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ON BIFURCATIONS OF THREE-DIMENSIONAL DIFFEOMORPHISMS WITH A HOMOCLINIC TANGENCY TO A "NEUTRAL" SADDLE FIXED POINT

ABSTRACT. Bifurcations of single-round periodic orbits of three-dimensional diffeomorphisms with a quadratic homoclinic tangency of manifolds of a saddle fixed point with the saddle value equal to 1 are studied.

We consider C^r-smooth ($r \geq 5$) three-dimensional diffeomorphism f_0 having a fixed saddle point O with multipliers $\lambda_1, \lambda_2, \gamma$ where $0 < \lambda_2 < \lambda_1 < 1 < \gamma$. We assume that the following conditions are fulfilled

A. $\lambda_1 \lambda_2 \gamma < 1$,

B. $\lambda_1 \gamma = 1$,

C. $W^u(O)$ and $W^s(O)$ have a quadratic tangency at the points of a homoclinic orbit Γ_0,

D. the extended unstable manifold of O and the strong stable foliation on $W^s(O)$ intersect transversely at the points of Γ_0.

Condition (B) means that the saddle point is of neutral type. Conditions (C) and (D) mean that the homoclinic tangency is simple (see [1]).

Diffeomorphisms satisfying (A)–(D) and close to f_0 compose a bifurcation surface $\mathcal{H}$ of codimension two in the space of three-dimensional diffeomorphisms. For study of bifurcations we consider two-parameter families f_μ, where $\mu = (\mu_1, \mu_2)$, which are transverse to $\mathcal{H}$. The transversality means that we can take μ_1 as a parameter of splitting of manifolds $W^s(O)$ and $W^u(O)$ with respect to some homoclinic point; and we can take μ_2 as a deviation of the saddle value $\lambda_1\gamma$ from unity, i.e., $\mu_2 = \lambda_1(\mu)\gamma(\mu) - 1$.

We consider a sufficiently small neighbourhood $U(O \cup \Gamma_0)$, which is a union of a neighbourhood U_0; of point O and neighbourhoods of those

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points of Γ_0 that lie outside U_0. We are interested in bifurcations of *single-round periodic orbits*. Such an orbit lies entirely in U and has exactly one intersection point with each of the neighborhoods of the set U/U_0. Every point of such an orbit is a fixed point of the corresponding *first return map*.

Let $M^+ \in W^s_{\rm loc}$ and $M^- \in W^u_{\rm loc}$ be two points of Γ_0 on U_0, and $\Pi^+ \subset U_0$ and $\Pi^- \subset U_0$ be sufficiently small neighbourhoods of M^+ and M^-, respectively. Evidently, $M^+ = f^{n_0}(M^-)$ for some integer n_0.

The first return maps $T_k \equiv f_\mu^{k+n_0}$ Π^+ into Π^+ (for a single-round periodic orbit of period $(k+n_0)$) can be represented in the form [2]

$$T_k \equiv T_1 T_0^k.$$

Here, T_0 is a *local map*, i.e., the restriction of diffeomorphism f_μ onto U_0 ($T_0 \equiv f_\mu|_{U_0}$), and $T_1 \equiv f_\mu^{n_0} : \Pi^- \to \Pi^+$ is a *global map* which is defined by orbits of f_μ close to a piece of Γ_0. Note, that maps $T_0^k : \Pi^+ \to \Pi^-$ are defined for every sufficiently large k (the domain of T_0^k is a "strip" σ_k^0 on Π^+, namely, $\sigma_k^0 = T_0^{-k}(\Pi^-) \cap \Pi^+$; and the range of T_0^k is a "strip" σ_k^1 on Π^-, namely, $\sigma_k^1 = T_0^k(\Pi^+) \cap \Pi^- = T_0^k(\sigma_k^0)$).

For construction of T_0^k for all sufficiently large k we will use coordinates (introduced in [3, 4]) such that T_0 has a form (the so-called *main normal form*)

$$\begin{aligned} \bar{x}_1 &= \lambda_1(\mu)x_1 + \widetilde{H}_2(y,\mu)x_2 + O(x^2y), \\ \bar{x}_2 &= \lambda_2(\mu)x_2 + \widetilde{R}_2(x,\mu) + \widetilde{H}_4(y,\mu)x_2 + O(x^2y), \\ \bar{y} &= \gamma(\mu)y + O(xy^2), \end{aligned} \tag{1}$$

where the right-hand side expressions are C^{r-1} in coordinates and C^{r-2} in parameters; $\widetilde{H}_{2,4}(0,\mu) = 0$, $\widetilde{R}_2(0,\mu) = 0$, $\partial\widetilde{R}_2(0,\mu)/\partial x = 0$. In these coordinates the map $T_0^k : (x_{01}, x_{02}, y_0) \to (x_{k1}, x_{k2}, y_k)$ can be written in the form (see for more details [4])

$$\begin{aligned} x_{k1} - \lambda_1^k(\mu)x_{01} &= \widehat{\lambda}^k \xi_{k1}(x_0, y_k, \mu), \\ x_{k2} - \lambda_2^k(\mu)x_{02} &= \widehat{\lambda}^k \xi_{k2}(x_0, y_k, \mu), \\ y_0 - \gamma^{-k}(\mu)y_k &= \widehat{\gamma}^{-k} \eta_k(x_0, y_k, \mu), \end{aligned} \tag{2}$$

where $0 < \widehat{\lambda} < \lambda_1$, $\widehat{\gamma} > \gamma$; and functions ξ_k and η_k are uniformly bounded in k along with all derivatives up to order $(r-2)$ and derivatives of order $(r-1)$ in coordinates tend to 0 as $k \to \infty$. Note, that in coordinates (1), $W^s_{\rm loc}(O)$ and $W^u_{\rm loc}(O)$ *have the equations* $y = 0$ *and* $x_1 = 0, x_2 = 0$, respectively. Thus, the point M^- is $(0,0,y^-)$ and M^+ is $(x_1^+, x_2^+, 0)$. We assume, without loss of generality, that $y^- > 0, x_1^+ > 0$. Hence, the global map $T_1(\mu) : \Pi^- \to \Pi^+$ can be represented in the following form

$$\begin{aligned}
\overline{x}_1 - x_1^+ &= a_{11}x_1 + a_{12}x_2 + b_1(y - y^-) + e_{02}(y - y^-)^2 + \dots , \\
\overline{x}_2 - x_2^+ &= a_{21}x_1 + a_{22}x_2 + b_2(y - y^-) + \dots , \\
\overline{y} &= \mu_1 + c_1x_1 + c_2x_2 + d(y - y^-)^2 + f_{11}x_1(y - y^-) + f_{03}y^3 + \dots ,
\end{aligned} \tag{3}$$

where all coefficients depend, in general, on μ and

$$d \neq 0 \quad \text{and} \quad b_1c_1 \neq 0.$$

(the first inequality follows from condition (C)) of the quadraticity of the tangency; the second one follows from condition (D), see for more details [1, 5]).

Consider now the first return map $T_k = T_1T_0^k : \sigma_k^0 \to \sigma_k^0$. Using formulas (2) and (3) we can write it in the coordinates (x_1, x_2, y). As in [5], we show that, at every sufficiently large k, the three-dimensional map T_k has on σ_k^0 a C^{r-1}-smooth nonlocal two-dimensional center manifold $\mathcal{M}_k^c$ of a form $x_2 = h_k(x_1, y, \mu)$ where $\|h_k\|_{C^{r-2}} \to 0$ as $k \to \infty$. The manifold $\mathcal{M}_k^c$ is asymptotically stable along x_2-direction and the map T_k in the restriction on $\mathcal{M}_k^c$ can be reduced to a special form.

Lemma 1 (Rescaling lemma)**.** *By means of C^{r-1}-change of variables $(x_1 \to X, y \to Y)$ and rescaling parameters $(\mu_1 \to M_1, \mu_2 \to M_2)$ the map $T_k|_{\mathcal{M}_k^c}$ can be transformed to the form*

$$\begin{aligned}
\overline{X} &= Y \\
\overline{Y} &= M_1 - M_2X - Y^2 + R\lambda_1^kX_1Y + S\gamma^{-k}Y^3 + O(\widehat{\lambda}^k),
\end{aligned} \tag{4}$$

where

$$R = 2a_{11} - \frac{2e_{02}c_1}{d} - \frac{f_{11}b_1}{d}, \quad S = \frac{f_{03}}{d^2}$$

and

$$\begin{aligned}
M_1 &= -d\gamma^{2k}[\mu_1 + \lambda_1^kc_1x_1^+ - \gamma^{-k}y^- + O(\widehat{\lambda}^k)], \\
M_2 &= -b_1c_1(1 + \mu_2)^k + O(\widehat{\lambda}^k)
\end{aligned} \tag{5}$$

and, in the limit $k \to \infty$, values of parameter M_1 and coordinates X, Y cover all finite values, whereas, parameter M_2 run either all positive *values in the "orientable case" $b_1 c_1 < 0$ or all negative values in the "nonorientable case" $b_1 c_1 > 0$.*

Map (4) is called *the generalized Hénon map*. Such a map was found in [6, 7] as a rescaled first return map in the case of two-dimensional diffeomorphisms close to a diffeomorphism with a homoclinic tangency to a fixed saddle point of neutral type. Map (4) has bifurcation curves L_k^+, L_k^-, and L_k^φ on the parameter plane (M_1, M_2) corresponding to existence of fixed point with multiplier $+1$ (saddle-node bifurcation), with multiplier -1 (period-doubling bifurcation) and with pair of multipliers $e^{\pm i\psi}, 0 < \psi < \pi$ (Andronov-Hopf bifurcation), respectively. The equations of this curves are

$$
\begin{aligned}
L_k^+ &: M_1 = -\frac{(1+M_2)^2}{4}(1 + o(\lambda_1^k)),\\
L_k^- &: M_1 = \frac{3(1+M_2)^2}{4}(1 + O(\lambda_1^k)),\\
L_k^\varphi &: M_1 = cos^2\psi - 2\cos\psi + O(\lambda^k),\\
&\quad M_2 = 1 - R\lambda^k \cos\psi + o(\lambda^k),\\
&\quad 0 < \psi < \pi,
\end{aligned}
\tag{6}
$$

where the last curve is written in the parametric form where the parameter is the argument of the complex multiplier.

Note, when $R \neq 0$ the generalized Hénon map, in contrast with the standard Hénon map, demonstrates nondegenerate Andronov–Hopf bifurcation [6] when parameters M_1 and M_2 cross the curve L_k^φ.

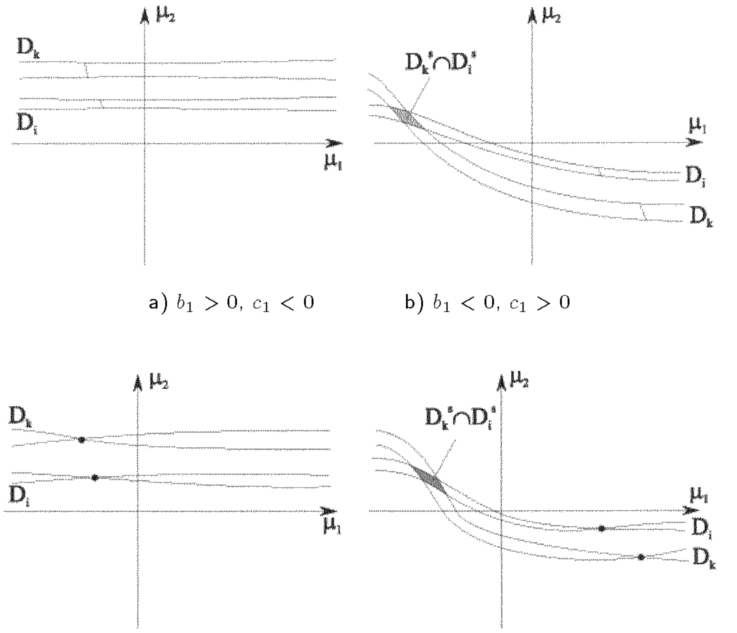
Formulas (5) and (6) allow to project the bifurcation curves L_k^+, L_k^-, and L_k^φ from the plane (M_1, M_2) to the plane (μ_1, μ_2) of the initial parameters. In this way we obtain the following result.

Theorem 1. (1) *There exist infinitely many domains D_k on the parameter plane (μ_1, μ_2) corresponding to the existence of a single-round periodic orbit of period n_0+k. Every domain D_k consist of two regions D_k^s and D_k^u; when $\mu \in D_k^s$ (resp., $\mu \in D_k^u$) the diffeomorphism f_μ has the stable periodic orbit (resp., the saddle periodic orbit with a two-dimensional unstable invariant manifold); as $k \to +\infty$ regions D_k^s accumulate on the half-axes $(\mu_2 = 0, \mu_1 < 0)$ and regions D_k^u accumulate on the half-axes $(\mu_2 = 0, \mu_1 > 0)$.*

(2) *When $R \neq 0$, there exist infinitely many domains D_k^φ on the parameter plane (μ_1, μ_2) accumulating at the origin as $k \to +\infty$ such that*

at $\mu \in D_k^{\varphi}$ diffeomorphism f_μ has a closed invariant periodic curve. This curve is stable in the "orientable case" $b_1 c_1 < 0$.

Note, that the domains D_k and D_i with $k \neq i$ can intersect. The following cases of intersection of domains D_k for different k are possible (see Fig. 1).



a) $b_1 > 0$, $c_1 < 0$ b) $b_1 < 0$, $c_1 > 0$

c) $b_1 < 0$, $c_1 < 0$ d) $b_1 > 0$, $c_1 > 0$

Fig. 1: a) and b) – "orientable case"; c) and d) – "nonorientable case.

A) If $c_1 < 0$, then in any sufficiently small fixed neighbourhood of the origin on the parameter plane (μ_1, μ_2) domains D_k and D_i do not intersect for sufficiently large $k \neq i$ (see Fig. 1a and 1c).

B) If $c_1 > 0$, then in any sufficiently small fixed neighbourhood of the origin on the parameter plane (μ_1, μ_2) domains D_k and D_i intersect always for sufficiently large k and i (see Fig. 1b and 1d).

Thus, in case B) *stable* single-round periodic orbits of different periods can coexist when $D_k^s \cap D_i^s \neq \emptyset$. Following [8], we can show that it is the case when, for example, $c_1 x_1^+ > y^-$, $-1 < b_1 c_1 < 0$. Note that in general case (of codimension one homoclinic tangency) intervals of existence of single-round stable periodic orbits do not intersect [2].

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