resp., $\tilde{g}_{2k}^{i_0,\,i_1,\,\dots,i_{k-1},k} = 0$) both with local maximal meanders.

Conjecture 4.4. *Every chart of M-smoothing a D_μ-germ with local maximal meander is meanderingly equivalent to one of the charts of M-smoothing provided by Theorem* 4.2.

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