Записки научных
семинаров ПОМИ
Том 300, 2003 г.

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ON TWO-DIMENSIONAL AREA-PRESERVING MAPS WITH HOMOCLINIC TANGENCIES THAT HAVE INFINITELY MANY GENERIC ELLIPTIC PERIODIC POINTS

ABSTRACT. Semi-local dynamics of two-dimensional symplectic diffeomorphisms with homoclinic tangencies are studied. Conditions when infinitely many generic elliptic periodic orbits exist of successive periods beginning with some integer are found.

The problem of existence of infinitely many generic elliptic periodic points was solved in [1, 2] for two-dimensional symplectic diffeomorphisms with simplest nontransversal heteroclinic cycles. A peculiarity of such diffeomorphisms is the presence of Ω-moduli (i.e., continuous invariants of local topological conjugacy on the set of nonwandering orbits) that allows to give effective criteria of the existence of many elliptic points.

General two-dimensional diffeomorphisms with homoclinic tangencies possess also Ω-moduli: such basic moduli are θ and τ [3, 4]. Here, $\theta = -\ln|\lambda|/\ln|\gamma|$ where λ and γ are the multipliers of the saddle fixed point with $0 < |\lambda| < 1 < |\gamma|$; the formula for τ includes also some coefficients characterizing a behavior of orbits near the homoclinic orbit. The existence of infinitely many stable (if $\theta > 1$) or unstable (if $\theta < 1$) periodic orbit is connected directly with arithmetic properties of moduli θ and τ [5, 6]. In the case of two-dimensional symplectic diffeomorphisms with homoclinic tangency, $\theta = 1$ and τ are not moduls, although they are invariant. Therefore, conditions of existence of infinitely many generic elliptic periodic orbits must have another nature, see [7] for more details. In particular, we find here a case of codim 2 homoclinic tangency (the quadratic homoclinic tangency plus $\tau = 0$ at the moment of the tangency) when infinitely many generic elliptic points exist and periods of

This work was supported in part by grant of INTAS No. 2000-221, grants of RFBR No. 02-01-00273 and No. 01-01-00905 and grant of "Russian Universities", project No. 1905.

these points compose a sequence $\{k_0, k_0+1, \dots,\}$ beginning with some sufficiently large integer k_0.

1. Properties of local and global maps

Consider a C^r-smooth ($r \geq 7$) two-dimensional area-preserving diffeomorphism f having a fixed saddle point O with multipliers λ and λ^{-1}, where $|\lambda| < 1$, and a nontransversal homoclinic orbit Γ at whose points the stable and unstable invariant manifolds of the saddle O have a quadratic tangency. In this paper, we consider the cases where $\lambda > 0$.

Let U be a sufficiently small neighborhood of the set $O \cup \Gamma$. It consists of a small disk U_0 containing the point O and a number of small disks containing those points of Γ that do not lie in U_0. Let T_0 be the restriction of f onto U_0. The point O is a fixed saddle point for T_0. It is well known that, in some (canonical) C^r-coordinates on U_0, the map T_0 can be written in the form

$$\bar{x} = \lambda x + f(x,y)x, \quad \bar{y} = \lambda^{-1} y + g(x,y)y,$$

where $f(0,0) = g(0,0) = 0$. In these coordinates the equations of W^s_{loc} and W^u_{loc} have the forms $y = 0$ and $x = 0$, respectively. In the case of C^r-smooth diffeomorphisms, where $r \leq \infty$, we will use finitely smooth normal forms. Namely, the following lemma is valid.

Lemma 1. *For any integer n such that $n < r/2$ (if $r = \infty$, then n is arbitrary), there exists a canonical change of coordinates of class C^{r-1} (if $n = 1$) or of class C^{r-2n} (if $n \geq 2$) under which T_0 takes the form*

$$\begin{aligned} \bar{x} &= \lambda x(1 + \beta_1 \cdot xy + \ldots + \beta_n \cdot (xy)^n) + O[(|x|^{n+1}|y|^n(|x|+|y|)], \\ \bar{y} &= \lambda^{-1} y(1 + \tilde{\beta}_1 \cdot xy + \ldots + \tilde{\beta}_n \cdot (xy)^n) + O[|x|^n|y|^{n+1}(|x|+|y|)], \end{aligned} \tag{1}$$

where the coefficients β_i and $\tilde{\beta}_i$ are invariants of smooth canonical changes of coordinates preserving form (1) of map T_0.

Using the method of boundary value problem [8], we can find formulae for the map $T_0^k \colon (x_0, y_0) \to (x_k, y_k)$ at all sufficiently large integers k, where $(x_i, y_i) \in U_0$, $i = 0, \dots, k$, and $(x_{i+1}, y_{i+1}) = T_0(x_i, y_i)$.

Lemma 2. *If T_0 has form (1), then T_0^k can be represented in the form*

$$\begin{aligned} x_k &= \lambda^k x_0 \cdot R_n^{(k)}(x_0 y_k) + \lambda^{(n+1)k} P_n^{(k)}(x_0, y_k), \\ y_0 &= \lambda^k y_k \cdot R_n^{(k)}(x_0 y_k) + \lambda^{(n+1)k} Q_n^{(k)}(x_0, y_k), \end{aligned} \tag{2}$$

where

$$R_n^{(k)}(x_0 y_k) \equiv$$
$$\equiv 1 + \hat\beta_1(k)\lambda^k x_0 y_k + \hat\beta_2(k)\lambda^{2k}(x_0 y_k)^2 + \ldots + \hat\beta_n(k)\lambda^{nk}(x_0 y_k)^n, \quad (3)$$

$\hat\beta_i(k)$ are ith degree polynomials in k with coefficients depending on $\beta_1, \ldots, \beta_n$, and the functions $P_n^{(k)}$, $Q_n^{(k)} = o[(x_0 y_k)^n]$ are uniformly bounded with respect to k along with all their derivatives up to the order $(r-2)$, if $n = 1$, or $(r - 2n - 1)$, if $n \geq 2$.

Naturally, these results can be applied to the analytic case too, but here, when $\lambda > 0$, we can use also the Birkhoff–Moser normal form [9]. Lemmas 1 and 2 remain valid when T_0 depends on parameters (see [2] for more details).

Let us choose in U_0 a pair of points $M^+(x^+, 0) \in W^s_{\rm loc}$ and $M^-(0, y^-) \in W^u_{\rm loc}$ of the orbit Γ. Without loss of generality, we can assume that $x^+ > 0$ and $y^- > 0$. Let Π^+ and Π^- be sufficiently small neighborhoods of the homoclinic points M^+ and M^- such that $T_0(\Pi^+) \cap \Pi^+ = \emptyset$, $T_0^{-1}(\Pi^-) \cap \Pi^- = \emptyset$. Then, a map Π^+ into Π^- along the orbits of T_0 is defined, as usual (see, e.g., [10, 11]), on the countable set of nonintersecting strips $\sigma_k^0 \equiv \Pi^+ \cap T_0^{-k}\Pi^-$. The images of the strips σ_k^0 under the maps T_0^k are the strips $\sigma_k^1 \equiv \Pi^- \cap T_0^k \Pi^+$ on Π^- and $\sigma_k^1 = T_0^k(\sigma_k^0)$. As $k \to \infty$, the strips σ_k^0 and σ_k^1 accumulate to $W^s_{\rm loc}(O)$ and to $W^u_{\rm loc}(O)$, respectively. For a given sufficiently large positive integer $\bar k$, we assume that the neighbourhoods Π^+ and Π^- contain strips σ_k^0 and σ_k^1, respectively only with numbers $k \geq \bar k$. The set of orbits lying entirely in such a neighbourhood U is accordingly denoted by $N_{\bar k} \equiv N_{\bar k}(f)$.

Obviously, $M^+ = f^q(M^-)$ for some positive integer q. The map $T_1 \equiv f^q : \Pi^- \to \Pi^+$ can be written in the form

$$\overline{x} - x^+ = F(x, y - y^-), \quad \overline{y} = G(x, y - y^-), \quad (4)$$

where $F(0,0) = G(0,0) = 0$, because $T_1(M^-) = M^+$; and $G_y(0,0) = 0$, $G_{yy}(0,0) = 2d \neq 0$, because the tangency of the manifolds $W^s(O)$ and $W^u(O)$ at the points of the orbit Γ is quadratic. We can write for functions F and G some terms of their Tailor expansions near the point $M^-(0, y^-))$ in order to show explicitly linear and essential quadratic and cubic terms:

$$F(x, y - y^-) \equiv ax + b(y - y^-) + e_{02}y^2 + \ldots, \quad (5)$$
$$G(x, y - y^-) \equiv cx + d(y - y^-)^2 + f_{20}x^2 + f_{11}xy + f_{12}xy^2 + f_{03}y^3 + \ldots,$$

(where the dots stand for both higher order terms and for nonessential quadratic and cubic terms). Note that the Jacobian $J(T_1)$ of map T_1 is equal to unity identically since T_1 is area-preserving. In particular, this implies that

$$\begin{aligned} &bc \equiv -1, \\ &R = 2ad - bf_{11} - 2e_{02}c \equiv 0, \end{aligned} \tag{6}$$

since $-bc = J(T_1)_{|M^-}$ and $R = \partial J(T_1)/\partial y_{|M^-}$.

2. Three classes of symplectic diffeomorphisms with homoclinic tangencies

Formulas (1) and (4) imply that the curve $l_u = T_1(W^u_{\mathrm{loc}} \cap \Pi^-)$ has the form of a parabola, i.e.,

$$l_u : y = \frac{d}{b^2}(x - x^+)^2(1 + O(x - x^+)).$$

Hence, the images $T_1(\sigma_j^1)$ of the strips σ_j^1 have forms of horseshoes accumulating to l_u as $j \to \infty$. Obviously, orbits from $N_{\bar{k}}$ must intersect Π^+ at the intersection points of the horseshoes $T_1(\sigma_j^1)$ and the strips σ_i^0 at all possible $i, j \geq \bar{k}$. Therefore, the structure of the set $N_{\bar{k}}$ has to depend essentially on a character of these intersections.

We say that a horseshoe $T_1(\sigma_j^1)$ has the regular intersection with a strip σ_i^0 if (i) the set $T_1(\sigma_j^1) \cap \sigma_i^0$ consists of two connected components Δ_{ij}^1 i Δ_{ij}^2; (ii) the map $T_1T_0^j$ is restricted to the preimage $(T_1T_0^j)^{-1}\Delta_{ij}^\alpha \subset \sigma_j^0$ of the component Δ_{ij}^α, where $\alpha = 1, 2$, is a saddle map in the sense of [8] (roughly speaking, this map is (exponentially) expanding along one of coordinates, y, and contracting along the another, x).

Lemma 3 [6, 12]**.** *There exist a positive constant S_1 and a sufficiently large integer $\bar{k}$ such that, for any $i, j \geq \bar{k}$, the following assertions hold.*

(i) *If*

$$d(\lambda^i y^- - c\lambda^j x^+) > S_{ij}(\bar{k}), \tag{7}$$

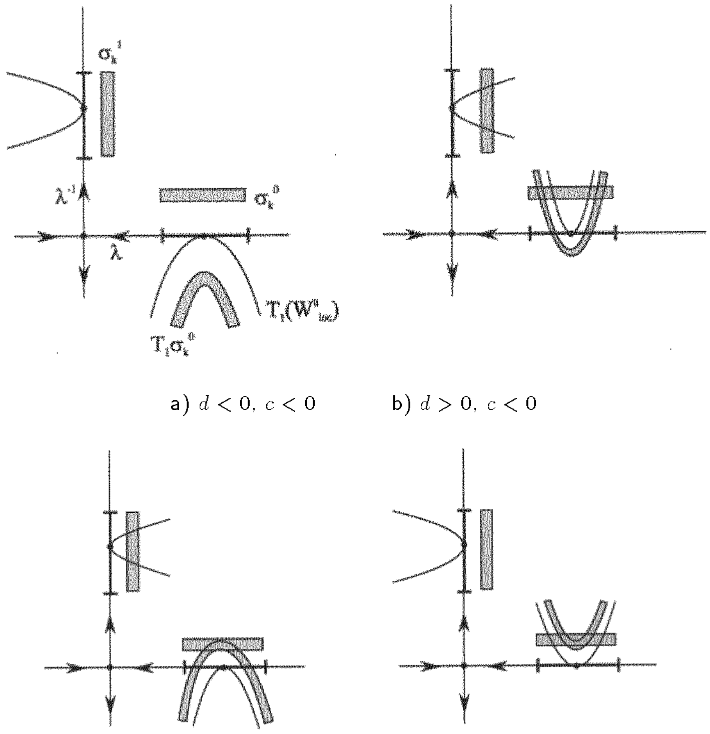
where $S_{ij} = S_1(|\lambda|^i + |\lambda|^j)|\lambda|^{\bar{k}/2}$, then the intersection of the horseshoe $T_1(\sigma_j^1)$ with the strip σ_i^0 is regular.

(ii) *If*

$$d(\lambda^i y^- - c\lambda^j x^+) < -S_{ij}(\bar{k}), \tag{8}$$

then $T_1(\sigma_j^1) \cap \sigma_i^0 = \emptyset$.

Since $\lambda > 0$, $x^+ > 0$, $y^- > 0$, the structure of solutions of inequalities (7)–(8) depends mainly on the signs of the coefficients c and d. We divide the diffeomorphisms corresponding to different signs of these coefficients into three classes as follows.



a) $d < 0$, $c < 0$ b) $d > 0$, $c < 0$

c) $d < 0$, $c > 0$ d) $d > 0$, $c > 0$

Fig. 1. Symplectic diffeomorphisms with a homoclinic tangency a) of the first class; b) of the second class; c) and d) of the third class.

The first class consists of the diffeomorphisms for which $c < 0$ and $d < 0$ (see Fig. 1a). Here, inequality (8) is valid for any $i, j \geq \bar{k}$. It means that $T_1(\sigma_j^1) \cap \sigma_i^0 = \emptyset$ always and, therefore, *the set* $N_{\bar{k}}$ *for a sufficiently large* $\bar{k}$ *has the trivial structure:* $N_{\bar{k}} = \{O, \Gamma\}$.

The second class consists of the diffeomorphisms for which $c < 0$ and $d > 0$ (see Fig. 1b). In this case, inequality (7) holds for all $i, j \geq \bar{k}$. It means that all the horseshoes $T_1(\sigma_j^1)$ and the strips σ_i^0 have the regular intersection. Therefore, the set $N_{\bar{k}}$ has a nonuniformly hyperbolic structure: all orbits from $N_{\bar{k}}$, except for Γ, are saddles. Moreover, in this case, the set $N_{\bar{k}}$ can be completely described by means of the topological Bernoulli scheme with three symbols (for more details, see [7]).

The diffeomorphisms for which either $c>0$ and $d>0$ or $c>0$ and $d<0$ belong to *the third class.* Note at once, that the case $d<0$, $c>0$ (Fig. 1c) is reduced to the case $c>0$, $d>0$ (Fig. 1d) for diffeomorphism f^{-1}. For this reason, in what follows, we assume that $c>0$ and $d>0$. In this case, the set $N_{\bar{k}}$ can have either trivial structure ($N_{\bar{k}}=\{O,\Gamma\}$) or nontrivial structure ($N_{\bar{k}}$ contains nontrivial hyperbolic subsets) depending on the sign of quantity

$$\tau=\frac{1}{\ln\lambda}\ln\frac{cx^+}{y^-}. \tag{9}$$

(Note that τ is an invariant of f since τ does not depend: (i) on a choice of the pair of homoclinic points of the orbit Γ and (ii) on smooth changes of coordinates preserving form (1) of T_0.)

Consider inequality (8). Taking the logarithms of the both sides, we obtain the inequality

$$j-i+\tau<-s|\lambda|^{\bar{k}/2}, \tag{10}$$

where s is a positive constant (independent of i, j and $\bar{k}$). By Lemma 3, if $i\geq\bar{k}$ and $j\geq\bar{k}$ satisfy (10), then the horseshoe $T_1(\sigma_j^1)$ has empty intersection with the strip σ_i^0. Note that in the case $\tau<0$ inequality (10) has solutions of form $j>i$ only. It means, in particular, that, for all $i\geqslant\bar{k}$, the horseshoes $T_1\sigma_i^1$ lie above the "own" strips σ_i^0, and therefore, all orbits, except for O and Γ, leave the neighborhood U under positive iterations of f. Thus,

if $\tau<0$, then there exists $\bar{k}=\bar{k}(\tau)$, $\bar{k}(\tau)\to\infty$ as $\tau\to 0$, such that the set $N_{\bar{k}}$ has the trivial structure: $N_{\bar{k}(\tau)}=\{O,\Gamma\}$.

The case $\tau>0$ is rather different. Here, inequality (7), which can be written in the form

$$j-i+\tau>s|\lambda|^{\bar{k}/2}, \tag{11}$$

has at sufficiently large $\bar{k}$ infinitely many integer solutions of the form $j\leq i$. In particular, (11) has always infinitely many solutions of the form $j=i$. By Lemma 3, it means that, for every sufficiently large i, the horseshoe $T_1\sigma_i^1$ has a regular intersection with the "own" strip σ_i^0. Thus,

if $\tau>0$, the set $N_{\bar{k}}$ contains nontrivial hyperbolic subsets.

Moreover, at a transition of τ from negative to positive values bifurcations occur connected with the creation of infinitely many Smale horseshoes. Below we will study main properties of these bifurcations and consider corresponding dynamical phenomena (see also [7]).

3. Proof of the existence of infinitely many generic elliptic points

We consider here one parameter families f_τ of diffeomorphisms of the third class (with $\lambda > 0$, $c > 0$, $d > 0$) when *the homoclinic tangency is not splitted and τ varies near zero.* In order to study basic bifurcations we, first of all, examine the first return maps $T_k \equiv T_1 T_0^k : \sigma_k^0 \to \sigma_k^0$.

Lemma 4 (Rescaling lemma). *For every sufficiently large k the map $T_k : \sigma_k^0 \to \sigma_k^0$, can be transformed, by linear transformation of coordinates and parameters, to the following form*

$$\begin{aligned} \bar{X} &= Y + k\lambda^{2k}\varepsilon_k^1, \\ \bar{Y} &= M - X - Y^2 + \frac{f_{03}}{d^2}\lambda^k Y^3 + k\lambda^{2k}\varepsilon_k^2, \end{aligned} \tag{12}$$

where functions $\varepsilon_k^{1,2}(X,Y,M)$ are defined on asymptotically large domain covering in the limit $k \to +\infty$ all finite values of X, Y, and M, and these functions are uniformly bounded in "k" along with all derivatives up to order $(r-4)$). Besides,

$$M = -\frac{d}{y^-}\lambda^{-k}\left(\frac{cx^+}{y^-} - 1\right)(1 + k\beta_1\lambda^k x^+ y^-)(1 + \nu_k^1) - s_0 + \nu_k^2, \tag{13}$$

where

$$s_0 = dx^+(ac + f_{11}x^+) + f_{12}x^+\left(1 - \frac{1}{4}f_{12}x^+\right) \tag{14}$$

is a quantity depending on coefficients of the global map T_1 only (see formulas (4) and (5)) and $\nu_k^{1,2} \to 0$ as $k \to +\infty$.

Proof. We will use the representation of the map T_0 in the "second normal form," i.e., in form (2) with $n = 2$. Then, by (2), (4), and (5), the map $T_k : \sigma_k^0 \to \sigma_k^0$ can be written in the form

$$\begin{aligned} &\bar{x} - x^+ = a\lambda^k x + b(y - y^-) + O(k|\lambda|^{2k}|x| + (y-y^-)^2 + |\lambda|^k|x||y-y^-|), \\ &\lambda^k \bar{y}\left(1 + k\lambda^k\beta_1\bar{x}\bar{y}\right) + k\lambda^{3k}O(|\bar{x}| + |\bar{y}|) = c\lambda^k x\left(1 + k\lambda^k\beta_1 xy\right) + d(y-y^-)^2 + \\ &+\lambda^{2k} f_{02}\, x^2 + \lambda^k f_{11}(1 + k\lambda^k\beta_1 xy)\, x(y-y^-) + \lambda^k f_{12}\, x(y-y^-)^2 + f_{03}(y-y^-)^3 + \\ &\quad + O\left((y-y^-)^4 + \lambda^{2k}|x||y-y^-| + k|\lambda|^{3k}|x| + k\lambda^{2k}|x||y-y^-|^2\right), \end{aligned} \tag{15}$$

where we use the notation $x = x_0$, $y = y_k$.

Shift the coordinates

$$\eta = y - y^-,\ \xi = x - x^+ - \lambda^k x^+(a + \rho_k^1),$$

where $\rho_k^1 = O(k\lambda^k)$ is some small coefficient, in order to nullify constant terms (independent of coordinates) in the first equation of (15). We obtain

$$\bar{\xi} = a\lambda^k\xi + b\eta + e_{02}\eta^2 + O\left(k\lambda^{2k}|\xi| + \eta^3 + |\lambda|^k O(|\xi||\eta|)\right),$$

$$\lambda^k\bar{\eta}\left(1 + 2k\lambda^k\beta_1 x^+ y^-\right) + k\lambda^{2k}\beta_1\left((y^-)^2\bar{\xi} + y^-\bar{\xi}\bar{\eta} + x^+\bar{\eta}^2 + \bar{\xi}\bar{\eta}\right) +$$

$$+k\lambda^{3k}O(|\bar{\eta}| + |\bar{\xi}|) = M_1 + c\lambda^k\xi(1 + 2k\lambda^k\beta_1 x^+ y^- + 2\lambda^k f_{02}x^+) +$$

$$+\eta^2(d + \lambda^k f_{12}x^+) + \lambda^k\eta(f_{11}x^+ + k\lambda^k\beta_1(x^+)^2(y^- + c) + \lambda^k a f_{11}) +$$

$$+f_{03}\eta^3 + +O\left(\eta^4 + k|\lambda|^{3k}|\xi| + k\lambda^{2k}(\xi^2 + \eta^2) + \lambda^k|\xi|\eta^2\right), \qquad (16)$$

where

$$M_1 = \lambda^k(cx^+ - y^-)(1 + k\lambda^k\beta_1 x^+ y^-) + \lambda^{2k}x^+(ac + f_{02}x^+) + O(k\lambda^{3k})$$

Now, we rescale the variables:

$$\xi = -\frac{b}{d + \lambda^k f_{12}x^+}\lambda^k u, \quad \eta = -\frac{1}{d + \lambda^k f_{12}x^+}\lambda^k v. \qquad (17)$$

System (16) in coordinates (u, v) is rewritten in the following form

$$\bar{u} = v + a\lambda^k u - \frac{e_{02}}{bd}\lambda^k v^2 + O(k\lambda^{2k}),$$

$$\bar{v}(1 + k\lambda^k p_k^1) = M_2 - u(1 + k\lambda^k p_k^2) - v^2 +$$

$$+v(f_{11}x^+ + k\lambda^k p_k^3) - \frac{f_{11}b}{d}\lambda^k uv + \frac{f_{03}}{d^2}\lambda^k v^3 + O(k\lambda^{2k}), \qquad (18)$$

where $M_2 = -(d+\lambda^k f_{12}x^+)\lambda^{-k}M_1$, and we denote by p_k^j some coefficients which are uniformly bounded in k. Next, we make the following simple linear change of coordinates: 1) the shift $u_{new} = u - \frac{1}{2}(f_{11}x^+ + k\lambda^k p_k^3)$, $v_{new} = v - \frac{1}{2}(f_{11}x^+ + k\lambda^k p_k^3)$, in order the linear in v term in the second equation of (18) vanish; 2) divide the second equation by the factor $(1 + k\lambda^k p_k^1)$; 3) denote by u_{new} the new linear in u term in the second equation of (18); and 4) rescale coordinates (u, v) with the same factor of form $(1 + O(k\lambda^k))$ in order to make the coefficient in front of v^2 in the second equation equal to -1 again. After this, system (18) is rewritten in the following form

$$\begin{aligned}\bar{u} &= v + a\lambda^k u - \frac{e_{02}}{bd}\lambda^k v^2 + O(k\lambda^{2k}),\\ \bar{v} &= M_3 - u - v^2 - \frac{f_{11}b}{d}\lambda^k uv + \frac{f_{03}}{d^2}\lambda^k v^3 + O(k\lambda^{2k}),\end{aligned} \tag{19}$$

where $M_3 = M_2(1 + k\lambda^k p_k^5) + f_{12}x^+(1 - \frac{1}{4}f_{12}x^+)$.

Now, we make the following linear change of coordinates

$$x = u + \tilde{\nu}_k^1 v, \quad y = v - \tilde{\nu}_k^2 u, \tag{20}$$

where

$$\tilde{\nu}_k^1 = -\frac{e_{02}}{bd}\lambda^k, \quad \tilde{\nu}_k^2 = -\frac{e_{02}}{bd}\lambda^k - a\lambda^k. \tag{21}$$

Then, system (19) is rewritten as

$$\begin{aligned}\bar{x} &= y + M_3\tilde{\nu}_k^1 + O(k\lambda^{2k}),\\ \bar{y} &= M_3 - x - y^2 + a\lambda^k y - \tilde{R}\lambda^k xy + \frac{f_{03}}{d^2}\lambda^k y^3 + O(k\lambda^{2k}),\end{aligned} \tag{22}$$

where $\tilde{R} = (2a + 2e_{02}/bd - bf_{11}/d) \equiv 0$ by (6). Hence, map (22) takes the form

$$\begin{aligned}\bar{x} &= y + M_3\tilde{\nu}_k^1 + O(k\lambda^{2k}),\\ \bar{y} &= M_3 - x - y^2 + a\lambda^k y + \frac{f_{03}}{d^2}\lambda^k y^3 + O(k\lambda^{2k}),\end{aligned} \tag{23}$$

Finally, under one more shift of coordinates

$$X = x - \frac{1}{2}a\lambda^k - \nu_k^1 M, \quad Y = y - \frac{1}{2}a\lambda^k,$$

in nullifying in (23) the constant term in the first equation and the linear in y term in the second equation. After this, we obtain the final form (12) of map T_k in the rescaled coordinates where formula (13) for parameter M holds. The lemma is proved. □

Map (12) is asymptotically close to the conservative Hénon map at $k \to \infty$

$$\bar{x} = Y, \quad \bar{y} = M - Y^2 - X, \tag{24}$$

in any bounded domain. Thus, we can recover bifurcations of single-round periodic orbits in the family f_τ by studying bifurcations of fixed points of Hénon map (24) or generalized conservative Hénon map of the form

$$\bar{x} = Y, \quad \bar{y} = M - Y^2 - X + \frac{f_{03}}{d^2}\lambda^k Y^3. \tag{25}$$

(The last is rather important when we want to study bifurcations of single-round periodic orbits with multipliers $e^{\pm i\pi/2}$, because in the Hénon map this bifurcation is degenerate.)

Bifurcations of fixed points in the conservative Hénon family are well-known. For example, map (24) has a generic elliptic fixed point for all values $M \in (-1; 3)$ except two values: (i) $M = 0$, when $\psi = \pi/2$, and (ii) $M = 5/4$, when $\psi = 2\pi/3$. When $M = -1$ the Hénon map has a fixed parabolic point (with multipliers $\nu_1 = \nu_2 = +1$), and when $M = 3$ the fixed point with multipliers $\nu_1 = \nu_2 = -1$ exists. It is also known (see, e.g., [13]) that, if $M > 5 + 2\sqrt{5}$ (this is a sufficient condition), then the nonwandering set of map is an invariant hyperbolic set on which (24) is conjugate to the topological Bernoulli scheme with two symbols.

It is clear that, at sufficiently large k, dynamics and bifurcations of map T_k (taking form (12) after rescaling) are described in a similar way. Note that the parameter M depends on invariant $\alpha \equiv (cx^+/-y^- - 1)$ and, by (13), bifurcations in (12) at large k may occur only for small α (the larger is k, the smaller is α). On the other hand, α is a single-valued function of τ:

$$\tau = \frac{1}{\ln\lambda}\ln(1+\alpha) \;\Leftrightarrow\; \alpha = \lambda^\tau - 1$$

and $\alpha = 0$ corresponds to $\tau = 0$. Thus, we can consider τ as the controlling parameter (in the class of diffeomorphisms with the homoclinic tangency) and, then, by (12),

$$M = -\frac{d}{y^-}\lambda^{-k}(\lambda^\tau - 1)(1 + k\beta_1\lambda^k x^+ y^-)(1 + \nu_k^1) - s_0 + \nu_k^2. \tag{26}$$

Hence, we can calculate directly bifurcation values of τ corresponding to bifurcations of fixed points of first return maps T_k at all large k. Since the Hénon map (24) has a fixed point (parabolic) with multipliers $\nu_1 = \nu_2 = +1$ at $M = -1$ and a fixed point with multipliers $\nu_1 = \nu_2 = -1$ at $M = 3$, we find from (26) that

$$\tau_k^{+1} = -\frac{s_0 - 1 + \dots}{dy^- \ln\lambda}\,\lambda^k, \quad \tau_k^{-1} = -\frac{3 + s_0 + \dots}{dy^- \ln\lambda}\,\lambda^k.$$

Besides,

$$\tau_k^{\pi/2} = -\frac{s_0 + \dots}{dy^- \ln\lambda}\,\lambda^k, \quad \tau_k^{2\pi/3} = -\frac{5/4 + s_0 + \dots}{dy^- \ln\lambda}\,\lambda^k.$$

Thus, we can formulate the following result of existence of single-round elliptic periodic orbits at diffeomorphisms of the family f_τ.

In any neighbourhood of the point $\tau = 0$ there is a countable set of intervals $\delta_k = (\tau_k^{+1}, \tau_k^{-1})$ such that, for $\tau \in \delta_k$, the diffeomorphism f_τ has a single-round elliptic periodic orbit of period $(k+q)$. This point is generic elliptic if $\tau \notin \{\tau_k^{\pi/2}, \tau_k^{2\pi/3}\}$.

We pay special attention to the cases where infinitely many generic elliptic periodic orbits exist. Thus, from the previous statement we deduce immediately the following

Theorem. *Let $\tau = 0$, $-3 < s_0 < 1$, and $s_0 \notin \{0; -5/4\}$. Then, there exists such k_0 that the diffeomorphism f_0 has a countable set of KAM-stable*[1] *single-round elliptic periodic orbits of all successive periods beginning with $k_0 + q$.*

It follows from rescaling lemma and (26) that at $\tau = 0$ all maps T_k for suffifciently large k demonstrate "the same" dynamics which is, in reality, the dynamics of the Hénon map with $M = -s_0$. We can deduce from here the following

Corollaries. (1) *The single-round elliptic periodic orbits from the Theorem have "the same multipliers" $e^{\pm i\psi_k}$ such that $\cos\psi_k = 1 - \sqrt{1 - s_0} + \alpha_k$ where $\alpha_k \to 0$ as $k \to \infty$.*

(2) *The diameter of the elliptic island sorrounding the elliptic fixed point of the Hénon map is of order one (except for the unstable case $\psi = 2\pi/3$). Then, from formulas (17) and (2) we obtain that, in the initial coordinates on the strip σ_k^0, the corresponding elliptic island will have sizes of order λ^k in x-coordinate and of order λ^{2k} in y-coordinate.*

About some other dynamical properties of the family f_τ see [7]. In particular, we note that bifurcations of the appearance of "bi-horseshoes" occur also in the family f_τ when τ varies near $\tau \in Z^+$, and these bifurcations give a rise of elliptic (two-round) periodic points.

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[1]At $s_0 = -9/16$, it is necessary to require smoothness $r \geq 9$, because the first Birkhoff coefficient of an elliptic periodic point with multipliers $e^{\pm i\psi}$ with $\psi = arccos(-1/4)$ equals to zero and the second one is nonzero [14].

References

1. S. V. Gonchenko, L. P. Shilnikov, *On two-dimensional analytic area-preserving diffeomorphisms with infinitely many stable elliptic periodic points.* — Regular and Chaotic Dynamics **2**, Nos. 3/4 (1997), 106–123.
2. S. V. Gonchenko, L. P. Shilnikov, *On two-dimensional area-preserving diffeomorphisms with infinitely many elliptic islands.* — J. Stat. Phys. **101**, Nos. 1/2 (2000), 321–356.
3. S. V. Gonchenko, L. P. Shilnikov, *Invariants of Ω-conjugacy of diffeomorphisms with a structurally unstable homoclinic trajectory.* — Ukrainian Math. J. **42**, No. 2 (1990), 134–140.
4. S. V. Gonchenko, L. P. Shilnikov, *On moduli of systems with a structurally unstable homoclinic Poincare curve.* Izv. RAN **41** (1992), 417–445.
5. S. V. Gonchenko, L. P. Shilnikov, *On dynamical systems with structurally unstable homoclinic curves.* — Soviet Math. Dokl. **33**, No. 1 (1986), 234–238.
6. S. V. Gonchenko, L. P. Shilnikov, *Arithmetic properties of topological invariants of systems with a structurally unstable homoclinic trajectory.* — Ukr. Math. J.**39**, No. 1 (1987), 21–28.
7. S. V. Gonchenko, L. P. Shilnikov, *On two-dimensional area-preserving mappings with homoclinic tangencies.* — Dokl. Mathem. **63**, No. 3 (2001), 395–399.
8. L. P. Shilnikov, *On a Poincaré-Birkhoff problem," Math. Sb. USSR* **3** *(1967), 353–371.*
9. J. Moser, *The analytic invariants of an area-preserving mapping near a hyperbolic fixed point.* — Comm. of Pure and Appl. Math. **9** (1956), 673–692.
10. N. K. Gavrilov, L. P. Shilnikov, *On three-dimensional dynamical systems close to systems with a structurally unstable homoclinic curve.* I. — Math. Sb. USSR **17** (1972), 467–485; II, **19** (1973), 139–156.
11. L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev, L. O. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics, Part I, World Scientific (1998).*
12. S. V. Gonchenko, L. P. Shilnikov, *On geometrical properties of two-dimensional diffeomorphisms with homoclinic tangencies.* Int. J. of Bifurcation and Chaos **5**, No. 3 (1995), 819–829.
13. V. S. Afraimovich, L. P. Shilnikov, in: Nonlinear Dynamics and Turbulence, eds. G. I. Barenblatt, G. Iooss, and D. D. Joseph (Boston, Pitmen) (1983), pp. 1–34.
14. V. S. Biragov, *Bifurcations in a two-parameter family of conservative mappings that are close to the Henon map.* — Selecta Math. Sov. **9** (1990), 273–282; Originally published in: *Methods of qualitative theory of differential equations*, Gorky State University (1987), pp. 10–24.

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Поступило 30 ноября 2002 г.